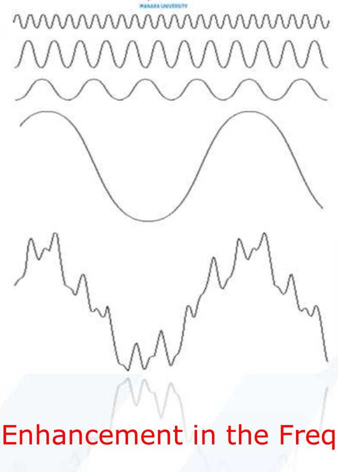


Chapter 4

Image Enhancement in the Frequency Domain

Fundamentals

- *Fourier: a periodic function can be represented by the sum of sines/cosines of different frequencies multiplied by a different coefficient (Fourier series)*

- *Non-periodic functions can also be represented as the integral of sines/cosines multiplied by weighing function (Fourier transform)*

Introduction to the Fourier Transform

- *f(x): continuous function of a real variable x*
- *Fourier transform of f(x):*

$$\Im\{f(x)\} = F(u) = \int_{-\infty}^{\infty} f(x)\exp[-j2\pi ux]dx \quad \text{Eq. 1}$$

where $j = \sqrt{-1}$

Introduction to the Fourier Transform

- *(u) is the frequency variable.*
- *The integral of Eq. 1 shows that F(u) is composed of an infinite sum of sine and cosine terms and…*
- *Each value of u determines the frequency of its corresponding sine-cosine pair.*

Introduction to the Fourier Transform

- *Given F(u), f(x) can be obtained by the inverse Fourier transform:*

$$\Im^{-1}\{F(u)\} = f(x)$$
$$= \int_{-\infty}^{\infty} F(u)\exp[j2\pi ux]du$$

- The above two equations are the Fourier transform pair.

Introduction to the Fourier Transform

- *Fourier transform pair for a function f(x,y) of two variables:*

$$\Im\{f(x,y)\} = F(u,v) = \int\int_{-\infty}^{\infty} f(x,y)\exp[-j2\pi(ux+vy)]dxdy$$

and

$$\Im^{-1}\{F(u,v)\} = f(x,y) = \int\int_{-\infty}^{\infty} F(u,v)\exp[j2\pi(ux+vy)]dudv$$

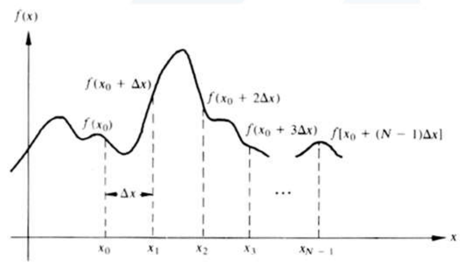
where u,v are the frequency variables.

Discrete Fourier Transform

- *A continuous function f(x) is discretized into a sequence:*

$$\{f(x_0), f(x_0+\Delta x), f(x_0+2\Delta x),\ldots, f(x_0+[N-1]\Delta x)\}$$

by taking N or M samples □x units apart.



Discrete Fourier Transform

- *Where x assumes the discrete values (0,1,2, 3, …, M-1) then*

$$f(x) = f(x_0 + x\Delta x)$$

- The sequence {f(0),f(1),f(2),...f(M-1)} denotes any M uniformly spaced samples from a corresponding continuous function.

Discrete Fourier Transform

- *The discrete Fourier transform pair that applies to sampled functions is given by:*

$$F(u)=\frac{1}{M}\sum_{x=0}^{M-1}f(x)\exp[-j2\pi ux/M]$$ For u=0,1,2,...,M-1

and

$$f(x)=\sum_{u=0}^{M-1}f(u)\exp[j2\pi ux/M]$$ For x=0,1,2,...,M-1

Discrete Fourier Transform

- *To compute F(u) we substitute u=0 in the exponential term and sum for all values of x*
- *We repeat for all M values of u*
- *It takes M*M summations and multiplications*

$$F(u)=\frac{1}{M}\sum_{x=0}^{M-1}f(x)\exp[-j2\pi ux/M]$$ For u=0,1,2,...,M-1

- *The Fourier transform and its inverse always exist!*

Discrete Fourier Transform

- *The values u = 0, 1, 2, …, M-1 correspond to samples of the continuous transform at values 0, Δu, 2Δu, …, (M-1)Δu.*
- *i.e. F(u) represents F(uΔu), where:*

$$\Delta u = \frac{1}{M\Delta x}$$

Details

$$e^{j\theta} = \cos\theta = j\sin\theta$$

$$\cos(-\theta) = \cos(\theta)$$

$$F(u) = \frac{1}{M}\sum_{x=0}^{M-1} f(x)[\cos 2\pi ux/M - j\sin 2\pi ux/M]$$

- *Each term of the FT (F(u) for every u) is composed of the sum of all values of f(x)*

Introduction to the Fourier Transform

- *The Fourier transform of a real function is generally complex and we use polar coordinates:*

$$F(u) = R(u) + jI(u)$$

$$F(u) = |F(u)| e^{j\phi(u)}$$

$$|F(u)| = [R^2(u) + I^2(u)]^{1/2}$$

$$\phi(u) = \tan^{-1}\left[\frac{I(u)}{R(u)}\right]$$

Introduction to the Fourier Transform

- *|F(u)| (magnitude function) is the Fourier spectrum of f(x) and ϕ(u) its phase angle.*
- *The square of the spectrum*

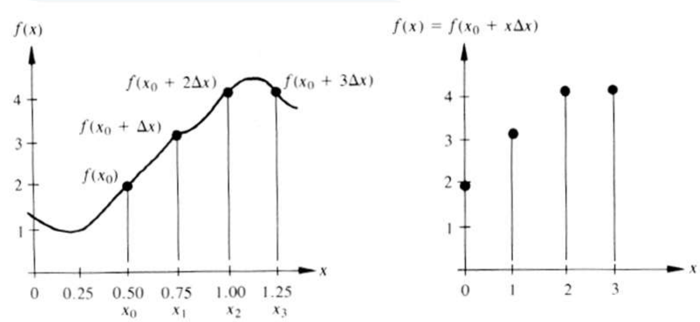
$$P(u) = |F(u)|^2 = R^2(u) + I^2(u)$$

is referred to as the power spectrum of f(x) (spectral density).

Introduction to the Fourier Transform

- *Fourier spectrum:* $|F(u,v)| = \left[R^2(u,v) + I^2(u,v)\right]^{1/2}$
- Phase: $\phi(u,v) = \tan^{-1}\left[\frac{I(u,v)}{R(u,v)}\right]$
- Power spectrum: $P(u,v) = |F(u,v)|^2 = R^2(u,v) + I^2(u,v)$

Discrete Fourier Transform



Discrete Fourier Transform

- *In a 2-variable case, the discrete FT pair is:*

$$F(u,v) = \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)\exp[-j2\pi(ux/M + vy/N)]$$

For u=0,1,2,...,M-1 and v=0,1,2,...,N-1

AND: $$f(x,y) = \sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u,v)\exp[j2\pi(ux/M + vy/N)]$$

For x=0,1,2,...,M-1 and y=0,1,2,...,N-1

Discrete Fourier Transform

- *Sampling of a continuous function is now in a 2-D grid (Δx, Δy divisions).*
- *The discrete function f(x,y) represents samples of the function $f(x_0+x\Delta x, y_0+y\Delta y)$ for x=0,1,2,…,M-1 and y=0,1,2,…,N-1.*

$$\Delta u = \frac{1}{M\Delta x}, \qquad \Delta v = \frac{1}{N\Delta y}$$

Discrete Fourier Transform

- *When images are sampled in a square array, M=N and the FT pair becomes:*

$$F(u,v)=\frac{1}{N}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} f(x,y)\exp[-j2\pi(ux+vy)/N]$$

For u,v=0,1,2,...,N-1

AND: $$f(x,y)=\frac{1}{N}\sum_{u=0}^{N-1}\sum_{v=0}^{N-1} F(u,v)\exp[j2\pi(ux+vy)/N]$$

For x,y=0,1,2,...,N-1

Basic Properties

- *Common practice:*

$$\Im\left[f(x,y)(-1)^{x+y}\right]=F(u-m/2,v-N/2)$$

- *F(0,0) is at u=M/2 and v=N/2*
- *Shifts the origin of F(u,v) to (M/2, N/2), i.e. the center of M x N of the 2-D DFT (frequency rectangle)*
- *Frequency rectangle:*
  from u=0 to u=M-1, and v=0 to v=N-1 (u,v integers, M,N even numbers)
- *In computers:*
  summations are from u=1 to M and v=1 to N
  center of transform: u=(M/2) +1 and v=(N/2) +1

Basic Properties

- *Value of transform at (u,v)=(0,0):*

$$F(0,0)=\frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1}f(x,y)$$

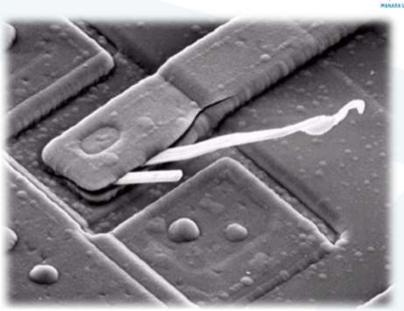
which means that the value of FT at the origin = the average gray level of the image

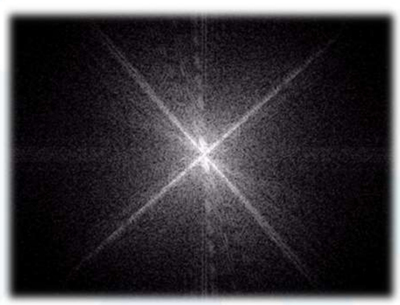
- *FT is also conjugate summetric:*

F(u,v)=F(-u,-v)*

so |F(u,v)|=|F(-u,-v)|

which means that the FT spectrum is symmetric.

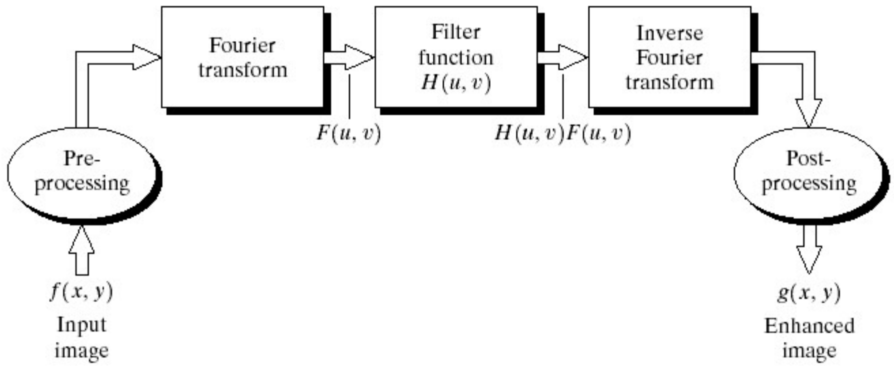




(a) SEM image of a damaged integrated circuit. (b) Fourier spectrum of (a).

https://manara.edu.sy/

Frequency domain filtering operation



Basic steps for filtering in the frequency domain.

Filtering in the Frequency Domain

- *Compute Fourier transform of image*
- *Multiply the result by a filter transfer function (or simply filter).*
- *Take the inverse transform to produce the enhanced image.*
- *Summary:*
  $G(u,v) = H(u,v)\, F(u,v)$
  Filtered Image =

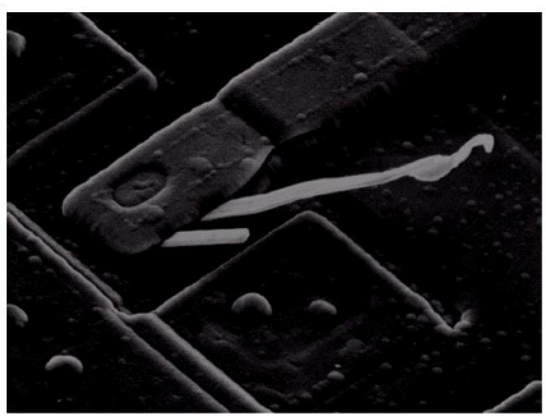
$$\Im^{-1}[G(u,v)]$$

Basic Filters

- *To force the average value of an image to 0:*
  - F(0,0) gives the average value of an image
  - then, since F(0,0)=0, take the inverse

- *Notch filter*

$$H(u,v) = \begin{cases} 0 & \text{if } (u,v) = M/2, N/2 \\ 1 & \text{otherwise} \end{cases}$$



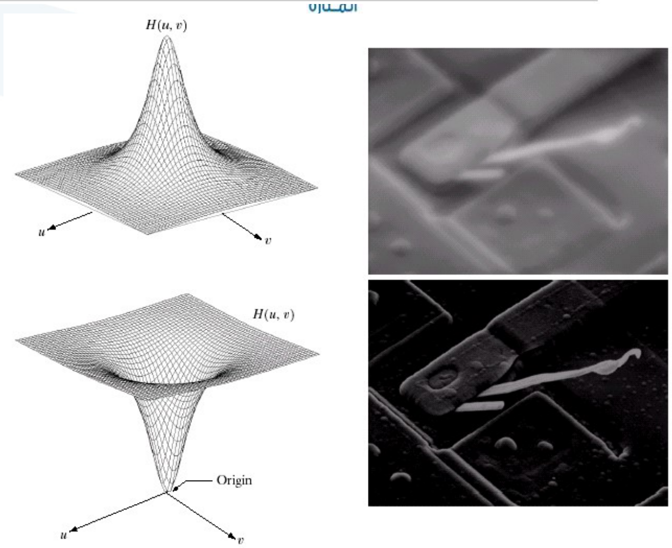
Result of filtering the image in Fig. 4.4(a) with a notch filter that set to 0 the $F(0, 0)$ term in the Fourier transform.

Enhancement in the Frequency Domain

- *Types of enhancement that can be done:*
  - Lowpass filtering: reduce the high-frequency content -- blurring or smoothing
  - Highpass filtering: increase the magnitude of high-frequency components relative to low-frequency components -- sharpening.

a b
c d

FIGURE 4.7 (a) A two-dimensional lowpass filter function. (b) Result of lowpass filtering the image in Fig. 4.4(a). (c) A two-dimensional highpass filter function. (d) Result of highpass filtering the image in Fig. 4.4(a).



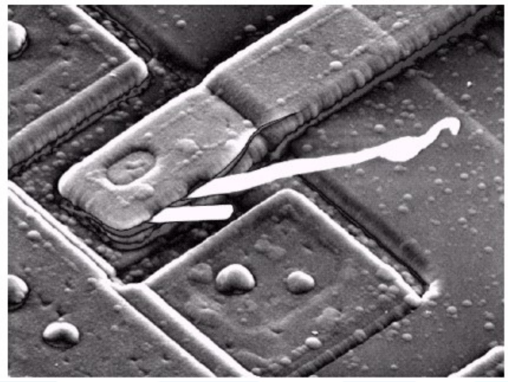


FIGURE 4.8 Result of highpass filtering the image in Fig. 4.4(a) with the filter in Fig. 4.7(c), modified by adding a constant of one-half the filter height to the filter function. Compare with Fig. 4.4(a).

Spatial and Frequency Domain

- *Convolution Theorem*

- *Discrete convolution of two functions (MxN)*

$$f(x,y) * h(x,y) = \frac{1}{MN}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f(m,n)h(x-m,y-n)$$

$$g(x,y) = \sum_{s=-a}^{a}\sum_{t=-b}^{b} w(s,t)f(x+s,y+t)$$

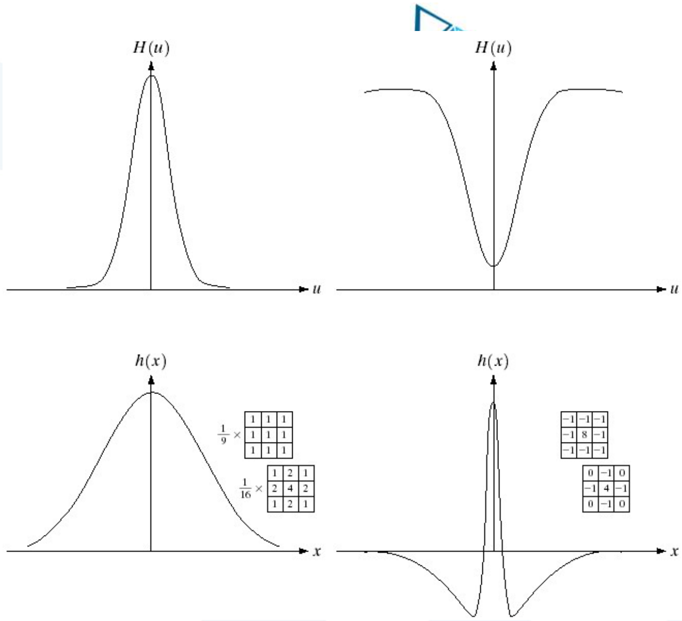
- $f(x,y)*h(x,y) \Leftrightarrow F(u,v)H(u,v)$

- $f(x,y)h(x,y) \Leftrightarrow F(u,v)*H(u,v)$

Spatial & Frequency Domain

$$f(x,y) * h(x,y) \Leftrightarrow F(u,v)\,H(u,v)$$
$$\delta(x,y) * h(x,y) \Leftrightarrow \Im[\delta(x,y)]\,H(u,v)$$
$$h(x,y) \Leftrightarrow H(u,v)$$

Filters in the spatial and frequency domain form a FT pair, i.e. given a filter in the frequency domain we can get the corresponding one in the spatial domain by taking its inverse FT



a b
c d

FIGURE 4.9 (a) Gaussian frequency domain lowpass filter. (b) Gaussian frequency domain highpass filter. (c) Corresponding lowpass spatial filter. (d) Corresponding highpass spatial filter. The masks shown are used in Chapter 3 for lowpass and highpass filtering.

Enhancement in the Frequency Domain

- *Types of enhancement that can be done:*
  - Lowpass filtering: reduce the high-frequency content -- blurring or smoothing
  - Highpass filtering: increase the magnitude of high-frequency components relative to low-frequency components -- sharpening.

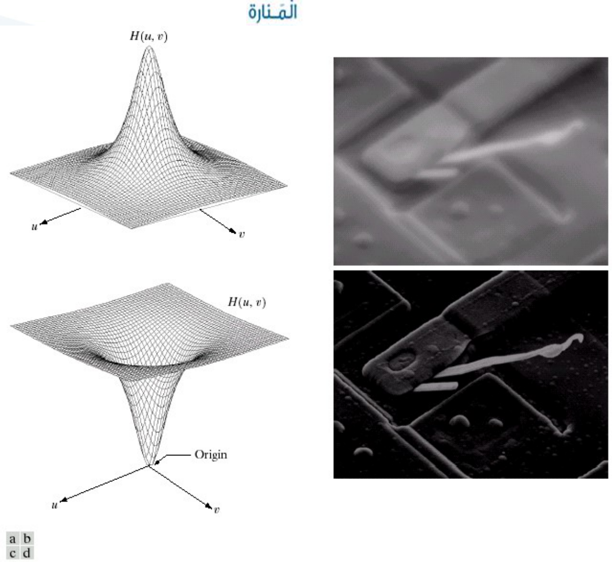


FIGURE 4.7 (a) A two-dimensional lowpass filter function. (b) Result of lowpass filtering the image in Fig. 4.4(a). (c) A two-dimensional highpass filter function. (d) Result of highpass filtering the image in Fig. 4.4(a).

Lowpass Filtering in the Frequency Domain

- *Edges, noise contribute significantly to the high-frequency content of the FT of an image.*
- *Blurring/smoothing is achieved by reducing a specified range of high-frequency components:*

$$G(u,v) = H(u,v)F(u,v)$$

Smoothing in the Frequency Domain

$$G(u,v) = H(u,v)\ F(u,v)$$

- *Ideal*
- *Butterworth (parameter: filter order)*
- *Gaussian*

Ideal Filter (Lowpass)

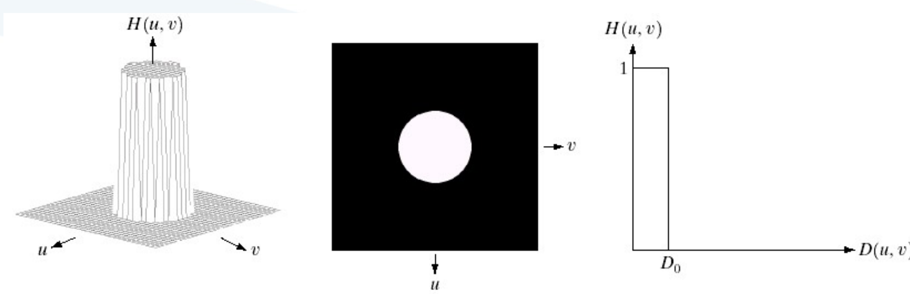
- *A 2-D ideal low-pass filter:*

$$H(u,v) = \begin{cases} 1 \text{ if } D(u,v) \le D_0 \\ 0 \text{ if } D(u,v) > D_0 \end{cases}$$

where D_0 is a specified nonnegative quantity and D(u,v) is the distance from point (u,v) to the center of the frequency rectangle.

- Center of frequency rectangle: (u,v)=(M/2,N/2)
- Distance from any point to the center (origin) of the FT:

$$D(u,v) = (u^2 + v^2)^{1/2}$$



a b c

FIGURE 4.10 (a) Perspective plot of an ideal lowpass filter transfer function. (b) Filter displayed as an image. (c) Filter radial cross section.

Ideal Filter (Lowpass)

- *Ideal*:
  - all frequencies inside a circle of radius D_0 are passed with no attenuation
  - all frequencies outside this circle are completely attenuated.

Ideal Filter (Lowpass)

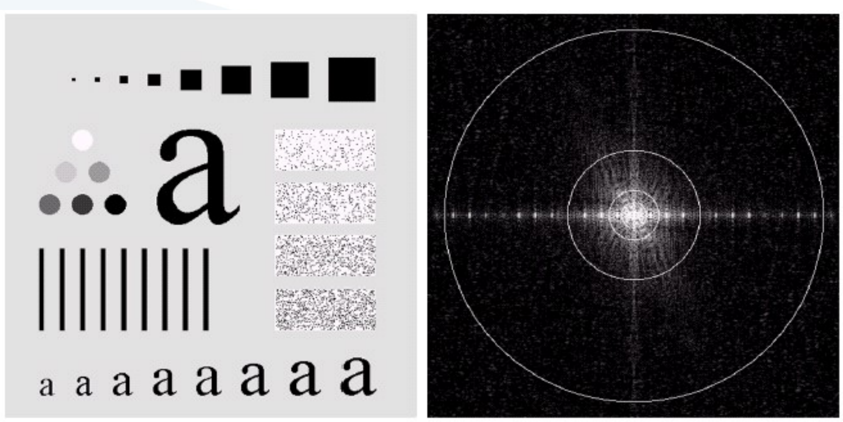
- *Cutoff-frequency: the point of transition between H(u,v)=1 and H(u,v)=0 (D_0)*
- *To establish cutoff frequency loci, we typically compute circles that enclose specified amounts of total image power P_T.*

Ideal Filter *(cont.)*

- *P_T is obtain by summing the components of power spectrum P(u,v) at each point for u up to M-1 and v up to N-1.*
- *A circle with radius r, origin at the center of the frequency rectangle encloses a percentage of the power which is given by the expression*

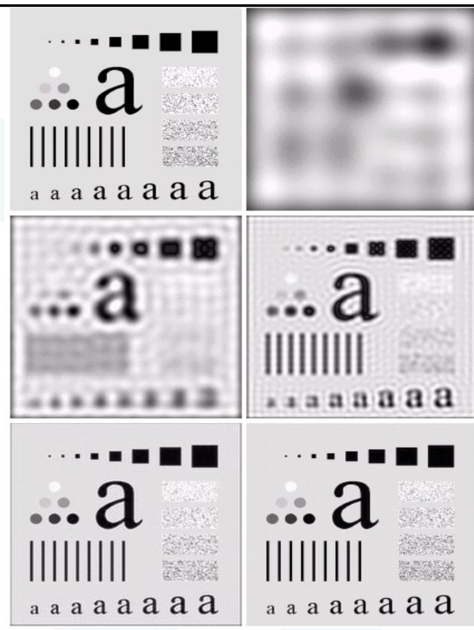
$$100[\sum_{u}\sum_{v} P(u,v)/P_T$$

- *The summation is taken within the circle r*

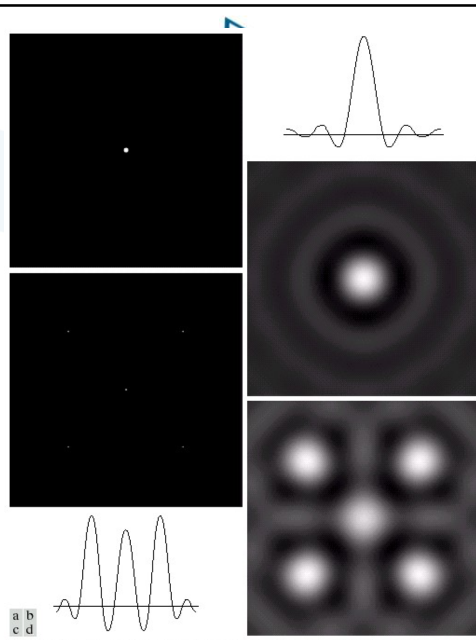


a b

FIGURE 4.11 (a) An image of size 500 × 500 pixels and (b) its Fourier spectrum. The superimposed circles have radii values of 5, 15, 30, 80, and 230, which enclose 92.0, 94.6, 96.4, 98.0, and 99.5% of the image power, respectively.



a b c d e f **FIGURE 4.12** (a) Original image. (b)–(f) Results of ideal lowpass filtering with cutoff frequencies set at radii values of 5, 15, 30, 80, and 230, as shown in Fig. 4.11(b). The power removed by these filters was 8, 5.4, 3.6, 2, and 0.5% of the total, respectively.



a b c d **FIGURE 4.13** (a) A frequency-domain ILPF of radius 5. (b) Corresponding spatial filter (note the ringing). (c) Five impulses in the spatial domain, simulating the values of five pixels. (d) Convolution of (b) and (c) in the spatial domain.

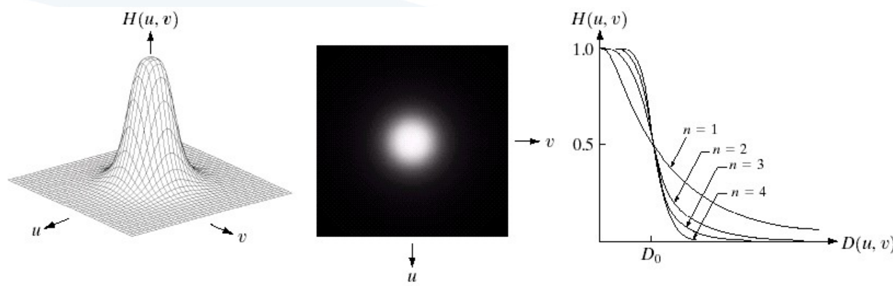
Butterworth Filter (Lowpass)

- *This filter does not have a sharp discontinuity establishing a clear cutoff between passed and filtered frequencies.*

$$H(u,v) = \frac{1}{1 + [D(u,v) / D_0]^{2n}}$$

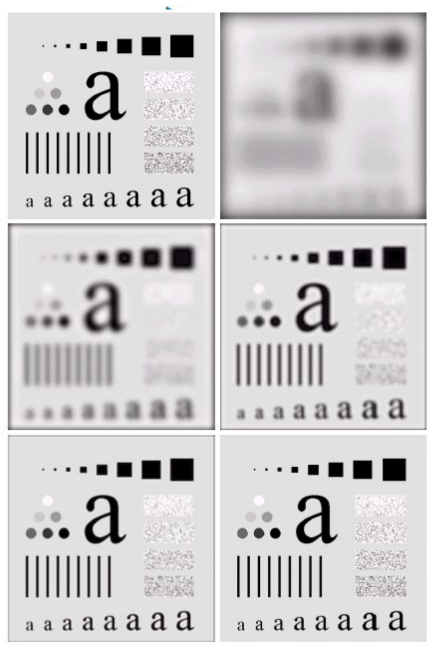
Butterworth Filter (Lowpass)

- *To define a cutoff frequency locus: at points for which H(u,v) is down to a certain fraction of its maximum value.*
- *When* $D(u,v) = D_0$, $H(u,v) = 0.5$
  - i.e. down 50% from its maximum value of 1.



a b c

FIGURE 4.14 (a) Perspective plot of a Butterworth lowpass filter transfer function. (b) Filter displayed as an image. (c) Filter radial cross sections of orders 1 through 4.



a b
c d
e f

FIGURE 4.15 (a) Original image. (b)–(f) Results of filtering with BLPFs of order 2, with cutoff frequencies at radii of 5, 15, 30, 80, and 230, as shown in Fig. 4.11(b). Compare with Fig. 4.12.

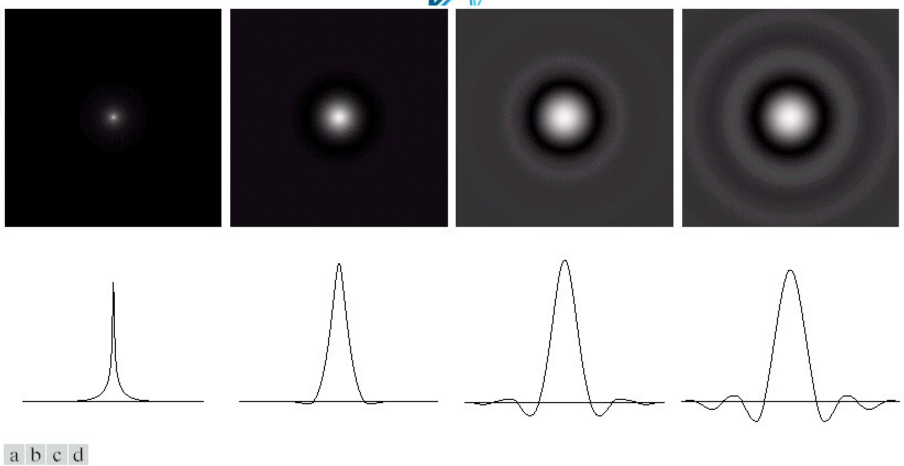
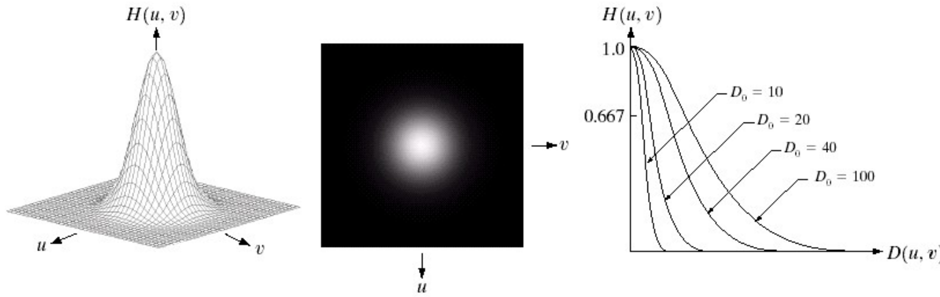


FIGURE 4.16 (a)–(d) Spatial representation of BLPFs of order 1, 2, 5, and 20, and corresponding gray-level profiles through the center of the filters (all filters have a cutoff frequency of 5). Note that ringing increases as a function of filter order.

Gaussian Lowpass Filter

$$H(u,v) = e^{-D^2(u,v)/2\sigma^2}$$

- *D(u,v): distance from the origin of FT*
- *Parameter: σ=D_0 (cutoff frequency)*
- *The inverse FT of the Gaussian filter is also a Gaussian*



a b c

FIGURE 4.17 (a) Perspective plot of a GLPF transfer function. (b) Filter displayed as an image. (c) Filter radial cross sections for various values of D_0.

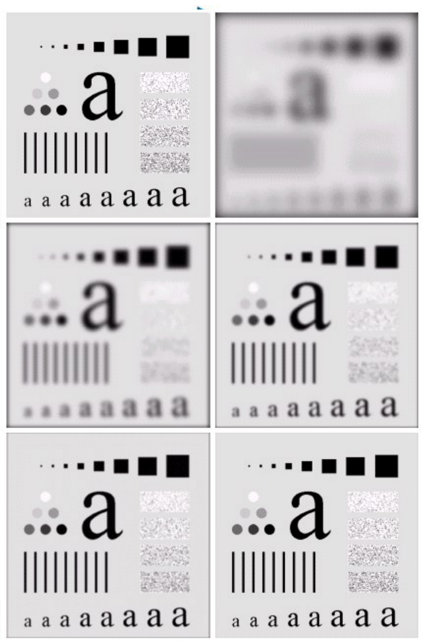
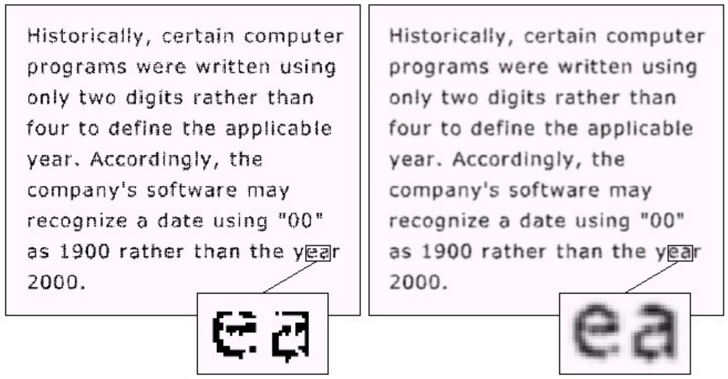


FIGURE 4.18 (a) Original image. (b)–(f) Results of filtering with Gaussian lowpass filters with cutoff frequencies set at radii values of 5, 15, 30, 80, and 230, as shown in Fig. 4.11(b). Compare with Figs. 4.12 and 4.15.

a b
c d
e f

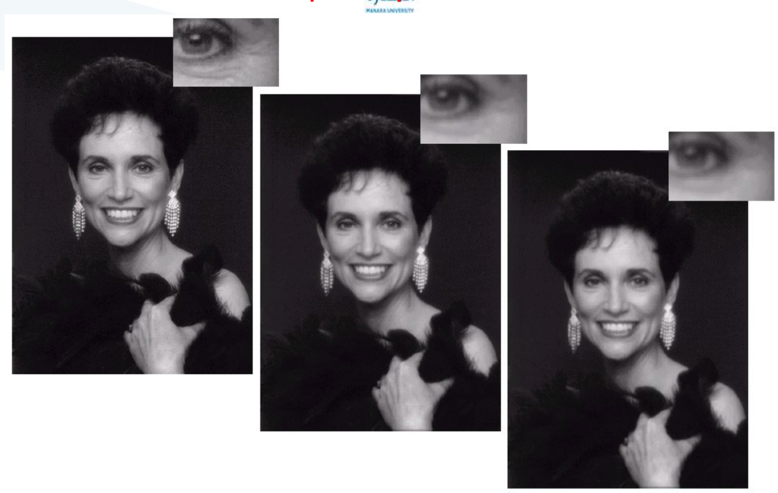
Image Enhancement in the Frequency Domain



a b

FIGURE 4.19 (a) Sample text of poor resolution (note broken characters in magnified view). (b) Result of filtering with a GLPF (broken character segments were joined).

Image Enhancement in the Frequency Domain



a b c

FIGURE 4.20 (a) Original image (1028 × 732 pixels). (b) Result of filtering with a GLPF with $D_0 = 100$. (c) Result of filtering with a GLPF with $D_0 = 80$. Note reduction in skin fine lines in the magnified sections of (b) and (c).

Image Enhancement in the Frequency Domain

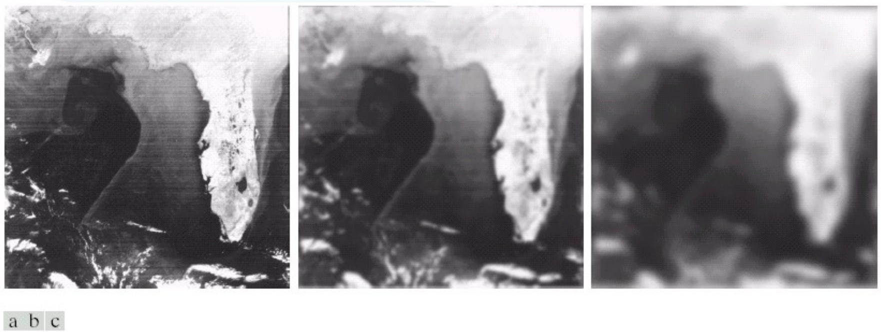


FIGURE 4.21 (a) Image showing prominent scan lines. (b) Result of using a GLPF with $D_0 = 30$. (c) Result of using a GLPF with $D_0 = 10$. (Original image courtesy of NOAA.)

Sharpening (Highpass) Filtering

- *Image sharpening can be achieved by a highpass filtering process, which attenuates the low-frequency components without disturbing high-frequency information.*
- *Zero-phase-shift filters: radially symmetric and completely specified by a cross section.*

$$H_{hp}(u,v) = 1 - H_{lp}(u,v)$$

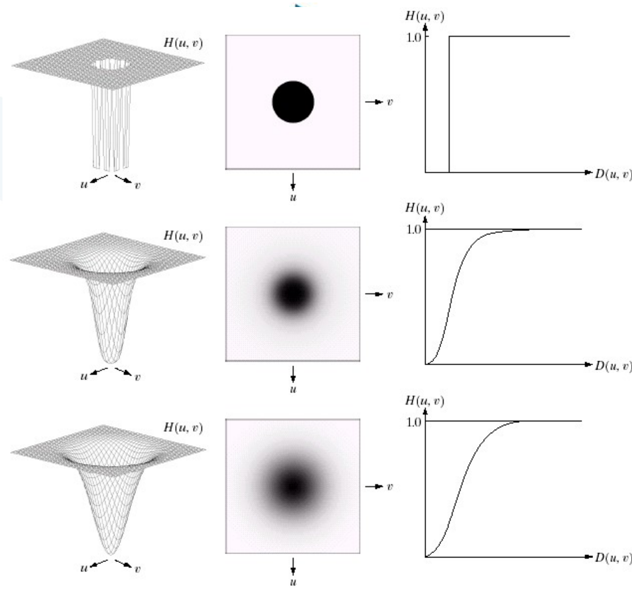


FIGURE 4.22 Top row: Perspective plot, image representation, and cross section of a typical ideal highpass filter. Middle and bottom rows: The same sequence for typical Butterworth and Gaussian highpass filters.

Ideal Filter (Highpass)

$$H(u,v) = \begin{cases} 0 & \text{if } D(u,v) \le D_0 \\ 1 & \text{if } D(u,v) > D_0 \end{cases}$$

- *This filter is the opposite of the ideal lowpass filter.*

Butterworth Filter (Highpass)

$$H(u,v) = \frac{1}{1+[D_0 / D(u,v)]^{2n}}$$

- *High-frequency emphasis: Adding a constant to a highpass filter to preserve the low-frequency components.*

Gaussian Highpass Filter

$$H(u,v) = 1 - e^{-D^2(u,v)/2\sigma^2}$$

- *D(u,v): distance from the origin of FT*
- *Parameter: $\sigma = D_0$ (cutoff frequency)*

Laplacian (recap)

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

$$\frac{\partial^2 f}{\partial^2 x^2} = f(x+1,y) + f(x-1,y) - 2f(x,y)$$

$$\frac{\partial^2 f}{\partial^2 y^2} = f(x,y+1) + f(x,y-1) - 2f(x,y)$$

$$\nabla^2 f = [f(x+1,y) + f(x-1,y) + f(x,y+1) + f(x,y-1)] - 4f(x,y)$$

Laplacian in the FD

- *It can be shown that:*

$$\Im\left[\nabla^2 f(x,y)\right] = -(u^2 + v^2)F(u,v)$$

- *The Laplacian can be implemented in the FD by using the filter*
- *FT pair:*

$$H(u,v) = -(u^2 + v^2)$$

$$\nabla^2 f(x,y) \Leftrightarrow -[(u - M/2)^2 + (v - N/2)^2]F(u,v)$$

Laplacian in the Frequency Domain

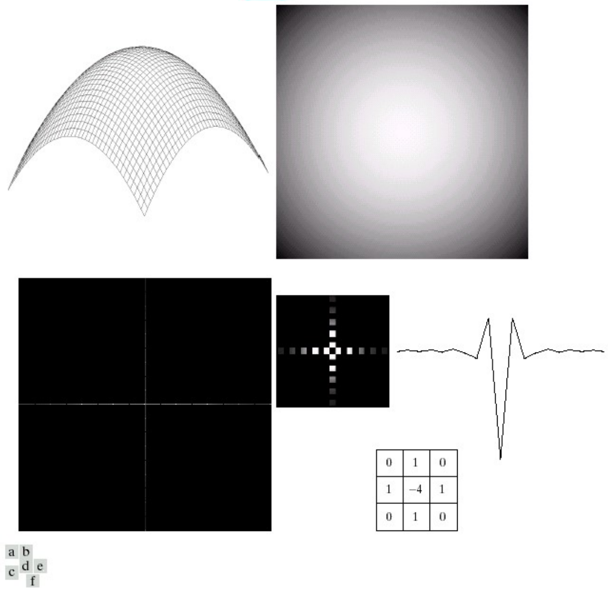


FIGURE 4.27 (a) 3-D plot of Laplacian in the frequency domain. (b) Image representation of (a). (c) Laplacian in the spatial domain obtained from the inverse DFT of (b). (d) Zoomed section of the origin of (c). (e) Gray-level profile through the center of (d). (f) Laplacian mask used in Section 3.7.