الرياضيات

Dr. Yamar Hamwi

Al-Manara University

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التكامل غير المحدد

التابع الأصلي

نقول عن التابع F أنه تابع أصلي للتابع f على المجال I إذا تحقق: $F'(x) = f(x)$ من أجل كل x من I.

$$F_1 = x^3 \quad F_2 = x^3 + 2 \quad F_3 = x^3 + 97 \quad \ldots\ldots$$

مبرهنة (تمثيل التوابع الأصلية):

إذا كان F تابعاً أصلياً للتابع f على مجال I عندئذ, يكون G تابعاً أصلياً للتابع f على المجال I إذا وفقط إذا كان $G(x) = F(x) + c$ من أجل كل x من I، حيث c ثابت.

$$\int f(x)dx = F(x) + C$$

$$\int f(x)dx = F(x) \quad \Leftrightarrow \quad F'(x) = f(x)$$

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خواص التكامل غير المحدد

$$1)\quad \int c f(x)dx = c\int f(x)dx \quad ; \quad c = \text{constant}$$

$$2)\quad \int [f_1(x) + f_2(x) - f_3(x)]dx = \int f_1(x)dx + \int f_2(x)dx - \int f_3(x)dx$$

$$3)\quad \left[\int f(x)dx\right]' = f(x)$$

(1) Table of Indefinite Integrals

$$\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$$

$$\int cf(x)\,dx = c\int f(x)\,dx$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$$

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int e^x\,dx = e^x + C$$

$$\int e^{kx}\,dx = \frac{1}{k}e^{kx} + C$$

$$\int a^x\,dx = \frac{a^x}{\ln a} + C$$

$$\int \sin x\,dx = -\cos x + C$$

$$\int \cos x\,dx = \sin x + C$$

$$\int \sec^2 x\,dx = \tan x + C$$

$$\int \csc^2 x\,dx = -\cot x + C$$

$$\int \sec x \tan x\,dx = \sec x + C$$

$$\int \csc x \cot x\,dx = -\csc x + C$$

$$\int \frac{1}{x^2+1}\,dx = \tan^{-1} x + C$$

مثال

احسب التكامل الآتي

$$\int\left(\frac{1}{\sqrt[3]{x}}+\frac{1}{x^{3}}-\frac{4}{\sqrt{x}}-\frac{3}{\sqrt[5]{x^{3}}}\right)dx$$

الحل

$$\int\left(\frac{1}{\sqrt[3]{x}}+\frac{1}{x^{3}}-\frac{4}{\sqrt{x}}-\frac{3}{\sqrt[5]{x^{3}}}\right)dx=\int\left(x^{-\frac{1}{3}}+x^{-3}-4x^{-\frac{1}{2}}-3x^{-\frac{3}{5}}\right)dx$$

$$=\int x^{-\frac{1}{3}}dx+\int x^{-3}dx-4\int x^{-\frac{1}{2}}dx-3\int x^{-\frac{3}{5}}dx$$

$$=\frac{x^{-\frac{1}{3}+1}}{-\frac{1}{3}+1}+\frac{x^{-3+1}}{-3+1}-4\frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}-3\frac{x^{-\frac{3}{5}+1}}{-\frac{3}{5}+1}+c$$

$$=\frac{3}{2}\sqrt[3]{x^{2}}-\frac{1}{2}\frac{1}{x^{2}}-8\sqrt{x}-\frac{15}{2}\sqrt[5]{x^{2}}+c$$

مثال

احسب التكامل الآتي

$$\int \left(10x^4 - 2\sec^2 x\right)dx$$

الحل

$$\int\left(10x^4 - 2\sec^2 x\right)dx = \int 10x^4 dx - \int 2\sec^2 x\,dx = \int 10x^4 dx - \int 2\frac{1}{\cos^2 x}dx$$

$$= 10\frac{x^5}{5} - 2\tan x + C = 2x^5 - 2\tan x + C$$

مثال

احسب التكامل الآتي

$$\int (t^2+1)^2 dt$$

الحل

$$\int (t^2+1)^2 dt = \int (t^4 + 2t^2 + 1)dt = \frac{1}{5}t^5 + \frac{2}{3}t^3 + t + C$$

مثال

احسب التكامل الآتي $\int \frac{x+1}{\sqrt{x}} dx$

الحل

$$\int \frac{x+1}{\sqrt{x}} dx = \int \left(\frac{x}{\sqrt{x}} + \frac{1}{\sqrt{x}} \right) dx = \int (x^{1/2} + x^{-1/2}) dx = \frac{2}{3} x^{3/2} + 2x^{1/2} + C$$

أمثلة

EXAMPLE 3 Find the general antiderivative of each of the following functions.

(a) $f(x) = x^5$ **(b)** $g(x) = \dfrac{1}{\sqrt{x}}$ **(c)** $h(x) = \sin 2x$

(d) $i(x) = \cos \dfrac{x}{2}$ **(e)** $j(x) = e^{-3x}$ **(f)** $k(x) = 2^x$

Solution In each case, we can use one of the formulas listed in Table 4.2.

(a) $F(x) = \frac{x^6}{6} + C$ — Formula 1 with $n = 5$

(b) $g(x) = x^{-1/2}$, so

$$G(x) = \frac{x^{1/2}}{1/2} + C = 2\sqrt{x} + C$$

Formula 1 with $n = -1/2$

(c) $H(x) = \frac{-\cos 2x}{2} + C$ — Formula 2 with $k = 2$

(d) $I(x) = \frac{\sin (x/2)}{1/2} + C = 2\sin \frac{x}{2} + C$ — Formula 3 with $k = 1/2$

(e) $J(x) = -\frac{1}{3}e^{-3x} + C$ — Formula 8 with $k = -3$

(f) $K(x) = \left(\frac{1}{\ln 2}\right)2^x + C$ — Formula 13 with $a = 2$, $k = 1$ ■

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مثال

احسب التكامل الآتي

$$\int \left(10x^4 - 2\sec^2 x\right)dx$$

الحل

$$\int \left(10x^4 - 2\sec^2 x\right)dx = \int 10x^4 dx - \int 2\sec^2 x\,dx = \int 10x^4 dx - \int 2\frac{1}{\cos^2 x}dx$$

$$= 10\frac{x^5}{5} - 2\tan x + C = 2x^5 - 2\tan x + C$$

مثال

احسب التكامل الآتي

$$\int (t^2+1)^2 dt$$

الحل

$$\int (t^2+1)^2 dt = \int (t^4 + 2t^2 + 1)dt = \frac{1}{5}t^5 + \frac{2}{3}t^3 + t + C$$

(4) The Substitution Rule If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))\, g'(x)\, dx = \int f(u)\, du$$

EXAMPLE Find $\int \sec^2(5x + 1) \cdot 5\, dx$

Solution We substitute $u = 5x + 1$ and $du = 5\, dx$. Then,

$$\int \sec^2(5x + 1) \cdot 5\, dx = \int \sec^2 u\, du \qquad \text{Let } u = 5x + 1,\ du = 5\, dx.$$

$$= \tan u + C \qquad \frac{d}{du}\tan u = \sec^2 u$$

$$= \tan(5x + 1) + C. \qquad \text{Substitute } 5x + 1 \text{ for } u.$$

indefinite integrals and the Substitution method

EXAMPLE $\int x^2 \cos x^3 \, dx$

$$\int x^2 \cos x^3 \, dx = \int \cos x^3 \cdot x^2 \, dx$$

$$= \int \cos u \cdot \frac{1}{3} \, du$$ Let $u = x^3$, $du = 3x^2 \, dx$, $(1/3) \, du = x^2 \, dx$.

$$= \frac{1}{3} \int \cos u \, du$$

$$= \frac{1}{3} \sin u + C$$ Integrate with respect to u.

$$= \frac{1}{3} \sin x^3 + C$$ Replace u by x^3. ■

indefinite integrals and the Substitution method

EXAMPLE Evaluate $\int \frac{2z\,dz}{\sqrt[3]{z^2+1}}$.

Substitute $u = z^2 + 1$.

$$\int \frac{2z\,dz}{\sqrt[3]{z^2+1}} = \int \frac{du}{u^{1/3}} \qquad \text{Let } u = z^2+1,\ du = 2z\,dz.$$

$$= \int u^{-1/3}\,du \qquad \text{In the form } \int u^n\,du$$

$$= \frac{u^{2/3}}{2/3} + C \qquad \text{Integrate.}$$

$$= \frac{3}{2}u^{2/3} + C$$

$$= \frac{3}{2}(z^2+1)^{2/3} + C \qquad \text{Replace } u \text{ by } z^2+1.$$

EXAMPLE 1 | Find $\int x^3 \cos(x^4 + 2)\, dx$.

SOLUTION We make the substitution $u = x^4 + 2$ because its differential is $du = 4x^3\, dx$, which, apart from the constant factor 4, occurs in the integral. Thus, using $x^3\, dx = \frac{1}{4}\, du$ and the Substitution Rule, we have

$$\int x^3 \cos(x^4 + 2)\, dx = \int \cos u \cdot \tfrac{1}{4}\, du = \tfrac{1}{4} \int \cos u\, du$$

$$= \tfrac{1}{4} \sin u + C$$

$$= \tfrac{1}{4} \sin(x^4 + 2) + C$$

Notice that at the final stage we had to return to the original variable x.

EXAMPLE 2 | Evaluate $\int \sqrt{2x+1}\,dx$.

SOLUTION 1 Let $u = 2x + 1$. Then $du = 2\,dx$, so $dx = \frac{1}{2}\,du$. Thus the Substitution Rule gives

$$\int \sqrt{2x+1}\,dx = \int \sqrt{u} \cdot \tfrac{1}{2}\,du = \tfrac{1}{2}\int u^{1/2}\,du$$

$$= \frac{1}{2} \cdot \frac{u^{3/2}}{3/2} + C = \tfrac{1}{3}u^{3/2} + C$$

$$= \tfrac{1}{3}(2x+1)^{3/2} + C$$

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EXAMPLE 3 | Find $\int \frac{x}{\sqrt{1 - 4x^2}} \, dx$.

SOLUTION Let $u = 1 - 4x^2$. Then $du = -8x \, dx$, so $x \, dx = -\frac{1}{8} \, du$ and

$$\int \frac{x}{\sqrt{1 - 4x^2}} \, dx = -\frac{1}{8} \int \frac{1}{\sqrt{u}} \, du = -\frac{1}{8} \int u^{-1/2} \, du$$

$$= -\frac{1}{8}(2\sqrt{u}) + C = -\frac{1}{4}\sqrt{1 - 4x^2} + C$$ ■

EXAMPLE 5 | Calculate $\int \tan x \, dx$.

SOLUTION First we write tangent in terms of sine and cosine:

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx$$

This suggests that we should substitute $u = \cos x$, since then $du = -\sin x \, dx$ and so $\sin x \, dx = -du$:

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx = -\int \frac{1}{u} \, du$$

$$= -\ln |u| + C = -\ln |\cos x| + C$$ ■

Since $-\ln|\cos x| = \ln(|\cos x|^{-1}) = \ln(1/|\cos x|) = \ln|\sec x|$, the result of Example 5 can also be written as

$$\int \tan x \, dx = \ln |\sec x| + C$$

التكامل بالتجزئة

$$\int u\,dv = uv - \int v\,du$$

التكامل بالتجزئة

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$$

الحالة ١ : حالة تكامل ذو حد واحد

نختار دوماً $f(x)$ التابع المكامل، و $g'(x) = 1$.

مثال أوجد قيمة التكامل الآتي $\int \ln x \, dx$

الحل

$$\left.\begin{matrix} f(x) = \ln x \\ g'(x) = 1 \end{matrix}\right\} \Rightarrow \left.\begin{matrix} f'(x) = \dfrac{1}{x} \\ g(x) = x \end{matrix}\right\}$$

$$\int \ln x \, dx = f(x)g(x) - \int f'(x)g(x)dx = x \ln x - \int x \frac{1}{x} dx = x \ln x - \int dx = x \ln x - x + C$$

مثال أوجد قيمة التكامل الآتي $\int \tan^{-1} x \, dx$

الحل

$$\left.\begin{array}{l} f(x) = \tan^{-1} x \\ g'(x) = 1 \end{array}\right\} \Rightarrow \left.\begin{array}{l} f'(x) = \dfrac{1}{x^2+1} \\ g(x) = x \end{array}\right\}$$

$$\int \tan^{-1} x \, dx = f(x)g(x) - \int f'(x)g(x)dx = x \tan^{-1} x - \int x \frac{1}{x^2+1} dx$$

$$= x \tan^{-1} x - \frac{1}{2}\int \frac{2x}{x^2+1} dx = x \tan^{-1} x - \frac{1}{2}\ln(x^2+1) + C$$

1) $\int \ln x dx$

$$u = \ln x \Rightarrow du = \frac{dx}{x}$$

$$dv = dx \Rightarrow v = x$$

عندئذٍ وحسب الصيغة (7) نجد:

$$\int \ln x dx = x \ln x - \int x \frac{dx}{x} = x \ln x - x + c$$

2) $\int x \cos x dx$

$$u = x \Rightarrow du = dx$$

$$dv = \cos x dx \Rightarrow v = \int \cos x dx = \sin x$$

عندئذٍ وحسب الصيغة (7) نجد:

$$\int x \cos x dx = x \sin x - \int \sin x dx = x \sin x + \cos x + c$$

3) $\int x \arctan x dx$

$$u = \arctan x \Rightarrow du = \frac{dx}{1 + x^2}$$

$$dv = x dx \Rightarrow v = \frac{x^2}{2}$$

EXAMPLE 3 | Find $\int t^2 e^t \, dt$.

SOLUTION Notice that t^2 becomes simpler when differentiated (whereas e^t is unchanged when differentiated or integrated), so we choose

$$u = t^2 \qquad dv = e^t \, dt$$

Then

$$du = 2t \, dt \qquad v = e^t$$

Integration by parts gives

$$\int t^2 e^t \, dt = t^2 e^t - 2 \int t e^t \, dt \tag{3}$$

The integral that we obtained, $\int t e^t \, dt$, is simpler than the original integral but is still not obvious. Therefore we use integration by parts a second time, this time with $u = t$ and $dv = e^t \, dt$. Then $du = dt$, $v = e^t$, and

$$\begin{aligned} \int t e^t \, dt &= t e^t - \int e^t \, dt \\ &= t e^t - e^t + C \end{aligned}$$

Putting this in Equation 3, we get

$$\begin{aligned} \int t^2 e^t \, dt &= t^2 e^t - 2 \int t e^t \, dt \\ &= t^2 e^t - 2(t e^t - e^t + C) \\ &= t^2 e^t - 2t e^t + 2e^t + C_1 \qquad \text{where } C_1 = -2C \end{aligned}$$

■

التكامل المحدد

التكامل المحدد

النظرية الأساسية في التحليل الرياضي:

ليكن f تابع مستمر على المجال المغلق $[a,b]$، وكان F التابع الأصلي ل f على المجال $[a,b]$، عندئذ

$$\int_b^a f(x)dx = F(a) - F(b)$$

التكامل المحدد

خواص التكامل المحدد

1. إذا كان التابع f معرف عند النقطة $x = a$ عندئذ $\int_a^a f(x)\,dx = 0$

2. إذا كان التابع f قابل للمكاملة على المجال $[a, b]$ عندئذ $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx.$

3. إذا كان التابع f قابل للمكاملة على المجال $[a, b]$ وكانت $c \in [a, b]$ عندئذ

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

4. $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$

5. $\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

خواص التكامل المحدد

6. إذا كان التابع f قابل للمكاملة غير سالب على المجال المغلق $[a,b]$ عندئذ

$$\int_a^b f(x)dx \geq 0$$

7. إذا كان التابعين f, g قابلين للاشتقاق على المجال $[a,b]$ وكان $f(x) \leq g(x) \quad \forall x \in [a,b]$

$$\int_a^b f(x)dx \leq \int_a^b g(x)dx$$

EXAMPLE 6 | Evaluate $\displaystyle\int_1^9 \frac{2t^2 + t^2\sqrt{t} - 1}{t^2}\,dt$.

SOLUTION First we need to write the integrand in a simpler form by carrying out the division:

$$\int_1^9 \frac{2t^2 + t^2\sqrt{t} - 1}{t^2}\,dt = \int_1^9 (2 + t^{1/2} - t^{-2})\,dt$$

$$= 2t + \frac{t^{3/2}}{\frac{3}{2}} - \frac{t^{-1}}{-1}\Bigg]_1^9 = 2t + \tfrac{2}{3}t^{3/2} + \frac{1}{t}\Bigg]_1^9$$

$$= \left(2 \cdot 9 + \tfrac{2}{3} \cdot 9^{3/2} + \tfrac{1}{9}\right) - \left(2 \cdot 1 + \tfrac{2}{3} \cdot 1^{3/2} + \tfrac{1}{1}\right)$$

$$= 18 + 18 + \tfrac{1}{9} - 2 - \tfrac{2}{3} - 1 = 32\tfrac{4}{9}$$ ■

مثال   احسب قيمة كل من التكاملا ت التالية:

$$\text{a. } \int_1^2 (x^2-3)\,dx \qquad \text{b. } \int_1^4 3\sqrt{x}\,dx \qquad \text{c. } \int_0^{\pi/4} \sec^2 x\,dx$$

الحل

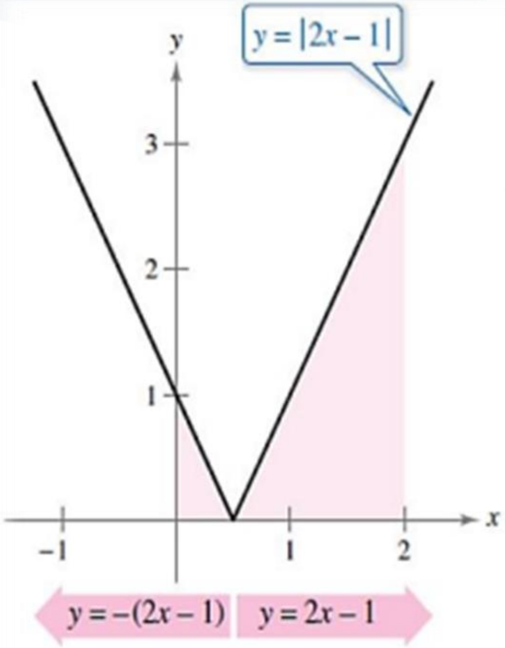
$$\text{a. } \int_1^2 (x^2-3)\,dx = \left[\frac{x^3}{3}-3x\right]_1^2 = \left(\frac{8}{3}-6\right)-\left(\frac{1}{3}-3\right) = -\frac{2}{3}$$

$$\text{b. } \int_1^4 3\sqrt{x}\,dx = 3\int_1^4 x^{1/2}\,dx = 3\left[\frac{x^{3/2}}{3/2}\right]_1^4 = 2(4)^{3/2}-2(1)^{3/2} = 14$$

$$\text{c. } \int_0^{\pi/4} \sec^2 x\,dx = \tan x\Big]_0^{\pi/4} = 1-0 = 1$$

مثال احسب قيمة التكامل الآتي $\int_0^2 |2x-1|\,dx.$

الحل باستخدام خواص القيمة المطلقة:



$$|2x-1| = \begin{cases} -(2x-1), & x < \frac{1}{2} \\ 2x-1, & x \geq \frac{1}{2} \end{cases}$$

$$\int_0^2 |2x-1|dx = \int_0^{1/2} -(2x-1)dx + \int_{1/2}^2 (2x-1)dx$$

$$= \Big[-x^2 + x\Big]_0^{1/2} + \Big[x^2 - x\Big]_{1/2}^2$$

$$= \left(-\frac{1}{4} + \frac{1}{2}\right) - (0+0) + (4-2) - \left(\frac{1}{4} - \frac{1}{2}\right) = \frac{5}{2}$$

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1 احسب التكاملات الآتية:

- $\int \left(\sqrt{x} + \sqrt[3]{x}\right) dx$
- $\int \frac{t\sqrt{t} + \sqrt{t}}{t^2} dt$
- $\int (4 \sec x \tan x - 2 \sec^2 x)\, dx$
- $\int \frac{1 + \cos 4t}{2} dt$

الحل

$\int \left(\sqrt{x} + \sqrt[3]{x}\right) dx$

$$\int\left(\sqrt{x}+\sqrt[3]{x}\right)dx=\int\left(x^{1/2}+x^{1/3}\right)dx=\frac{x^{3/2}}{\frac{3}{2}}+\frac{x^{4/3}}{\frac{4}{3}}+C=\frac{2}{3}x^{3/2}+\frac{3}{4}x^{4/3}+C$$

$\int \frac{t\sqrt{t} + \sqrt{t}}{t^2} dt$

$$\int\frac{t\sqrt{t}+\sqrt{t}}{t^2}\,dt=\int\left(\frac{t^{3/2}}{t^2}+\frac{t^{1/2}}{t^2}\right)dt=\int\left(t^{-1/2}+t^{-3/2}\right)dt=\frac{t^{1/2}}{\frac{1}{2}}+\left(\frac{t^{-1/2}}{-\frac{1}{2}}\right)+C=2\sqrt{t}-\frac{2}{\sqrt{t}}+C$$

$\int (4 \sec x \tan x - 2 \sec^2 x)\, dx$

$$\int(4\sec x\tan x-2\sec^2 x)\,dx=4\sec x-2\tan x+C$$

$\int \frac{1 + \cos 4t}{2} dt$

$$\int\frac{1+\cos 4t}{2}\,dt=\int\left(\frac{1}{2}+\frac{1}{2}\cos 4t\right)dt=\frac{1}{2}t+\frac{1}{2}\left(\frac{\sin 4t}{4}\right)+C=\frac{t}{2}+\frac{\sin 4t}{8}+C$$

تمارين

2 احسب التكاملات الآتية:

- $\int_1^8 \frac{(x^{1/3}+1)(2-x^{2/3})}{x^{1/3}}$
- $\int_{\pi/2}^0 \frac{1+\cos 2t}{2} dt$
- $\int_0^{\pi/3} 4\frac{\sin u}{\cos^2 u} du$
- $\int_0^1 (x^2+\sqrt{x})\, dx$

الحل

$\int_0^1 (x^2+\sqrt{x})\, dx$

$$\int_0^1 \left(x^2+\sqrt{x}\right) dx = \left[\frac{x^3}{3}+\frac{2}{3}x^{3/2}\right]_0^1 = \left(\frac{1}{3}+\frac{2}{3}\right)-0=1$$

$\int_0^{\pi/3} 4\frac{\sin u}{\cos^2 u} du$

$$\int_0^{\pi/3} 4\frac{\sin u}{\cos^2 u} du = \frac{4}{\cos u}\Big]_0^{\pi/3} = \left(\frac{4}{(1/2)}-\frac{4}{1}\right)=4$$

$\int_{\pi/2}^0 \frac{1+\cos 2t}{2} dt$

$$\int_{\pi/2}^0 \frac{1+\cos 2t}{2} dt = \int_{\pi/2}^0 \left(\frac{1}{2}+\frac{1}{2}\cos 2t\right) dt = \left[\frac{1}{2}t+\frac{1}{4}\sin 2t\right]_{\pi/2}^0 = \left(\frac{1}{2}(0)+\frac{1}{4}\sin 2(0)\right)-\left(\frac{1}{2}\left(\frac{\pi}{2}\right)+\frac{1}{4}\sin 2\left(\frac{\pi}{2}\right)\right) = -\frac{\pi}{4}$$

مثال أوجد قيمة التكامل الآتي $\int_{-2}^{2}(x^6+1)dx$

الحل

التابع المكامل زوجي

$$f(-x)=(-x)^6+1=x^6+1=f(x)$$

$$\int_{-2}^{2}(x^6+1)dx=2\int_{0}^{2}(x^6+1)dx=2\left[\frac{1}{7}x^7+x\right]_0^2=\frac{284}{7}$$

مثال أوجد قيمة التكامل الآتي $\int_{-1}^{1}\frac{\tan x}{1+x^2+x^4}dx$

الحل

التابع المكامل فردي

$$f(-x)=\frac{\tan(-x)}{1+(-x)^2+(-x)^4}=\frac{-\tan(x)}{1+x^2+x^4}=-f(x)$$

$$\int_{-1}^{1}\frac{\tan x}{1+x^2+x^4}dx=0$$

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1 احسب التكاملات الآتية:

- $\int 2x(x^2+5)^{-4}\,dx,$
- $\int (3x+2)(3x^2+4x)^4\,dx,$
- $\int \sqrt{x}\sin^2(x^{3/2}-1)\,dx,$
- $\int \frac{1}{x^2}\sqrt{2-\frac{1}{x}}\,dx$

الحل

$\int 2x(x^2+5)^{-4}\,dx,$ $\quad u=x^2+5 \Rightarrow du=2x\,dx \Rightarrow \frac{1}{2}du = x\,dx$

$$\int 2x(x^2+5)^{-4}dx = \int 2u^{-4}\frac{1}{2}du = \int u^{-4}du = -\frac{1}{3}u^{-3}+C = -\frac{1}{3}(x^2+5)^{-3}+C$$

$\int (3x+2)(3x^2+4x)^4\,dx,$ $\quad u=3x^2+4x \Rightarrow du=(6x+4)dx = 2(3x+2)dx \Rightarrow \frac{1}{2}du=(3x+2)dx$

$$\int (3x+2)(3x^2+4x)^4dx = \int u^4\frac{1}{2}du = \frac{1}{2}\int u^4du = \frac{1}{10}u^5+C = \frac{1}{10}(3x^2+4x)^5+C$$

تمارين

2 احسب التكاملات الآتية:

- $\int_0^1 t^3(1+t^4)^3\,dt$
- $\int_0^1 \frac{10\sqrt{v}}{(1+v^{3/2})^2}dv$
- $\int_0^1 \frac{x^3}{\sqrt{x^4+9}}dx$
- $\int_1^4 \frac{dy}{2\sqrt{y}(1+\sqrt{y})^2}$

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$\int_0^1 t^3(1+t^4)^3\,dt$ $\quad u=1+t^4 \Rightarrow du=4t^3dt \Rightarrow \frac{1}{4}du=t^3dt; t=0 \Rightarrow u=1, t=1 \Rightarrow u=2$

$$\int_0^1 t^3(1+t^4)^3\,dt=\int_1^2 \frac{1}{4}u^3\,du=\left[\frac{u^4}{16}\right]_1^2=\frac{2^4}{16}-\frac{1^4}{16}=\frac{15}{16}$$

$\int_0^1 \frac{10\sqrt{v}}{(1+v^{3/2})^2}dv$ $\quad u=1+v^{3/2} \Rightarrow du=\frac{3}{2}v^{1/2}\,dv \Rightarrow \frac{20}{3}du=10\sqrt{v}\,dv; v=0 \Rightarrow u=1, v=1 \Rightarrow u=2$

$$\int_0^1 \frac{10\sqrt{v}}{(1+v^{3/2})^2}dv=\int_1^2 \frac{1}{u^2}\left(\frac{20}{3}du\right)=\frac{20}{3}\int_1^2 u^{-2}\,du=-\frac{20}{3}\left[\frac{1}{u}\right]_1^2=-\frac{20}{3}\left[\frac{1}{2}-\frac{1}{1}\right]=\frac{10}{3}$$

$$\int_0^1 \frac{x^3}{\sqrt{x^4+9}}dx$$

$$u = x^4 + 9 \Rightarrow du = 4x^3\ dx \Rightarrow \tfrac{1}{4}du = x^3\ dx;\ x = 0 \Rightarrow u = 9,\ x = 1 \Rightarrow u = 10$$

$$\int_0^1 \frac{x^3}{\sqrt{x^4+9}}\,dx = \int_9^{10} \frac{1}{4} u^{-1/2}\ du = \left[\tfrac{1}{4}(2)u^{1/2}\right]_9^{10} = \tfrac{1}{2}(10)^{1/2} - \tfrac{1}{2}(9)^{1/2} = \frac{\sqrt{10}-3}{2}$$

$$\int_1^4 \frac{dy}{2\sqrt{y}(1+\sqrt{y})^2}$$

$$u = 1 + \sqrt{y} \Rightarrow du = \frac{dy}{2\sqrt{y}};\ y = 1 \Rightarrow u = 2,\ y = 4 \Rightarrow u = 3$$

$$\int_1^4 \frac{dy}{2\sqrt{y}(1+\sqrt{y})^2} = \int_2^3 \frac{1}{u^2}\,du = \int_2^3 u^{-2}\ du = [-u^{-1}]_2^3 = \left(-\tfrac{1}{3}\right) - \left(-\tfrac{1}{2}\right) = \tfrac{1}{6}$$

(6) Integrals of Symmetric Functions

Suppose f is continuous on $[-a, a]$.

(a) If f is even $[f(-x) = f(x)]$, then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$.

(b) If f is odd $[f(-x) = -f(x)]$, then $\int_{-a}^{a} f(x)\,dx = 0$.

EXAMPLE 9 | Since $f(x) = x^6 + 1$ satisfies $f(-x) = f(x)$, it is even and so

$$\int_{-2}^{2} (x^6 + 1)\,dx = 2\int_{0}^{2} (x^6 + 1)\,dx$$

$$= 2\left[\tfrac{1}{7}x^7 + x\right]_0^2 = 2\left(\tfrac{128}{7} + 2\right) = \tfrac{284}{7}$$

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EXAMPLE 10 | Since $f(x) = (\tan x)/(1 + x^2 + x^4)$ satisfies $f(-x) = -f(x)$, it is odd and so

$$\int_{-1}^{1} \frac{\tan x}{1 + x^2 + x^4}\,dx = 0$$

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Thank you for your attention