Calculus 1

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Calculus 1

Lecture 7

Some special functions

Chapter 4
Applications of Derivatives

4.5 Applied Optimization

4.6Exponential Function

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4.8Inverse Trigonometric Functions

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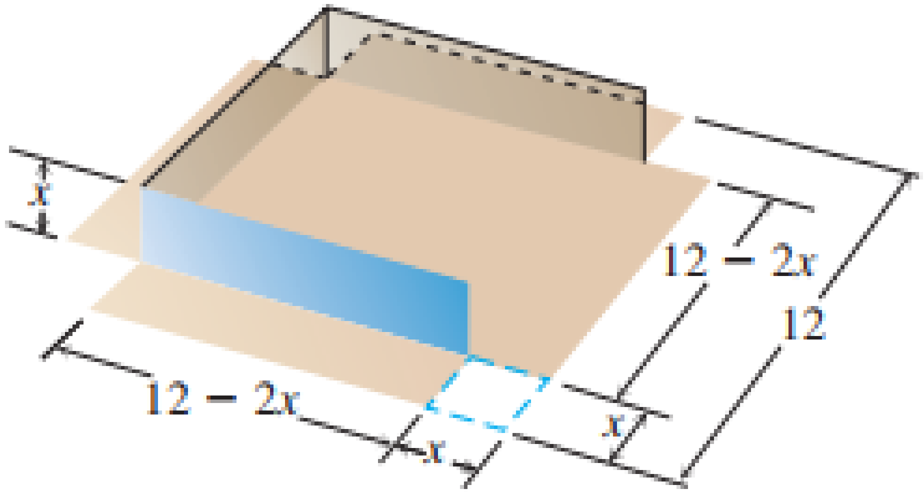
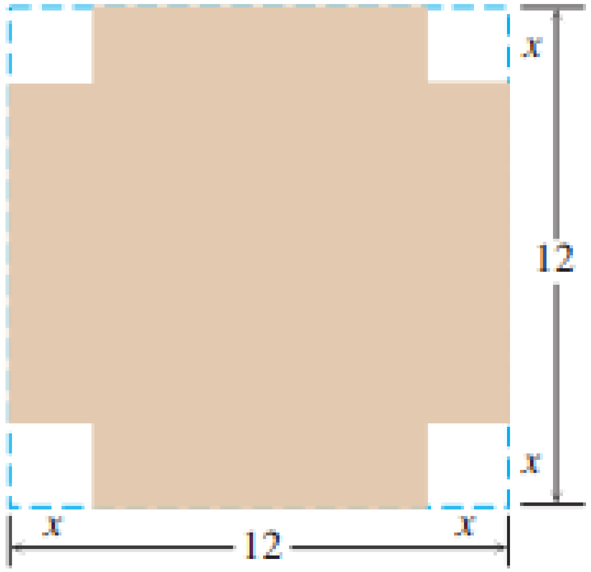
EXAMPLE 1 An open-top box is to be made by cutting small congruent squares from the corners of a 12-in.-by-12-in. sheet of tin and bending up the sides. How large should the squares cut from the corners be to make the box hold as much as possible?



(b)



(a)

Solution We start with a picture (Figure 4.36). In the figure, the corner squares are x in. on a side. The volume of the box is a function of this variable:

$$V(x) = x(12 - 2x)^2 = 144x - 48x^2 + 4x^3. \qquad V = hlw$$

Since the sides of the sheet of tin are only 12 in. long, $x \leq 6$ and the domain of V is the interval $0 \leq x \leq 6$.

A graph of V (Figure 4.37) suggests a minimum value of 0 at $x = 0$ and $x = 6$ and a maximum near $x = 2$. To learn more, we examine the first derivative of V with respect to x:

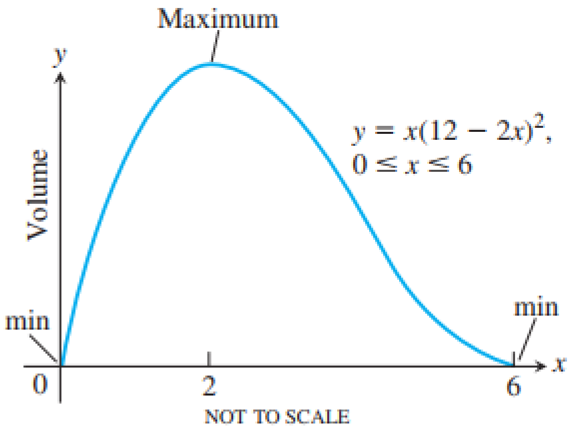
$$\frac{dV}{dx} = 144 - 96x + 12x^2 = 12(12 - 8x + x^2) = 12(2 - x)(6 - x).$$

Of the two zeros, $x = 2$ and $x = 6$, only $x = 2$ lies in the interior of the function's domain and makes the critical-point list. The values of V at this one critical point and two endpoints are

Critical point value: $V(2) = 128$

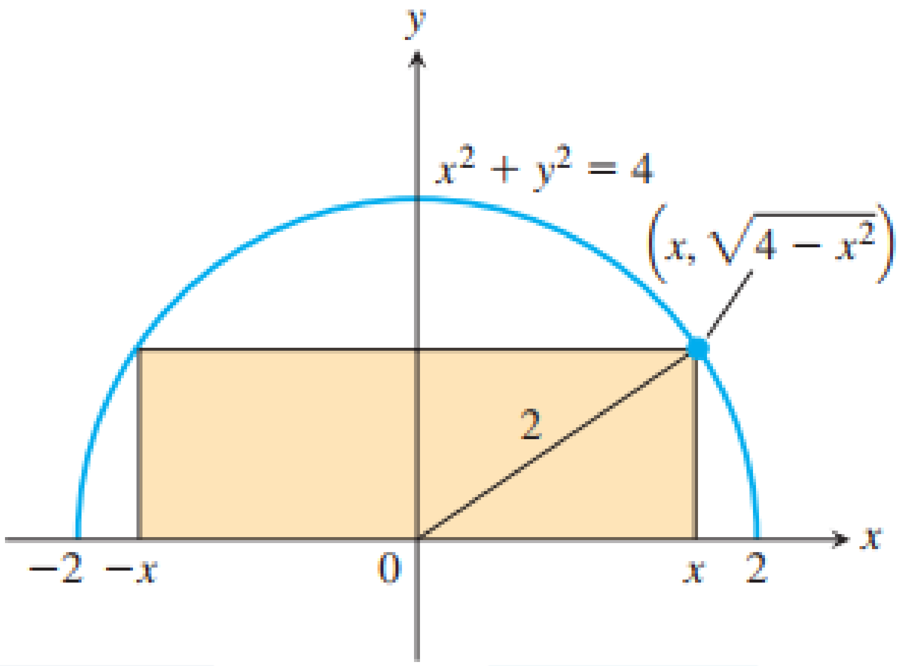
Endpoint values: $V(0) = 0, \qquad V(6) = 0.$

The maximum volume is 128 in^3. The cutout squares should be 2 in. on a side. ■



Exercices

A rectangle is to be inscribed in a semicircle of radius 2. What is the largest area the rectangle can have, and what are its dimensions?



Solution Let $\left(x, \sqrt{4 - x^2}\right)$

Length: $2x$, Height: $\sqrt{4 - x^2}$, Area: $2x\sqrt{4 - x^2}$.

$$A(x) = 2x\sqrt{4 - x^2}$$

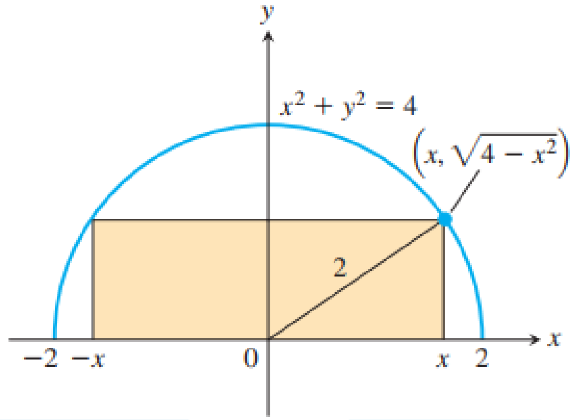
on the domain $[0, 2]$.

The derivative

$$\frac{dA}{dx} = \frac{-2x^2}{\sqrt{4 - x^2}} + 2\sqrt{4 - x^2}$$

is not defined when $x = 2$ and is equal to zero when

$$\frac{-2x^2}{\sqrt{4 - x^2}} + 2\sqrt{4 - x^2} = 0$$
$$-2x^2 + 2(4 - x^2) = 0$$
$$8 - 4x^2 = 0$$
$$x^2 = 2$$
$$x = \pm\sqrt{2}.$$



Of the two zeros, $x = \sqrt{2}$ and $x = -\sqrt{2}$, only $x = \sqrt{2}$ lies in the interior of A's domain and makes the critical-point list. The values of A at the endpoints and at this one critical point are

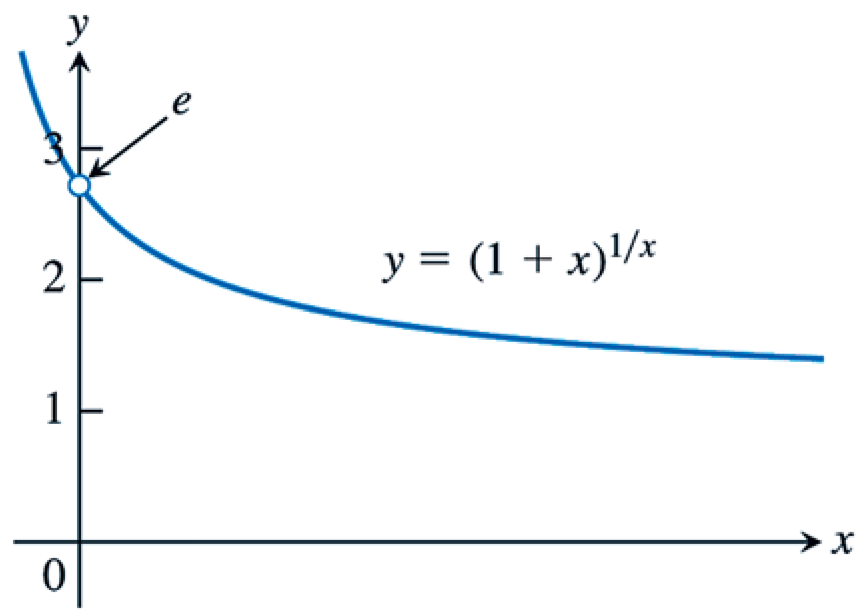
$$\text{Critical point value:} \quad A\left(\sqrt{2}\right) = 2\sqrt{2}\sqrt{4-2} = 4$$
$$\text{Endpoint values:} \qquad A(0) = 0, \qquad A(2) = 0.$$

The area has a maximum value of 4 when the rectangle is $\sqrt{4 - x^2} = \sqrt{2}$ units high and $2x = 2\sqrt{2}$ units long. ■

Exponential Function

THEOREM 4—The Number *e* as a Limit The number e can be calculated as the limit

$$e = \lim_{x \to 0} (1 + x)^{1/x}.$$



Exponential Function

Derivative of $y = e^x$

$$\frac{d}{dx}(e^x) = 1 \cdot e^x = e^x.$$

$$\frac{d}{dx}(3c^x) = 3\frac{d}{dx}c^x = 3c^x$$

$$\frac{d}{dx}(x^2c^x) = 2xc^x + x^2c^x = xc^x(x+2)$$

Exponential Function

Theorem: $\frac{d}{dx}c^{u} = c^{u}\frac{du}{dx}$

$$\frac{d}{dx}(c^{x^2-5x}) = c^{x^2-5x}(x^2-5x)' = c^{x^2-5x}(2x-5)$$

$$\frac{d}{dx}c^{\sqrt{x^2-3}} = c^{\sqrt{x^2-3}}(\sqrt{x^2-3})' = c^{\sqrt{x^2-3}}\tfrac{1}{2}(x^2-3)^{-1/2}\cdot 2x = \frac{xc^{\sqrt{x^2-3}}}{\sqrt{x^2-3}}$$

natural Logarithm function

DEFINITION Natural Logarithm

For $x > 0$, $\quad y = \ln x$ if, and only if, $x = e^y$

Property
(1) $\ln 1 = 0$; equivalently, $(1, 0)$ on graph of $y = \ln x$
(2) $\ln e = 1$; equivalently, $(e, 1)$ on graph of $y = \ln x$
(3) $\ln e^x = x$; equivalently, (e^x, x) on graph of $y = \ln x$
(4) $e^{\ln x} = x$; equivalently, $(\ln x, x)$ on graph of $y = e^x$

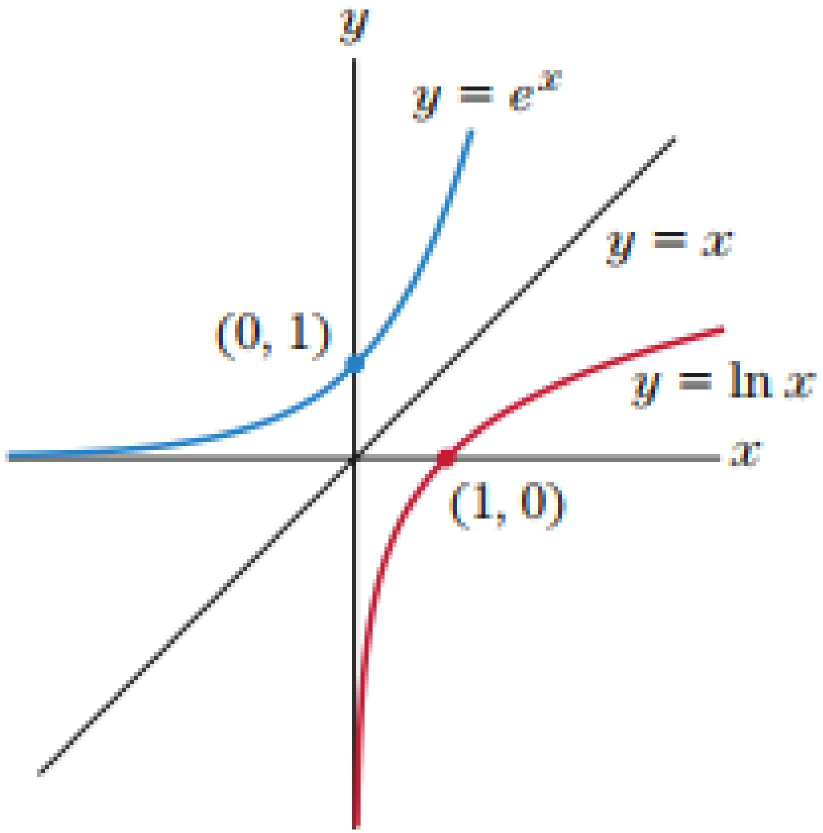
natural Logarithm function

Properties of natural logarithm functions

Property: For $x, y > 0$, and b any number	Examples	In words
(1) ln of a Product: $\ln(x \cdot y) = \ln x + \ln y$	$\ln(6) = \ln(2 \cdot 3) = \ln 2 + \ln 3$ $\ln(27) = \ln(3 \cdot 9) = \ln 3 + \ln 9$	ln of a product is the sum of ln's
(2) ln of an Inverse: $\ln\left(\frac{1}{x}\right) = -\ln x$	$\ln(\frac{1}{2}) = -\ln 2$ $\ln(\frac{1}{\sqrt{2}}) = -\ln(\sqrt{2})$	ln of $1/x$ is minus ln of x, or ln of an inverse is minus the ln
(3) ln of a Quotient: $\ln\left(\frac{x}{y}\right) = \ln x - \ln y$	$\ln(\frac{4}{3}) = \ln 4 - \ln 3$ $\ln\left(\frac{x}{5}\right) = \ln x - \ln 5$	ln of a quotient is the difference of ln's
(4) ln of a Power: $\ln(x^b) = b \ln x$	$\ln(2^3) = 3 \ln 2$ $\ln(x^2) = 2 \ln x$ $\ln \sqrt{x} = \ln(x^{1/2}) = \frac{1}{2} \ln x$	ln of x to the b is b times ln x

natural Logarithm function

Note: each function of e^x and $\ln x$ is the inverse of the other



$$\ln e^x = x, \text{ for all } x \in R$$

$$e^{\ln x} = x, \text{ for all } x > 0$$

natural Logarithm function

Theorem:

$\ln x$ exists only for positive numbers x. The domain is $(0, \infty)$

$\ln x < 0$ for $0 < x < 1$

$\ln x = 0$ when $x = 1$

$\ln x > 0$ for $x > 1$

Derivative of $y = \ln x$

$$\frac{d}{dx}(\ln x) = \frac{1}{x}, \quad x > 0. \tag{1}$$

$$\frac{d}{dx}(x^2 \ln x + 5x) = \frac{d}{dx}(x^2 \ln x) + \frac{d}{dx}(5x) = 2x \cdot \ln x + x^2 \frac{1}{x} + 5 = 2x \cdot \ln x + x + 5$$

Chain Rule for Log Function

$$\frac{d}{dx}[\ln g(x)] = \frac{1}{g(x)} \cdot \frac{d}{dx} g(x) = \frac{g'(x)}{g(x)}. \quad (2)$$

If $u = g(x)$, equation (4) can be written in the form

$$\frac{d}{dx}[\ln u] = \frac{1}{u}\frac{du}{dx}. \quad (2a)$$

natural Logarithm function

$$\frac{d}{dx}\ln(x^2-5x)=\frac{2x-5}{x^2-5x}$$

$$\ln(AB)=\ln A+\ln B \quad A,B>0$$

$$\ln\frac{A}{B}=\ln A-\ln B \quad A,B>0$$

$$\frac{d}{dx}\ln\frac{x^2-5}{x}=\frac{d}{dx}\left[\ln(x^2-5)-\ln x\right]=\frac{d}{dx}\ln(x^2-5)-\frac{d}{dx}\ln x=\frac{2x}{x^2-5}-\frac{1}{x}$$

Inverse Trigonometric Functions

The **inverse sine** function:

$$y = \arcsin x = \operatorname{asin} x = \sin^{-1} x \Leftrightarrow x = \sin y,\ -1 \le x \le 1 \text{ and } -\frac{\pi}{2} \le y \le \frac{\pi}{2}$$

The **inverse cosine** function:

$$y = \arccos x = \operatorname{acos} x = \cos^{-1} x \Leftrightarrow x = \cos y,\ -1 \le x \le 1 \text{ and } 0 \le y \le \pi$$

The **inverse tangent** function:

$$y = \arctan x = \operatorname{atan} x = \tan^{-1} x \Leftrightarrow x = \tan y,\ -\infty \le x \le \infty \text{ and } -\frac{\pi}{2} \le y \le \frac{\pi}{2}$$

$$\cos^{-1}\frac{1}{2} = \frac{\pi}{3} \qquad \cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3} \qquad \tan^{-1}\frac{1}{\sqrt{3}} = \frac{\pi}{6} \qquad \sin^{-1} -\frac{1}{2} = -\frac{\pi}{6}$$

Inverse Trigonometric Functions

TABLE 3.1 Derivatives of the inverse trigonometric functions

1. $$\frac{d(\sin^{-1} u)}{dx} = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}, \quad |u| < 1$$

2. $$\frac{d(\cos^{-1} u)}{dx} = -\frac{1}{\sqrt{1-u^2}}\frac{du}{dx}, \quad |u| < 1$$

3. $$\frac{d(\tan^{-1} u)}{dx} = \frac{1}{1+u^2}\frac{du}{dx}$$

4. $$\frac{d(\cot^{-1} u)}{dx} = -\frac{1}{1+u^2}\frac{du}{dx}$$

EXAMPLE

$$\frac{d}{dx}\arctan x^2 = \frac{2x}{1+x^4}$$

$$\frac{d}{dx}(\sin^{-1} x^2) = \frac{1}{\sqrt{1-(x^2)^2}} \cdot \frac{d}{dx}(x^2) = \frac{2x}{\sqrt{1-x^4}}.$$

Hyperbolic Function

Definitions: Hyperbolic Functions

$$\text{sh}x = \sinh x = \frac{e^x - e^{-x}}{2}$$

$$\text{ch}x = \cosh x = \frac{e^x + e^{-x}}{2}$$

$$\text{th}x = \tanh x = \frac{\sinh x}{\cosh x}$$

$$\text{csch}x = \frac{1}{\sinh x}, \quad x > 0$$

$$\text{sech}x = \frac{1}{\cosh x}$$

$$\text{coth}x = \frac{1}{\tanh x}, \quad x > 0$$

$$\cosh^2 x - \sinh^2 x = 1$$

$$\cos^2 x + \sin^2 x = 1$$

Exercices

Derivatives Involving e^x Differentiate

(a) $(1 + x^2)e^x$ and (b) $\dfrac{1 + e^x}{2x}$.

Derivative of Functions of the Form $e^{g(x)}$ Differentiate

(a) $y = e^{5x}$ (b) $y = e^{x^2-1}$ (c) $y = e^{x-1/x}$

SOLUTION

(a) $$\frac{d}{dx}[(1+x^2)e^x] = (1+x^2)\frac{d}{dx}[e^x] + e^x\frac{d}{dx}(1+x^2)$$ Product rule.

$$= (1+x^2)e^x + e^x(2x)$$

$$= e^x(x^2+2x+1) = e^x(1+x)^2.$$

(b) $$\frac{d}{dx}\left[\frac{1+e^x}{2x}\right] = \frac{(2x)\frac{d}{dx}[e^x+1] - (e^x+1)\frac{d}{dx}[2x]}{(2x)^2}$$ Quotient rule.

$$= \frac{2xe^x - (e^x+1)(2)}{4x^2} = \frac{2xe^x - 2e^x - 2}{4x^2}$$

$$= \frac{2(xe^x - e^x - 1)}{4x^2} = \frac{xe^x - e^x - 1}{2x^2}.$$

(a) Here $g(x) = 5x$, $g'(x) = 5$, so

$$\frac{d}{dx}(e^{5x}) = e^{5x}\frac{d}{dx}(5x) = e^{5x} \cdot 5 = 5e^{5x}.$$

(b) Here $g(x) = x^2 - 1$, $g'(x) = 2x$, so

$$\frac{d}{dx}(e^{x^2-1}) = e^{x^2-1}\frac{d}{dx}(x^2 - 1) = e^{x^2-1} \cdot (2x) = 2xe^{x^2-1}.$$

(c) Here $g(x) = x - \frac{1}{x}$, $g'(x) = 1 + \frac{1}{x^2}$, so

$$\frac{d}{dx}(e^{x-1/x}) = e^{x-1/x} \cdot \left(1 + \frac{1}{x^2}\right) = \left(1 + \frac{1}{x^2}\right)e^{x-1/x}.$$

Using Properties of Exponential and Logarithm Functions Simplify

(a) $e^{\ln 4+\ln 5}$ (b) $e^{\ln 4-\ln 3}$ (c) $e^{\ln 3+2\ln 4}$ (d) $\ln\left(\frac{1}{e^2}\right)$

SOLUTION

(a) $e^{\ln 4+\ln 5} = e^{\ln 4} \cdot e^{\ln 5}$

$= (4) \cdot (5)$

$= 20$

(b) $e^{\ln 4-\ln 3} = \dfrac{e^{\ln 4}}{e^{\ln 3}}$

$= \dfrac{4}{3}$

(c) $e^{\ln 3+2\ln 4} = e^{\ln 3} \cdot e^{2\ln 4}$

$= 3 \cdot e^{(\ln 4)(2)}$

$= 3 \cdot (e^{\ln 4})^2$

$= (3)(4)^2 = 48$

(d) $\ln\left(\dfrac{1}{e^2}\right) = \ln(e^{-2})$

$= -2$

Derivatives Involving $\ln x$ Differentiate

(a) $y = (\ln x)^5$ (b) $y = x \ln x$

SOLUTION (a) By the general power rule,

$$\frac{d}{dx}(\ln x)^5 = 5(\ln x)^4 \cdot \frac{d}{dx}(\ln x) = 5(\ln x)^4 \cdot \frac{1}{x} = \frac{5(\ln x)^4}{x}.$$

(b) By the product rule,

$$\frac{d}{dx}(x \ln x) = x \cdot \frac{d}{dx}(\ln x) + (\ln x) \cdot 1 = x \cdot \frac{1}{x} + \ln x = 1 + \ln x.$$

Derivatives of Functions of the Form $y = \ln[g(x)]$ Differentiate

(a) $y = \ln(2x + 1)$ **(b)** $y = \ln(4x^2 - 2x + 9)$ **(c)** $y = \ln(xe^x)$

(a) Here, $g(x) = 2x + 1$, $g'(x) = 2$, and so,

$$\frac{d}{dx}[\ln(2x + 1)] = \frac{1}{2x + 1}\frac{d}{dx}(2x + 1) = \frac{2}{2x + 1}.$$

(b) Here, $g(x) = 4x^2 - 2x + 9$, $g'(x) = 8x - 2$, and so,

$$\frac{d}{dx}[\ln(4x^2 - 2x + 9)] = \frac{1}{4x^2 - 2x + 9}\frac{d}{dx}(4x^2 - 2x + 9) = \frac{8x - 2}{4x^2 - 2x + 9}.$$

(c) To compute the derivative of $g(x) = xe^x$, we use the product rule: $g'(x) = xe^x + e^x = e^x(x + 1)$. So,

$$\frac{d}{dx}(xe^x) = \frac{1}{xe^x}\frac{d}{dx}(xe^x) = \frac{e^x(x + 1)}{xe^x} = \frac{x + 1}{x}.$$

Analyzing a Function Involving ln x The function $f(x) = (\ln x)/x$ has a relative extreme point for some $x > 0$. Find the point and determine whether it is a relative maximum or a relative minimum point.

SOLUTION By the quotient rule,

$$f'(x) = \frac{x \cdot \frac{1}{x} - (\ln x) \cdot 1}{x^2} = \frac{1 - \ln x}{x^2}$$

$$f''(x) = \frac{x^2 \cdot \left(-\frac{1}{x}\right) - (1 - \ln x)(2x)}{x^4}$$

$$= \frac{-x - (2x - 2x \ln x)}{x^4}$$

$$= \frac{-3x + 2x \ln x}{x^4} = \frac{x(-3 + 2 \ln x)}{x^4}$$

$$= \frac{-3 + 2 \ln x}{x^3}$$

If we set $f'(x) = 0$, then,

$$\frac{1 - \ln x}{x^2} = 0$$

$$1 - \ln x = 0 \qquad \text{Multiply by } x^2 \neq 0.$$

$$\ln x = 1 \qquad \text{Add } \ln x \text{ to each side.}$$

$$e^{\ln x} = e^1 = e \qquad \text{Take exponential of each side.}$$

$$x = e. \qquad e^{\ln x} = x.$$

Therefore, the only possible relative extreme point is at $x = e$. When $x = e$, $f(e) = (\ln e)/e = 1/e$. Furthermore,

$$f''(e) = \frac{2 \ln e - 3}{e^3} = -\frac{1}{e^3} < 0,$$

which implies that the graph of $f(x)$ is concave down at $x = e$. Therefore, $(e, 1/e)$ is a relative maximum point of the graph of $f(x)$. ( )

Derivative of $\ln|x|$ The function $y = \ln|x|$ is defined for all nonzero values of x. Its graph is sketched in Fig. 3. Compute the derivative of $y = \ln|x|$.

If x is positive, $|x| = x$, so,

$$\frac{d}{dx}\ln|x| = \frac{d}{dx}\ln x = \frac{1}{x}.$$

If x is negative, $|x| = -x$ and, by the chain rule,

$$\frac{d}{dx}\ln|x| = \frac{d}{dx}\ln(-x) = \frac{1}{-x}\cdot\frac{d}{dx}(-x) = \frac{1}{-x}\cdot(-1) = \frac{1}{x}.$$

Therefore, we have established the following useful fact:

$$\frac{d}{dx}\ln|x| = \frac{1}{x}, \qquad x \neq 0.$$

Thank you for your attention