



Calculus 1

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Calculus 1

Lecture 8

Integrals

Chapter 5

Integrals

3.1 Antiderivatives

3.2 Indefinite Integrals

3.3 Integration by Parts



More generally, starting with a function f , we want to find a function F whose derivative is f . If such a function F exists, it is called an *antiderivative* of f . Antiderivatives are the link connecting the two major elements of calculus: derivatives and definite integrals.

Finding Antiderivatives

DEFINITION A function F is an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all x in I .

EXAMPLE 1 Find an antiderivative for each of the following functions.

(a) $f(x) = 2x$ (b) $g(x) = \cos x$ (c) $h(x) = \frac{1}{x} + 2e^{2x}$

Solution We need to think backward here: What function do we know has a derivative equal to the given function?

(a) $F(x) = x^2$ (b) $G(x) = \sin x$ (c) $H(x) = \ln |x| + e^{2x}$

Each answer can be checked by differentiating. The derivative of $F(x) = x^2$ is $2x$. The derivative of $G(x) = \sin x$ is $\cos x$, and the derivative of $H(x) = \ln |x| + e^{2x}$ is $(1/x) + 2e^{2x}$. ■

The function $F(x) = x^2$ is not the only function whose derivative is $2x$. The function $x^2 + 1$ has the same derivative. So does $x^2 + C$ for any constant C . Are there others?



THEOREM 8 If F is an antiderivative of f on an interval I , then the most general antiderivative of f on I is

$$F(x) + C$$

where C is an arbitrary constant.

EXAMPLE 2 Find an antiderivative of $f(x) = 3x^2$ that satisfies $F(1) = -1$.

Solution Since the derivative of x^3 is $3x^2$, the general antiderivative

$$F(x) = x^3 + C$$

gives all the antiderivatives of $f(x)$. The condition $F(1) = -1$ determines a specific value for C . Substituting $x = 1$ into $F(x) = x^3 + C$ gives

$$F(1) = (1)^3 + C = 1 + C.$$

Since $F(1) = -1$, solving $1 + C = -1$ for C gives $C = -2$. So

$$F(x) = x^3 - 2$$

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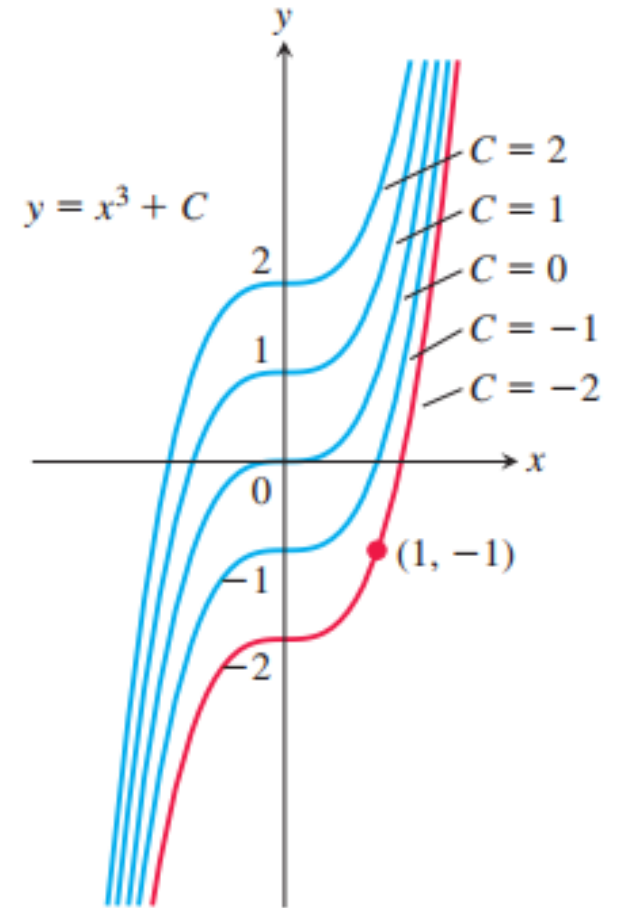




TABLE 4.2 Antiderivative formulas, k a nonzero constant

Function	General antiderivative	Function	General antiderivative
1. x^n	$\frac{1}{n+1}x^{n+1} + C, \quad n \neq -1$	8. e^{kx}	$\frac{1}{k}e^{kx} + C$
2. $\sin kx$	$-\frac{1}{k}\cos kx + C$	9. $\frac{1}{x}$	$\ln x + C, \quad x \neq 0$
3. $\cos kx$	$\frac{1}{k}\sin kx + C$	10. $\frac{1}{\sqrt{1-k^2x^2}}$	$\frac{1}{k}\sin^{-1} kx + C$
4. $\sec^2 kx$	$\frac{1}{k}\tan kx + C$	11. $\frac{1}{1+k^2x^2}$	$\frac{1}{k}\tan^{-1} kx + C$
5. $\csc^2 kx$	$-\frac{1}{k}\cot kx + C$	12. $\frac{1}{x\sqrt{k^2x^2-1}}$	$\sec^{-1} kx + C, \quad kx > 1$
6. $\sec kx \tan kx$	$\frac{1}{k}\sec kx + C$	13. a^{kx}	$\left(\frac{1}{k \ln a}\right)a^{kx} + C, \quad a > 0, a \neq 1$
7. $\csc kx \cot kx$	$-\frac{1}{k}\csc kx + C$		



EXAMPLE 3

Find the general antiderivative of each of the following functions.

(a) $f(x) = x^5$

(b) $g(x) = \frac{1}{\sqrt{x}}$

(c) $h(x) = \sin 2x$

(d) $i(x) = \cos \frac{x}{2}$

(e) $j(x) = e^{-3x}$

(f) $k(x) = 2^x$



Solution In each case, we can use one of the formulas listed in Table 4.2.

(a) $F(x) = \frac{x^6}{6} + C$

Formula 1
with $n = 5$

(b) $g(x) = x^{-1/2}$, so

$$G(x) = \frac{x^{1/2}}{1/2} + C = 2\sqrt{x} + C$$

Formula 1
with $n = -1/2$

(c) $H(x) = \frac{-\cos 2x}{2} + C$

Formula 2
with $k = 2$

(d) $I(x) = \frac{\sin(x/2)}{1/2} + C = 2 \sin \frac{x}{2} + C$

Formula 3
with $k = 1/2$

(e) $J(x) = -\frac{1}{3}e^{-3x} + C$

Formula 8
with $k = -3$

(f) $K(x) = \left(\frac{1}{\ln 2}\right)2^x + C$

Formula 13
with $a = 2, k = 1$ ■

TABLE 4.3 Antiderivative linearity rules

	Function	General antiderivative
1. <i>Constant Multiple Rule:</i>	$kf(x)$	$kF(x) + C$, k a constant
2. <i>Negative Rule:</i>	$-f(x)$	$-F(x) + C$
3. <i>Sum or Difference Rule:</i>	$f(x) \pm g(x)$	$F(x) \pm G(x) + C$

EXAMPLE 4 Find the general antiderivative of

$$f(x) = \frac{3}{\sqrt{x}} + \sin 2x.$$



$$\begin{aligned} F(x) &= 3G(x) + H(x) + C \\ &= 6\sqrt{x} - \frac{1}{2}\cos 2x + C \end{aligned}$$



Indefinite Integrals

Indefinite Integrals

A special symbol is used to denote the collection of all antiderivatives of a function f .

DEFINITION The collection of all antiderivatives of f is called the **indefinite integral** of f with respect to x , and is denoted by

$$\int f(x) dx.$$

The symbol \int is an **integral sign**. The function f is the **integrand** of the integral, and x is the **variable of integration**.



Indefinite Integrals

$$\int 2x \, dx = x^2 + C,$$

$$\int \cos x \, dx = \sin x + C,$$

$$\int \left(\sec^2 x + \frac{1}{2\sqrt{x}} \right) dx = \tan x + \sqrt{x} + C$$

EXAMPLE 6 Evaluate

$$\int (x^2 - 2x + 5) \, dx.$$



Indefinite Integrals

Solution If we recognize that $(x^3/3) - x^2 + 5x$ is an antiderivative of $x^2 - 2x + 5$, we can evaluate the integral as

$$\int (x^2 - 2x + 5) dx = \overbrace{\frac{x^3}{3} - x^2 + 5x}^{\text{antiderivative}} + \underbrace{C}_{\text{arbitrary constant}}$$

If we do not recognize the antiderivative right away, we can generate it term-by-term with the Sum, Difference, and Constant Multiple Rules:

$$\begin{aligned}\int (x^2 - 2x + 5) dx &= \int x^2 dx - \int 2x dx + \int 5 dx \\ &= \int x^2 dx - 2 \int x dx + 5 \int 1 dx \\ &= \left(\frac{x^3}{3} + C_1 \right) - 2 \left(\frac{x^2}{2} + C_2 \right) + 5(x + C_3) \\ &= \frac{x^3}{3} + C_1 - x^2 - 2C_2 + 5x + 5C_3.\end{aligned}$$



Indefinite Integrals

$$\begin{aligned}\int (x^2 - 2x + 5) dx &= \int x^2 dx - \int 2x dx + \int 5 dx \\ &= \frac{x^3}{3} - x^2 + 5x + C.\end{aligned}$$



Integration by Parts

Integration by Parts Formula

$$\int u dv = uv - \int v du \quad (2)$$

EXAMPLE 1 Find

$$\int x \cos x dx.$$

Solution We use the formula $\int u dv = uv - \int v du$ with

$$\begin{aligned} u &= x, & dv &= \cos x dx, \\ du &= dx, & v &= \sin x. \end{aligned}$$

Simplest antiderivative of $\cos x$

Then

$$\int x \cos x dx = x \sin x - \int \sin x dx = x \sin x + \cos x + C. \quad \blacksquare$$



Integration by Parts

EXAMPLE 2 Find $\int \ln x \, dx$.

Solution

$$u(x) = \ln x \quad \text{and} \quad v'(x) = 1.$$

We differentiate $u(x)$ and find an antiderivative of $v'(x)$,

$$u'(x) = \frac{1}{x} \quad \text{and} \quad v(x) = x.$$

Then

$$\begin{aligned} \int \underbrace{\ln x}_{u(x)} \cdot \underbrace{1}_{v'(x)} \, dx &= \underbrace{(\ln x)}_{u(x)} \underbrace{x}_{v(x)} - \int \underbrace{x}_{v(x)} \underbrace{\frac{1}{x}}_{u'(x)} \, dx && \text{Integration by parts formula} \\ &= x \ln x - x + C && \text{Simplify and integrate.} \end{aligned}$$





Integration by Parts

EXAMPLE 3 Evaluate

$$\int x^2 e^x dx.$$

Solution We use the integration by parts formula Equation (1) with

$$u(x) = x^2 \quad \text{and} \quad v'(x) = e^x.$$

We differentiate $u(x)$ and find an antiderivative of $v'(x)$,

$$u'(x) = 2x \quad \text{and} \quad v(x) = e^x.$$

We summarize this choice by setting $du = u'(x) dx$ and $dv = v'(x) dx$, so

$$du = 2x dx \quad \text{and} \quad dv = e^x dx.$$

We then have

$$\int \underbrace{x^2}_u \underbrace{e^x}_{dv} dx = \underbrace{x^2}_u \underbrace{e^x}_v - \int \underbrace{e^x}_v \underbrace{2x}_{du} dx. \quad \text{Integration by parts formula}$$

$u = x, dv = e^x dx$. Then $du = dx, v = e^x$, and

$$\int \underbrace{xe^x}_{u \, dv} dx = \underbrace{xe^x}_{u \, v} - \int \underbrace{e^x}_{v} \underbrace{dx}_{du} = xe^x - e^x + C.$$

Integration by parts Equation (2)

$$u = x, dv = e^x dx$$

$$v = e^x, du = dx$$

Using this last evaluation, we then obtain

$$\begin{aligned} \int x^2 e^x dx &= x^2 e^x - 2 \int x e^x dx \\ &= x^2 e^x - 2x e^x + 2e^x + C, \end{aligned}$$

where the constant of integration is renamed after substituting for the integral on the right. ■



Integration by Parts

EXAMPLE 4 Evaluate

$$\int e^x \cos x \, dx.$$

Solution Let $u = e^x$ and $dv = \cos x \, dx$. Then $du = e^x \, dx$, $v = \sin x$, and

$$\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx. \quad u(x) = e^x, \quad v(x) = \sin x$$

The second integral is like the first except that it has $\sin x$ in place of $\cos x$. To evaluate it, we use integration by parts with

$$u = e^x, \quad dv = \sin x \, dx, \quad v = -\cos x, \quad du = e^x \, dx.$$

Then

$$\int e^x \cos x \, dx = e^x \sin x - \left(-e^x \cos x - \int (-\cos x)(e^x \, dx) \right) \quad u(x) = e^x, \quad v(x) = -\cos x$$
$$= e^x \sin x + e^x \cos x - \int e^x \cos x \, dx.$$

The unknown integral now appears on both sides of the equation, but with opposite signs. Adding the integral to both sides and adding the constant of integration give

$$2 \int e^x \cos x \, dx = e^x \sin x + e^x \cos x + C_1.$$

Dividing by 2 and renaming the constant of integration give

$$\int e^x \cos x \, dx = \frac{e^x \sin x + e^x \cos x}{2} + C. \quad \blacksquare$$



Integration by Parts

EXAMPLE 5 Obtain a formula that expresses the integral

$$\int \cos^n x \, dx$$

Solution We may think of $\cos^n x$ as $\cos^{n-1} x \cdot \cos x$. Then we let

$$u = \cos^{n-1} x \quad \text{and} \quad dv = \cos x \, dx,$$

so that

$$du = (n - 1) \cos^{n-2} x (-\sin x \, dx) \quad \text{and} \quad v = \sin x.$$

Integration by parts then gives

$$\begin{aligned} \int \cos^n x \, dx &= \cos^{n-1} x \sin x + (n - 1) \int \sin^2 x \cos^{n-2} x \, dx \\ &= \cos^{n-1} x \sin x + (n - 1) \int (1 - \cos^2 x) \cos^{n-2} x \, dx \\ &= \cos^{n-1} x \sin x + (n - 1) \int \cos^{n-2} x \, dx - (n - 1) \int \cos^n x \, dx. \end{aligned}$$



Integration by Parts

If we add

$$(n - 1) \int \cos^n x \, dx$$

to both sides of this equation, we obtain

$$n \int \cos^n x \, dx = \cos^{n-1} x \sin x + (n - 1) \int \cos^{n-2} x \, dx.$$

We then divide through by n , and the final result is

$$\int \cos^n x \, dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} \int \cos^{n-2} x \, dx.$$

$$\begin{aligned} \int \cos^3 x \, dx &= \frac{\cos^2 x \sin x}{3} + \frac{2}{3} \int \cos x \, dx \\ &= \frac{1}{3} \cos^2 x \sin x + \frac{2}{3} \sin x + C. \end{aligned}$$

Finding Antiderivatives

In Exercises 1–24, find an antiderivative for each function. Do as many as you can mentally. Check your answers by differentiation.

1. a. $2x$

b. x^2

c. $x^2 - 2x + 1$

2. a. $6x$

b. x^7

c. $x^7 - 6x + 8$

3. a. $-3x^{-4}$

b. x^{-4}

c. $x^{-4} + 2x + 3$

4. a. $2x^{-3}$

b. $\frac{x^{-3}}{2} + x^2$

c. $-x^{-3} + x - 1$

5. a. $\frac{1}{x^2}$

b. $\frac{5}{x^2}$

c. $2 - \frac{5}{x^2}$

6. a. $-\frac{2}{x^3}$

b. $\frac{1}{2x^3}$

c. $x^3 - \frac{1}{x^3}$

7. a. $\frac{3}{2}\sqrt{x}$

b. $\frac{1}{2\sqrt{x}}$

c. $\sqrt{x} + \frac{1}{\sqrt{x}}$

8. a. $\frac{4}{3}\sqrt[3]{x}$

b. $\frac{1}{3\sqrt[3]{x}}$

c. $\sqrt[3]{x} + \frac{1}{\sqrt[3]{x}}$

9. a. $\frac{2}{3}x^{-1/3}$

b. $\frac{1}{3}x^{-2/3}$

c. $-\frac{1}{3}x^{-4/3}$



1. (a) x^2

(b) $\frac{x^3}{3}$

(c) $\frac{x^3}{3} - x^2 + x$

2. (a) $3x^2$

(b) $\frac{x^8}{8}$

(c) $\frac{x^8}{8} - 3x^2 + 8x$

3. (a) x^{-3}

(b) $-\frac{x^{-3}}{3}$

(c) $-\frac{x^{-3}}{3} + x^2 + 3x$

4. (a) $-x^{-2}$

(b) $-\frac{x^{-2}}{4} + \frac{x^3}{3}$

(c) $\frac{x^{-2}}{2} + \frac{x^2}{2} - x$

5. (a) $\frac{-1}{x}$

(b) $\frac{-5}{x}$

(c) $2x + \frac{5}{x}$

6. (a) $\frac{1}{x^2}$

(b) $\frac{-1}{4x^2}$

(c) $\frac{x^4}{4} + \frac{1}{2x^2}$

7. (a) $\sqrt{x^3}$

(b) \sqrt{x}

(c) $\frac{2}{3}\sqrt{x^3} + 2\sqrt{x}$

8. (a) $x^{4/3}$

(b) $\frac{1}{2}x^{2/3}$

(c) $\frac{3}{4}x^{4/3} + \frac{3}{2}x^{2/3}$

9. (a) $x^{2/3}$

(b) $x^{1/3}$

(c) $x^{-1/3}$

11. a. $\frac{1}{x}$ b. $\frac{7}{x}$ c. $1 - \frac{5}{x}$
12. a. $\frac{1}{3x}$ b. $\frac{2}{5x}$ c. $1 + \frac{4}{3x} - \frac{1}{x^2}$
13. a. $-\pi \sin \pi x$ b. $3 \sin x$ c. $\sin \pi x - 3 \sin 3x$
14. a. $\pi \cos \pi x$ b. $\frac{\pi}{2} \cos \frac{\pi x}{2}$ c. $\cos \frac{\pi x}{2} + \pi \cos x$
15. a. $\sec^2 x$ b. $\frac{2}{3} \sec^2 \frac{x}{3}$ c. $-\sec^2 \frac{3x}{2}$
16. a. $\csc^2 x$ b. $-\frac{3}{2} \csc^2 \frac{3x}{2}$ c. $1 - 8 \csc^2 2x$
17. a. $\csc x \cot x$ b. $-\csc 5x \cot 5x$ c. $-\pi \csc \frac{\pi x}{2} \cot \frac{\pi x}{2}$
18. a. $\sec x \tan x$ b. $4 \sec 3x \tan 3x$ c. $\sec \frac{\pi x}{2} \tan \frac{\pi x}{2}$
19. a. e^{3x} b. e^{-x} c. $e^{x/2}$
20. a. e^{-2x} b. $e^{4x/3}$ c. $e^{-x/5}$

11. (a) $\ln |x|$ (b) $7 \ln |x|$ (c) $x - 5 \ln |x|$
12. (a) $\frac{1}{3} \ln |x|$ (b) $\frac{2}{5} \ln |x|$ (c) $x + \frac{4}{3} \ln |x| + \frac{1}{x}$
13. (a) $\cos(\pi x)$ (b) $-3 \cos x$ (c) $\frac{-\cos(\pi x)}{\pi} + \cos(3x)$
14. (a) $\sin(\pi x)$ (b) $\sin\left(\frac{\pi x}{2}\right)$ (c) $\left(\frac{2}{\pi}\right) \sin\left(\frac{\pi x}{2}\right) + \pi \sin x$
15. (a) $\tan x$ (b) $2 \tan\left(\frac{x}{3}\right)$ (c) $-\frac{2}{3} \tan\left(\frac{3x}{2}\right)$
16. (a) $-\cot x$ (b) $\cot\left(\frac{3x}{2}\right)$ (c) $x + 4 \cot(2x)$
17. (a) $-\csc x$ (b) $\frac{1}{5} \csc(5x)$ (c) $2 \csc\left(\frac{\pi x}{2}\right)$
18. (a) $\sec x$ (b) $\frac{4}{3} \sec(3x)$ (c) $\frac{2}{\pi} \sec\left(\frac{\pi x}{2}\right)$
19. (a) $\frac{1}{3} e^{3x}$ (b) $-e^{-x}$ (c) $2e^{x/2}$
20. (a) $-\frac{1}{2} e^{-2x}$ (b) $\frac{3}{4} e^{4x/3}$ (c) $-5e^{-x/5}$

21. a. 3^x

b. 2^{-x}

c. $\left(\frac{5}{3}\right)^x$

22. a. $x^{\sqrt{3}}$

b. x^π

c. $x^{\sqrt{2}-1}$

21. (a) $\frac{1}{\ln 3} \cdot 3^x$

(b) $\frac{-1}{\ln 2} \cdot 2^{-x}$

(c) $\frac{1}{\ln(5/3)} \cdot \left(\frac{5}{3}\right)^x$

22. (a) $\frac{1}{\sqrt{3}+1} x^{\sqrt{3}+1}$

(b) $\frac{1}{\pi+1} x^{\pi+1}$

(c) $\frac{1}{\sqrt{2}} x^{\sqrt{2}}$

33. $\int x^{-1/3} dx$

35. $\int (\sqrt{x} + \sqrt[3]{x}) dx$

37. $\int \left(8y - \frac{2}{y^{1/4}}\right) dy$

39. $\int 2x(1 - x^{-3}) dx$

41. $\int \frac{t\sqrt{t} + \sqrt{t}}{t^2} dt$

43. $\int (-2 \cos t) dt$

45. $\int 7 \sin \frac{\theta}{3} d\theta$

47. $\int (-3 \csc^2 x) dx$

49. $\int \frac{\csc \theta \cot \theta}{2} d\theta$

34. $\int x^{-5/4} dx$

36. $\int \left(\frac{\sqrt{x}}{2} + \frac{2}{\sqrt{x}}\right) dx$

38. $\int \left(\frac{1}{7} - \frac{1}{y^{5/4}}\right) dy$

40. $\int x^{-3}(x + 1) dx$

42. $\int \frac{4 + \sqrt{t}}{t^3} dt$

44. $\int (-5 \sin t) dt$

46. $\int 3 \cos 5\theta d\theta$

48. $\int \left(-\frac{\sec^2 x}{3}\right) dx$

50. $\int \frac{2}{5} \sec \theta \tan \theta d\theta$

$$33. \int x^{-1/3} dx = \frac{x^{2/3}}{\frac{2}{3}} + C = \frac{3}{2} x^{2/3} + C$$

$$34. \int x^{-5/4} dx = \frac{x^{-1/4}}{-\frac{1}{4}} + C = \frac{-4}{\sqrt[4]{x}} + C$$

$$35. \int (\sqrt{x} + \sqrt[3]{x}) dx = \int (x^{1/2} + x^{1/3}) dx = \frac{x^{3/2}}{\frac{3}{2}} + \frac{x^{4/3}}{\frac{4}{3}} + C = \frac{2}{3} x^{3/2} + \frac{3}{4} x^{4/3} + C$$

$$36. \int \left(\frac{\sqrt{x}}{2} + \frac{2}{\sqrt{x}} \right) dx = \int \left(\frac{1}{2} x^{1/2} + 2x^{-1/2} \right) dx = \frac{1}{2} \left(\frac{x^{3/2}}{\frac{3}{2}} \right) + 2 \left(\frac{x^{1/2}}{\frac{1}{2}} \right) + C = \frac{1}{3} x^{3/2} + 4x^{1/2} + C$$

$$37. \int \left(8y - \frac{2}{y^{1/4}} \right) dy = \int \left(8y - 2y^{-1/4} \right) dy = \frac{8y^2}{2} - 2 \left(\frac{y^{3/4}}{\frac{3}{4}} \right) + C = 4y^2 - \frac{8}{3} y^{3/4} + C$$

$$38. \int \left(\frac{1}{7} - \frac{1}{y^{5/4}} \right) dy = \int \left(\frac{1}{7} - y^{-5/4} \right) dy = \frac{1}{7} y - \left(\frac{y^{-1/4}}{-\frac{1}{4}} \right) + C = \frac{y}{7} + \frac{4}{y^{1/4}} + C$$

$$39. \int 2x(1-x^{-3}) dx = \int (2x - 2x^{-2}) dx = \frac{2x^2}{2} - 2 \left(\frac{x^{-1}}{-1} \right) + C = x^2 + \frac{2}{x} + C$$

$$40. \int x^{-3}(x+1) dx = \int (x^{-2} + x^{-3}) dx = \frac{x^{-1}}{-1} + \left(\frac{x^{-2}}{-2} \right) + C = -\frac{1}{x} - \frac{1}{2x^2} + C$$

$$41. \int \frac{t\sqrt{t} + \sqrt{t}}{t^2} dt = \int \left(\frac{t^{3/2}}{t^2} + \frac{t^{1/2}}{t^2} \right) dt = \int (t^{-1/2} + t^{-3/2}) dt = \frac{t^{1/2}}{\frac{1}{2}} + \left(\frac{t^{-1/2}}{-\frac{1}{2}} \right) + C = 2\sqrt{t} - \frac{2}{\sqrt{t}} + C$$

$$42. \int \frac{4+\sqrt{t}}{t^3} dt = \int \left(\frac{4}{t^3} + \frac{t^{1/2}}{t^3} \right) dt = \int \left(4t^{-3} + t^{-5/2} \right) dt = 4 \left(\frac{t^{-2}}{-2} \right) + \left(\frac{t^{-3/2}}{-\frac{3}{2}} \right) + C = -\frac{2}{t^2} - \frac{2}{3t^{3/2}} + C$$

$$43. \int -2 \cos t dt = -2 \sin t + C$$

$$44. \int -5 \sin t dt = 5 \cos t + C$$

$$45. \int 7 \sin \frac{\theta}{3} d\theta = -21 \cos \frac{\theta}{3} + C$$

$$46. \int 3 \cos 5\theta d\theta = \frac{3}{5} \sin 5\theta + C$$

$$47. \int -3 \csc^2 x dx = 3 \cot x + C$$

$$48. \int -\frac{\sec^2 x}{3} dx = -\frac{\tan x}{3} + C$$

$$49. \int \frac{\csc \theta \cot \theta}{2} d\theta = -\frac{1}{2} \csc \theta + C$$

$$50. \int \frac{2}{5} \sec \theta \tan \theta d\theta = \frac{2}{5} \sec \theta + C$$

51. $\int (e^{3x} + 5e^{-x}) dx$

52. $\int (2e^x - 3e^{-2x}) dx$

53. $\int (e^{-x} + 4^x) dx$

54. $\int (1.3)^x dx$

61. $\int \left(\frac{1}{x} - \frac{5}{x^2 + 1} \right) dx$

55. $\int (4 \sec x \tan x - 2 \sec^2 x) dx$

56. $\int \frac{1}{2} (\csc^2 x - \csc x \cot x) dx$

57. $\int (\sin 2x - \csc^2 x) dx$

58. $\int (2 \cos 2x - 3 \sin 3x) dx$

59. $\int \frac{1 + \cos 4t}{2} dt$

60. $\int \frac{1 - \cos 6t}{2} dt$

63. $\int 3x^{\sqrt{3}} dx$

64. $\int x^{\sqrt{2}-1} dx$

$$51. \int (e^{3x} + 5e^{-x}) dx = \frac{e^{3x}}{3} - 5e^{-x} + C$$

$$52. \int (2e^x - 3e^{-2x}) dx = 2e^x + \frac{3}{2}e^{-2x} + C$$

$$53. \int (e^{-x} + 4^x) dx = -e^{-x} + \frac{4^x}{\ln 4} + C$$

$$54. \int (1.3)^x dx = \frac{(1.3)^x}{\ln(1.3)} + C$$

$$55. \int (4 \sec x \tan x - 2 \sec^2 x) dx = 4 \sec x - 2 \tan x + C$$

$$56. \int \frac{1}{2} (\csc^2 x - \csc x \cot x) dx = -\frac{1}{2} \cot x + \frac{1}{2} \csc x + C$$

$$57. \int (\sin 2x - \csc^2 x) dx = -\frac{1}{2} \cos 2x + \cot x + C$$

$$58. \int (2 \cos 2x - 3 \sin 3x) dx = \sin 2x + \cos 3x + C$$

$$59. \int \frac{1 + \cos 4t}{2} dt = \int \left(\frac{1}{2} + \frac{1}{2} \cos 4t \right) dt = \frac{1}{2}t + \frac{1}{2} \left(\frac{\sin 4t}{4} \right) + C = \frac{t}{2} + \frac{\sin 4t}{8} + C$$

$$60. \int \frac{1 - \cos 6t}{2} dt = \int \left(\frac{1}{2} - \frac{1}{2} \cos 6t \right) dt = \frac{1}{2}t - \frac{1}{2} \left(\frac{\sin 6t}{6} \right) + C = \frac{t}{2} - \frac{\sin 6t}{12} + C$$

$$61. \int \left(\frac{1}{x} - \frac{5}{x^2+1} \right) dx = \ln|x| - 5 \tan^{-1} x + C$$

$$63. \int 3x^{\sqrt{3}} dx = \frac{3x^{(\sqrt{3}+1)}}{\sqrt{3}+1} + C$$

$$64. \int x^{(\sqrt{2}-1)} dx = \frac{x^{\sqrt{2}}}{\sqrt{2}} + C$$

65. $\int (1 + \tan^2 \theta) d\theta$
(Hint: $1 + \tan^2 \theta = \sec^2 \theta$)

66. $\int (2 + \tan^2 \theta) d\theta$

67. $\int \cot^2 x dx$
(Hint: $1 + \cot^2 x = \csc^2 x$)

68. $\int (1 - \cot^2 x) dx$

69. $\int \cos \theta (\tan \theta + \sec \theta) d\theta$ 70. $\int \frac{\csc \theta}{\csc \theta - \sin \theta} d\theta$

$$65. \int (1 + \tan^2 \theta) d\theta = \int \sec^2 \theta d\theta = \tan \theta + C$$

$$66. \int (2 + \tan^2 \theta) d\theta = \int (1 + 1 + \tan^2 \theta) d\theta = \int (1 + \sec^2 \theta) d\theta = \theta + \tan \theta + C$$

$$67. \int \cot^2 x dx = \int (\csc^2 x - 1) dx = -\cot x - x + C$$

$$68. \int (1 - \cot^2 x) dx = \int (1 - (\csc^2 x - 1)) dx = \int (2 - \csc^2 x) dx = 2x + \cot x + C$$

$$69. \int \cos \theta (\tan \theta + \sec \theta) d\theta = \int (\sin \theta + 1) d\theta = -\cos \theta + \theta + C$$

$$70. \int \frac{\csc \theta}{\csc \theta - \sin \theta} d\theta = \int \left(\frac{\csc \theta}{\csc \theta - \sin \theta} \right) \left(\frac{\sin \theta}{\sin \theta} \right) d\theta = \int \frac{1}{1 - \sin^2 \theta} d\theta = \int \frac{1}{\cos^2 \theta} d\theta = \int \sec^2 \theta d\theta = \tan \theta + C$$

1. $\int x \sin \frac{x}{2} dx$

3. $\int t^2 \cos t dt$

10. $\int (x^2 - 2x + 1)e^{2x} dx$

4. $\int x^2 \sin x dx$

6. $\int_1^e x^3 \ln x dx$

22. $\int e^{-y} \cos y dy$

1. $u = x, du = dx; dv = \sin \frac{x}{2} dx, v = -2 \cos \frac{x}{2};$

$$\int x \sin \frac{x}{2} dx = -2x \cos \frac{x}{2} - \int \left(-2 \cos \frac{x}{2}\right) dx = -2x \cos \left(\frac{x}{2}\right) + 4 \sin \left(\frac{x}{2}\right) + C$$

$$\int t^2 \cos t dt = t^2 \sin t + 2t \cos t - 2 \sin t + C$$

$$\int x^2 \sin x dx = -x^2 \cos x + 2x \sin x + 2 \cos x + C$$

6. $u = \ln x, du = \frac{dx}{x}; dv = x^3 dx, v = \frac{x^4}{4};$

$$\int_1^e x^3 \ln x dx = \left[\frac{x^4}{4} \ln x \right]_1^e - \int_1^e \frac{x^4}{4} \frac{dx}{x} = \frac{e^4}{4} - \left[\frac{x^4}{16} \right]_1^e = \frac{3e^4 + 1}{16}$$

$$\int (x^2 - 2x + 1)e^{2x} dx = \frac{1}{2}(x^2 - 2x + 1)e^{2x} - \frac{1}{4}(2x - 2)e^{2x} + \frac{1}{4}e^{2x} + C$$

$$= \left(\frac{1}{2}x^2 - \frac{3}{2}x + \frac{5}{4}\right)e^{2x} + C$$

22. $I = \int e^{-y} \cos y dy; [u = \cos y, du = -\sin y dy; dv = e^{-y} dy, v = -e^{-y}]$

$$\Rightarrow I = -e^{-y} \cos y - \int (-e^{-y})(-\sin y) dy = -e^{-y} \cos y - \int e^{-y} \sin y dy;$$

$$[u = \sin y, du = \cos y dy; dv = e^{-y} dy, v = -e^{-y}] \Rightarrow I = -e^{-y} \cos y - \left(-e^{-y} \sin y - \int (-e^{-y}) \cos y dy\right)$$

$$= -e^{-y} \cos y + e^{-y} \sin y - I + C' \Rightarrow 2I = e^{-y} (\sin y - \cos y) + C' \Rightarrow I = \frac{1}{2}(e^{-y} \sin y - e^{-y} \cos y) + C, \text{ where}$$

$C = \frac{C'}{2}$ is another arbitrary constant



Thank you for your attention