# Tension and Compression in Bars

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Objectives: *Mechanics of Materials* investigates the stressing and the deformations of structures subjected to applied loads, starting by the simplest structural members, namely, bars in tension or compression.

يدرس ميكانيك المواد إجهادات وتشوهات الجمل الإنشائية (الهياكل الحاملة) الناتجة عن الحمولات الخارجية، مبتدأً بالعناصر الأبسط أي القضبان (العناصر الطولية) المشدودة أو المضغوطة.

In order to treat such problems, the kinematic relations and a constitutive law are needed to complement the equilibrium conditions which are known from Engineering Mechanics (Statics).

تقوم هذه الدراسة على:

(1) معادلات التوازن التي دُرست في الميكانيك الهندسي (علم السكون)

(2) العلاقات الكينماتيكية التي ستدرس وهي تصف التشوهات كمياً أي تحدد شكل ومقدار تغيرات الشكل الجيومتري.

(3) قوانين سلوك مادة الجملة وهي كما ستُعرض لاحقاً، قوانين تجريبية تعرّف السلوك الميكانيكي لمادة الهيكل الحامل.

The kinematic relations represent the geometry of the deformation, whereas the behavior of the material is described by the constitutive law. The students will learn how to apply these equations and how to solve determinate as well as statically indeterminate problems. يعالج الطلبة مسائل مقررة سكونياً وأخرى غير مقررة سكونياً

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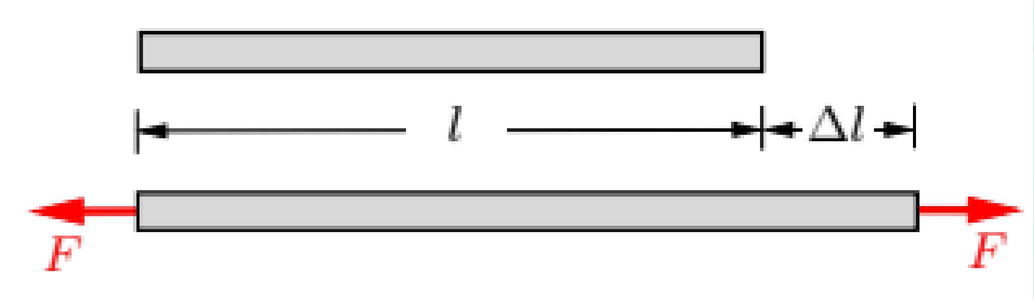
# 2 Strain التشوه

Let us first consider a bar with a constant cross-sectional area which has the undeformed length $l$.

نبدأ أولاً بالنظر إلى قضيب بمقطع ثابت وطول أولي غير مشوه قدره: $l$ .

Under the action of tensile forces (Fig.) it gets slightly longer.

تحت تأثير قوتي شد كما في الشكل سيتطاول القضيب قليلاً.




The elongation is denoted by $\Delta l$ and is assumed to be much smaller than the original length $l$.

نرمز لهذا التطاول $\Delta l$ ونفترض أنه مقدار صغير جداً مقارنة مع الطول $l$.

As a measure of the amount of deformation, it is useful to introduce, in addition to the elongation, the ratio between the elongation and the original (undeformed) length:

نقيس التشوه الطولي بالنسبة بين التطاول والطول الأصلي

$$\varepsilon = \frac{\Delta l}{l}$$

**The <u>dimensionless</u> quantity $\varepsilon$ is called *strain*.** التشوه الطولي النسبي وهنا اختصارا التشوه، مقدارلابعدي (دون واحدات)

Example: If, for example, a bar of the length $l = 1$ m undergoes an elongation of $\Delta l = 0.5$ mm then we have $\varepsilon = 0.5 \times 10^{-3}$. This is a strain of $0.05\%$.

If the bar gets longer ($\Delta l > 0$) the strain is positive; it is negative in the case of a shortening.

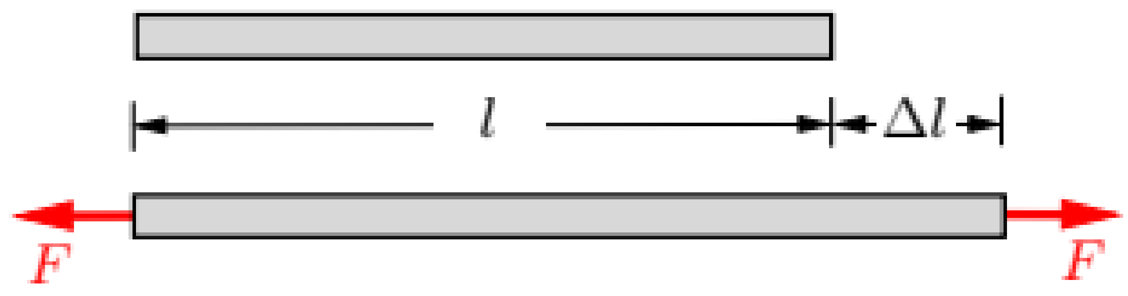
إذا ازداد الطول أي كان $(\Delta l > 0)$ يكون التشوه موجباً، وهذا يتوافق مع إجهاد الشد الموجب.

أما إذا نقص الطول أي كان $(\Delta l < 0)$ يكون التشوه سالباً، وهذا يتوافق مع إجهاد الضغط السالب..

In what follows we will consider only small deformations

فيما يلي سنفترض أن التشوهات صغيرة دوماً

$$|\Delta l| \ll l \text{ or } |\varepsilon| \ll 1.$$

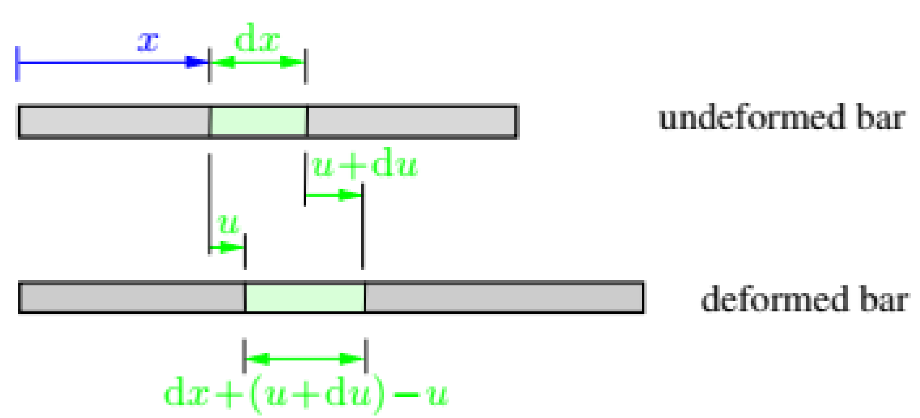



The above definition $\varepsilon = \frac{\Delta l}{l}$ for the strain is valid only if $\varepsilon$ is constant over the bar length.

يصح التعريف السابق فقط إذا كان التشوه ثابتاً على كامل الطول.

If the cross-sectional area is not constant or if the bar is subjected to volume forces acting along its axis, the strain may depend on the location.

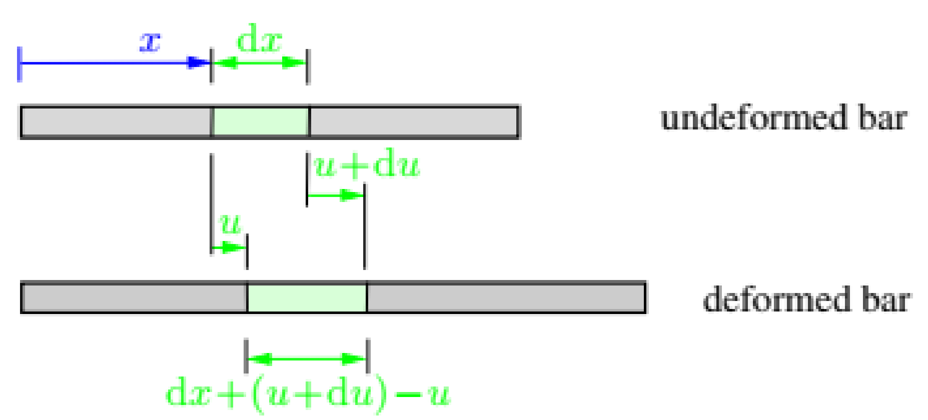
إذا كان المقطع متغير أو كانت القوى الخارجيةمتغيرة على طول المحور ، فإن التشوه يتغير من موضع إلى آخر على هذا المحور.




Instead of the whole we consider an element of the bar (Fig.). It has the length $dx$ in the undeformed state. Its left end is located at $x$, the right end at $x + dx$.

Instead of the whole we consider an element of the bar (Fig.). It has the length $dx$ in the undeformed state. Its left end is located at $x$, the right end at $x + dx$.

If the bar is elongated, the cross sections undergo displacements in the $x$-direction which are denoted by $u$. They depend on the location: $u = u(x)$.




Thus, the displacements are $u$ at the left end of the element and $u + du$ at the right end.

The length of the elongated element is $dx + (u + du) - u = dx + du$.

Hence, the elongation of the element is given by $du$. Now the local strain can be defined as the ratio between the elongation and the undeformed length of the element:

$$\varepsilon(x) = \frac{du}{dx}$$

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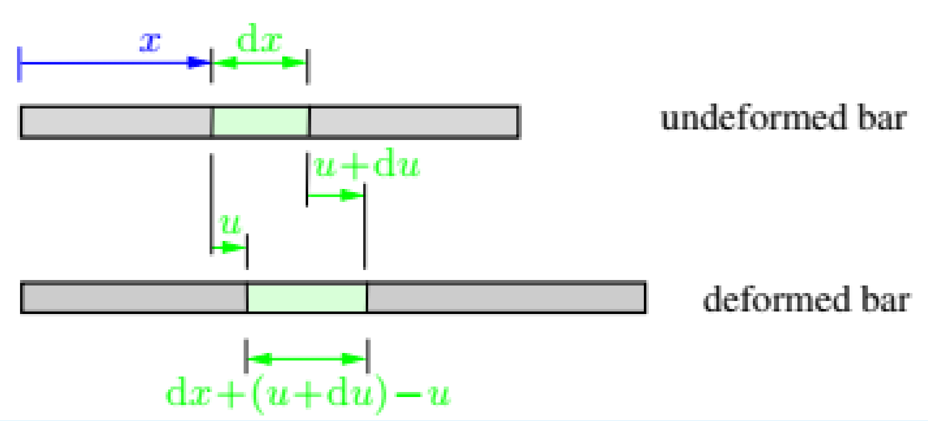



If the displacement $u(x)$ is known, the strain $\varepsilon(x)$ can be determined through differentiation. Reversely, if $\varepsilon(x)$ is known, the displacement $u(x)$ is obtained through integration.

The displacement $u(x)$ and the strain $\varepsilon(x)$ describe the geometry of the deformation. Therefore they are called *kinematic quantities*. So the equation $\varepsilon(x) = \frac{du}{dx}$ is referred to as a kinematic relation.

$$External\ Loads$$

Statics $\sigma(x) = \frac{N(x)}{A(x)}$ *constitutive law* $\varepsilon(x) = \frac{du}{dx}$, kinematic

**قانون السلوك المادة**

# 3 Constitutive Law قانون سلوك المادة

Stresses are quantities derived from statics; they are a measure for the stressing in the material .

On the other hand, strains are kinematic quantities; they measure the deformation of a body.

However, the deformation depends on the load which acts on the body. Therefore, the stresses and the strains are not independent.

*External Loads*

Statics $\sigma(x) = \frac{N(x)}{A(x)}$ $\varepsilon(x) = \frac{du}{dx}$, kinematic

*constitutive law*

قانون السلوك المادة

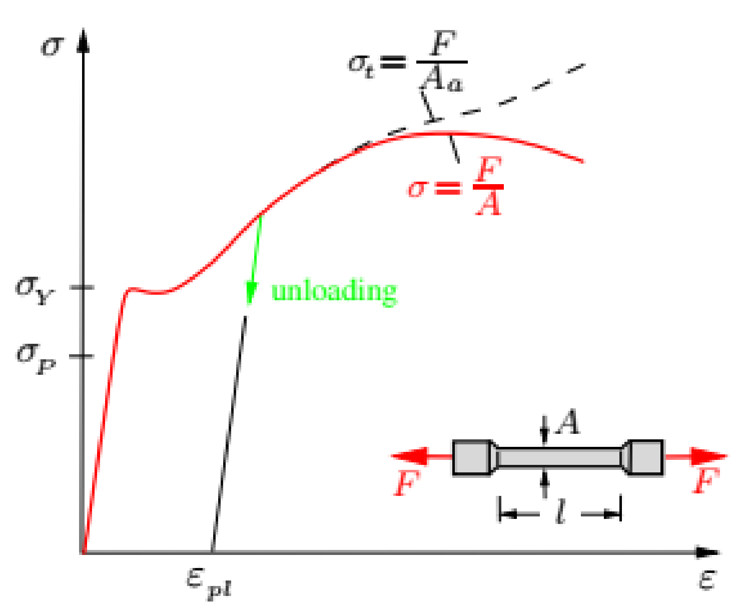
The physical relation that connects these quantities is called *constitutive law*.

It describes the behavior of the material of the body under a load. It depends on the material and can be obtained only with the aid of experiments.

One of the most important experiments to find the relationship between stress and strain is the tension or compression test. Here, a small specimen of the material is placed into a testing machine and elongated or shortened.

The force $F$ applied by the machine onto the specimen can be read on the dial of the machine; it causes the normal stress $\sigma = F/A$. The change $\Delta l$ of the length $l$ of the specimen can be measured and the strain $\varepsilon = \Delta l/l$ can be calculated.

The graph of the relationship between stress and strain is shown schematically (not to scale) for a steel specimen in Fig.




This graph is referred to as *stress-strain diagram*. One can see that for small values of the strain the relationship is linear (straight line) and the stress is proportional to the strain.

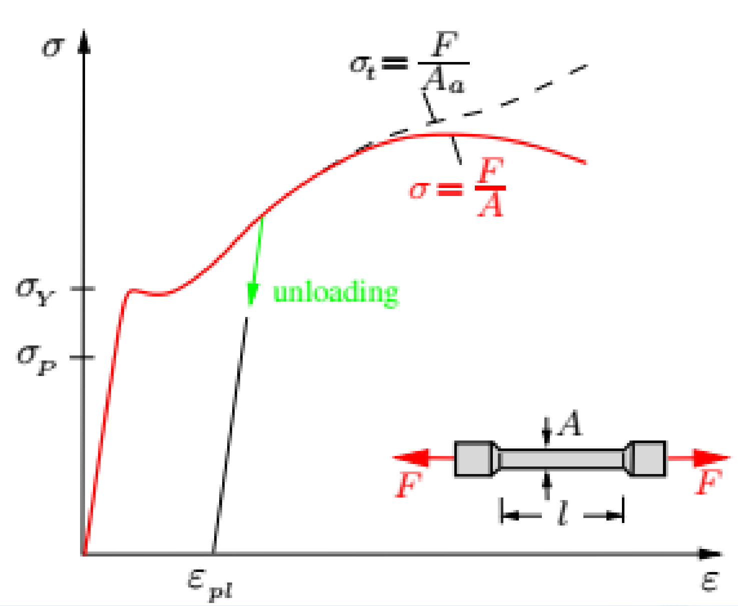
This behavior is valid until the stress reaches the *proportional limit* $\sigma_P$. If the stress exceeds the proportional limit the strain begins to increase more rapidly and the slope of the curve decreases.

This continues until the stress reaches the *yield stress* $\sigma_Y$ . From this point of the stress-strain diagram the strain increases at a practically constant stress: the material begins to *yield*. Note that many materials do not exhibit a pronounced yield point.

At the end of the yielding the slope of the curve increases again which shows that the material can sustain an additional load. This phenomenon is called *strain hardening*.

Experiments show that an elongation of the bar leads to a reduction of the cross-sectional area $A$. This phenomenon is referred to as *lateral contraction*.

Whereas the cross-sectional area decreases uniformly over the entire length of the bar in the case of small stresses, it begins to decrease locally at very high stresses.



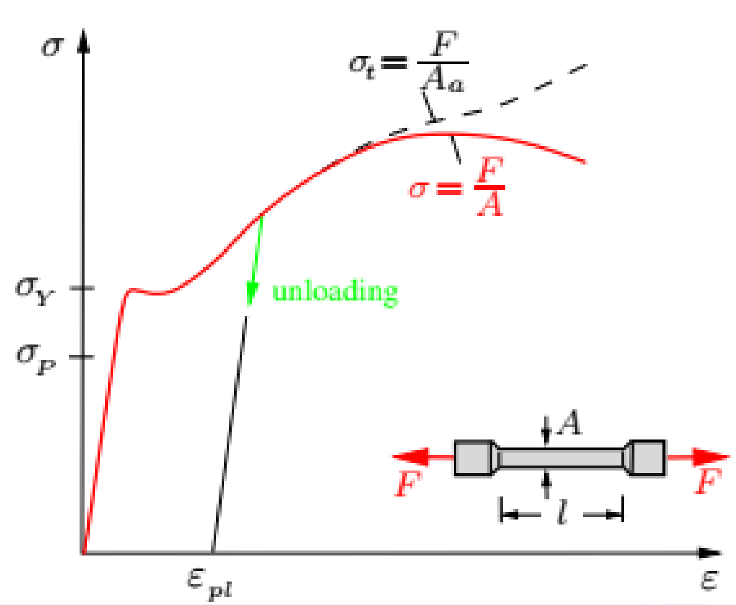

This phenomenon is called *necking*. Since the actual cross section $A_a$ may then be considerably smaller than the original cross section $A$, the stress $\sigma = F/A$ does not describe the real stress any more.

It is therefore appropriate to introduce the stress $\sigma_t = F/Aa$ which is called *true stress* or *physical stress*. It represents the true stress in the region where necking takes place. The stress $\sigma = F/A$ is referred to as *nominal* or *conventional* or *engineering stress*. The Fig. shows both stresses until fracture occurs.

Consider a specimen being first *loaded* by a force which causes the stress $\sigma$. Assume that $\sigma$ is smaller than the yield stress $\sigma_Y$ , i.e., $\sigma < \sigma_Y$.

Subsequently, the load is again removed. Then the specimen will return to its original length: the strain returns to zero.

In addition, the curves during the loading and the unloading coincide. This behavior of the material is called *elastic* ; the behavior in the region $\sigma \leq \sigma_P$ is referred to as *linearly elastic*.




Now assume that the specimen is loaded beyond the yield stress, i.e., until a stress $\sigma > \sigma_Y$ is reached. Then the curve during the unloading is a straight line which is parallel to the straight line in the linear-elastic region, see Fig. If the load is completely removed the strain does not return to zero: a *plastic strain* $\varepsilon_{pl}$ remains after the unloading. This material behavior is referred to as *plastic*.

In the following we will always restrict ourselves to a linearly elastic material behavior. For the sake of simplicity we will refer to this behavior shortly as elastic, i.e., Then we have the linear relationship between the stress and the strain.

$$\sigma = E\varepsilon$$

The proportionality factor $E$ is called *modulus of elasticity* or *Young's modulus* (Thomas Young, 1773–1829). The constitutive law $\sigma = E\varepsilon$ is called *Hooke's law* after Robert Hooke (1635–1703). Note that Robert Hooke could not present this law in this form since the notion of stress was introduced only in 1822 by Augustin Louis Cauchy.

The modulus of elasticity has the same value for tension and compression. But, $\sigma$ must be less than the proportional limit $\sigma_P$ which may be different for tension or compression.

The modulus of elasticity $E$ is a constant which depends on the material and which can be determined with the aid of a tension test. It has the dimension of force/area (which is also the dimension of stress); it is given, for example, in the unit MPa.

Next Table shows the values of $E$ for several materials at room temperature. Note that these values are just a guidance since the modulus of elasticity depends on the composition of the material and on the temperature.

A tensile or a compressive force, respectively, causes the strain: $\varepsilon = \sigma/E$

Changes of the length and thus strains are not only caused by forces but also by changes of the temperature. Experiments show that the *thermal strain* $\varepsilon_T$ is proportional to the change $\Delta T$ of the temperature if the temperature of the bar is changed uniformly across its section and along its length: $\varepsilon_T = \alpha_T \Delta T$

The proportionality factor $\alpha_T$ is called *coefficient of thermal expansion*. It is a material constant and is given in the unit $1/°\mathrm{C}$. Next Table shows several values of $\alpha_T$ *and* $E$.

If the change of the temperature is not the same along the entire length of the bar (if it depends on the location) then $\varepsilon_T = \alpha_T \Delta T$ represents the local strain $\varepsilon_T(x) = \alpha_T\, \Delta T(x)$.

If a bar is subjected to a stress $\sigma$ as well as to a change $\Delta T$ of the temperature, the total strain $\varepsilon$ is obtained through a superposition $\varepsilon = \frac{\sigma}{E} + \alpha_T \Delta T$

This relation can also be written in the form $\sigma = E(\varepsilon - \alpha_T \Delta T)$.

Table of Material Constants

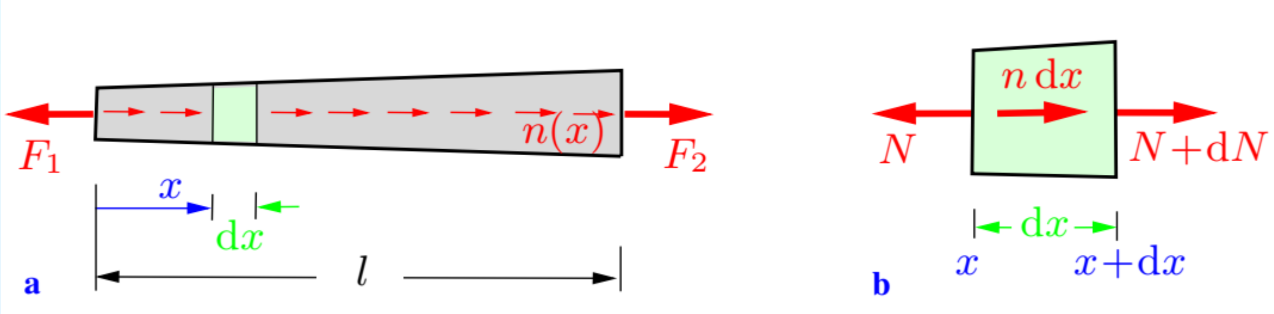
| Material | $E$ in MPa | $\alpha_T$ in 1/°C |
|---|---|---|
| Steel | $2.1 \cdot 10^5$ | $1.2 \cdot 10^{-5}$ |
| Aluminium | $0.7 \cdot 10^5$ | $2.3 \cdot 10^{-5}$ |
| Concrete | $0.3 \cdot 10^5$ | $1.0 \cdot 10^{-5}$ |
| Wood (in fibre direction) | $0.7 \ldots 2.0 \cdot 10^4$ | $2.2 \ldots 3.1 \cdot 10^{-5}$ |
| Cast iron | $1.0 \cdot 10^5$ | $0.9 \cdot 10^{-5}$ |
| Copper | $1.2 \cdot 10^5$ | $1.6 \cdot 10^{-5}$ |
| Brass | $1.0 \cdot 10^5$ | $1.8 \cdot 10^{-5}$ |

# 4 Single Bar under Tension or Compression

There are three different types of equations that allow us to determine the stresses & the strains in a bar: the equilibrium condition, the kinematic relation and Hooke's law.

Depending on the problem, the equilibrium condition may be formulated for the entire bar, a portion of the bar or for an element of the bar.

We will derive the equilibrium condition for an element. For this purpose we consider a bar which is subjected to two forces $F_1$ & $F_2$ at its ends and to a line load $n = n(x)$, see Fig.a.




The forces are assumed to be in equilibrium. We imagine a slice element of infinitesimal length $dx$ separated from the bar as shown in Fig.b.

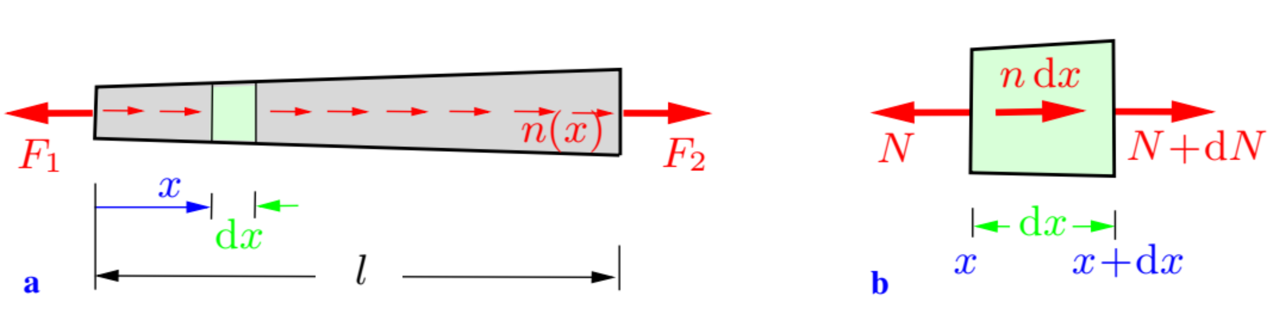
The F. B. D. shows the normal forces $N$ and $N + dN$, respectively, at the ends of the element; the line load is replaced by its resultant $ndx$ (note that $n$ may be considered to be constant over the length $dx$).

Equilibrium of the forces in the direction of the axis of the bar

$\rightarrow: N + dN + n\,dx - N = 0$

yields the *equilibrium condition*

$$\frac{dN}{dx} + n = 0$$




In the special case of a vanishing line load ($n \equiv 0$) $F_1 = F_2 = N$

The *kinematic relation* for the bar is

$$\varepsilon = \frac{du}{dx}$$

and Hooke's law is given by

$$\varepsilon = \frac{\sigma}{E}$$

If we insert the kinematic relation and $\sigma = N/A$ into Hooke's law we obtain

$$\varepsilon(x) = \frac{du}{dx} = \frac{N(x)}{EA(x)}$$

$$N(x) = EA(x)\varepsilon(x) = EA(x)\frac{du}{dx}$$

This differential equation relates the displacements $u(x)$ of the cross sections and the normal force $N(x)$.

It may be called the *constitutive law for the bar.*

The displacement $u$ of a cross section is found through integration of the strain:

$$\varepsilon = \frac{du}{dx} \rightarrow \int du = \int \varepsilon dx \rightarrow u(x) - u(0) = \int_0^x \varepsilon d\bar{x}\,.$$

The elongation $\Delta l$ follows as the difference of the displacements at the ends $x = l$ and $x = 0$ of the bar:

$$\Delta l = u(l) - u(0) = \int_0^l \varepsilon dx$$

With $\varepsilon = du/dx$ and the *constitutive law for the bar* this yields

$$\Delta l = \int_0^l \frac{N(x)}{EA(x)} dx$$

In the special case of a bar (length $l$) with constant axial rigidity ($EA = const$) which is subjected only to forces at its end ($n \equiv 0, N = F$) the elongation is given by

$$\Delta l = \frac{l}{EA} F \Leftrightarrow F = \frac{EA}{l} \Delta l$$

Quantity $\frac{EA}{l}$ is the *axial rigidity (Stiffness) of the bar.*

The Inverse $\frac{l}{EA}$ is the axial *flexibility* of the bar

If we want to apply these equations to specific problems, we have to distinguish between *statically determinate* and *statically indeterminate* problems.

In a *statically determinate* system we can always calculate the normal force $N(x)$ with the aid of the equilibrium condition.

Subsequently, the strain $\varepsilon(x)$ follows from $\sigma = N/A$ and Hooke's law $\varepsilon = \sigma/E$. Finally, integration yields the displacement $u(x)$ and the elongation $\Delta l$.

In a *statically indeterminate* problem, with the equilibrium condition alone the normal force cannot be calculated.

In such problems the basic equations (equilibrium condition, kinematic relation and Hooke´s law) are a system of *coupled* equations and have to be solved simultaneously.

Finally we will reduce the basic equations to a single equation for the displacement $u$.

By combining the two equations: $\varepsilon = \frac{du}{dx} = \frac{N}{EA}$ $\quad \frac{dN}{dx} + n = 0$ To get:

$$\frac{d}{dx}\left(EA\frac{du}{dx}\right) = \frac{d}{dx}(N) = -n$$

With, the primes denoting derivatives with respect to $x$

$$(EAu')' = -n$$

If the functions $EA(x), n(x)$, are given, the equation $(EAu')' = -n$

the displacement $u(x)$ of an arbitrary cross section can be determined by integration.

The constants of integration are calculated from the boundary conditions.

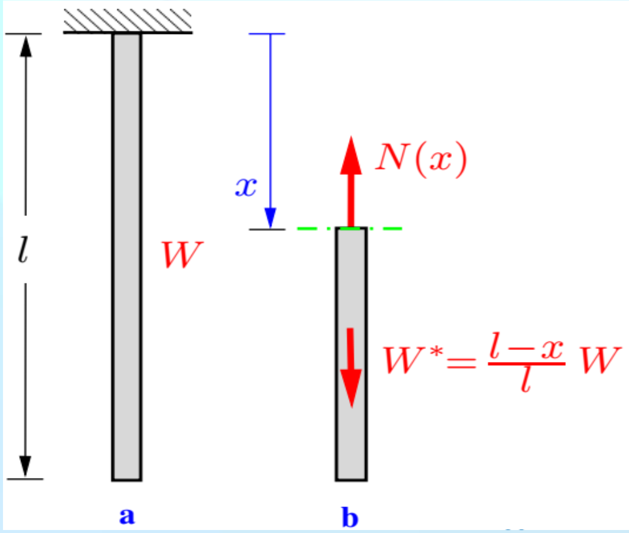
If, for example, one end of the bar is fixed then $u = 0$ at this end.

If, on the other hand, one end of the bar can move and is subjected to a force $F_e$, then applying d $N = F_e$ yields the boundary condition $u' = F_e/EA$ .

This reduces to the boundary condition $u' = 0$ in the special case of a stress-free end ($F_e = 0$) of a bar.

**illustrative example** As a statically determinate system let us consider a slender bar (weight $W$, cross-sectional area $A$) that is suspended from the ceiling (Fig.a).

First we determine the normal force caused by the weight of the bar. We cut the bar at an arbitrary position $x$ (Fig.b).




The normal force $N$ = the weight $W^*$ of the portion of the bar below the imaginary cut. Thus, it is given by $N(x) = W^*(x) = W(l - x)/l$.

Then the normal stress is $$\sigma(x) = \frac{N(x)}{A} = \frac{W}{A}\left(1 - \frac{x}{l}\right)$$

Accordingly, the normal stress in the bar varies linearly; it decreases from the value $\sigma(0) = W/A$ at the upper end to $\sigma(l) = 0$ at the free end.

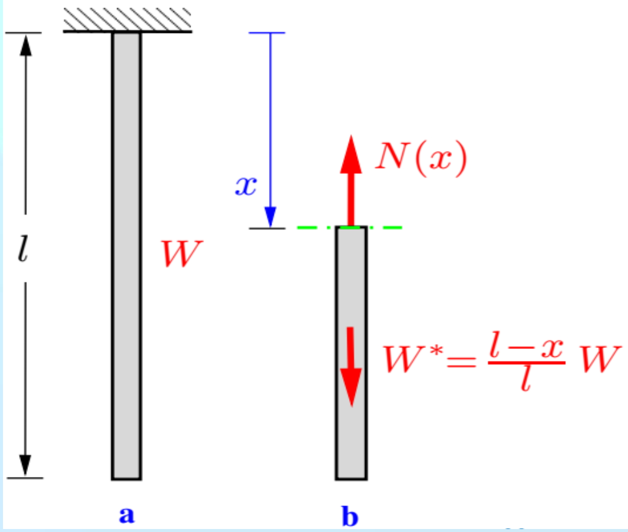
The elongation $\Delta l$ of the bar due to its own weight is obtained from

$$\Delta l = \int_0^l \frac{N}{EA} dx = \frac{W}{EA}\int_0^l \left(1 - \frac{x}{l}\right) dx = \frac{1}{2}\frac{Wl}{EA}$$

It is half the elongation of a bar with negligible weight which is subjected to the force $W$ at the free end.

**illustrative example** As a statically determinate system let us consider a slender bar (weight *W*, cross-sectional area *A*) that is suspended from the ceiling (Fig.a).

We may also solve the problem by applying the differential equation $(EAu')' = -n$ for the displacements $u(x)$ of the cross sections of the bar. Integration with the constant line load $n = W/l$ , yields




$$EAu'' = -W/l \quad \Rightarrow EAu' = -(W/l)x + C_1$$

$$\Rightarrow EAu = -(W/2l)x^2 + C_1x + C_2$$

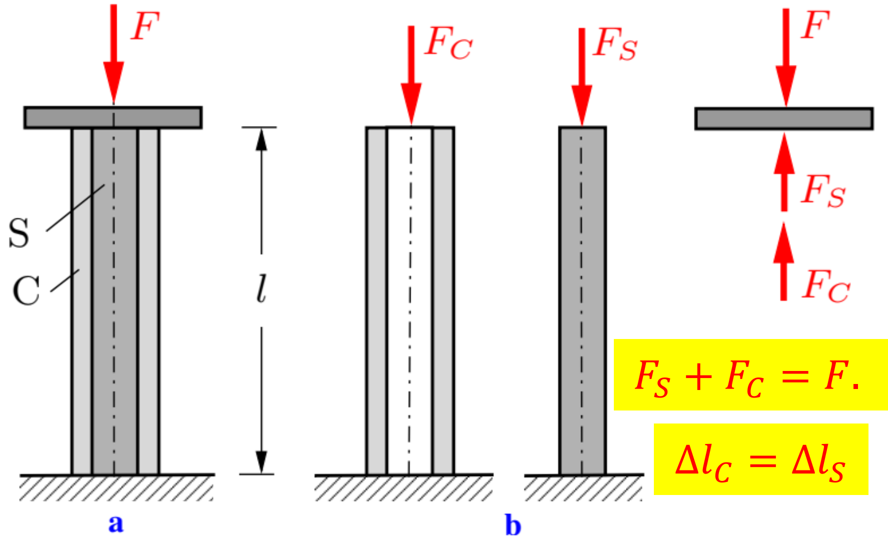
$C_1$ & $C_2$ , constants of integration, can be determined from the boundary conditions. The displacement of the cross section at the upper end of the bar is equal to zero: $u(0) = 0$. Since the stress $\sigma$ vanishes at the free end, we have $u'(l) = 0$. This leads to $C_2 = 0$ and $C_1 = W$.

Thus, the displacement and the normal force are given by $u(x) = \frac{1}{2}\frac{Wl}{EA}\left(2\frac{x}{l} - \frac{x^2}{l^2}\right)$

The bar elongation $\Delta l = u(l) = \frac{1}{2}\frac{Wl}{EA}$

and the normal force $N(x) = EAu' = W(1 - \frac{x}{l})$

**illustrative example** As an illustrative example of a statically indeterminate system let us consider a solid circular steel cylinder (cross-sectional area $A_S$, modulus of elasticity $E_S$, length $l$) is placed inside a copper tube (cross-sectional area $A_C$, modulus of elasticity $E_C$, length $l$). The assembly is compressed between a rigid plate and the rigid floor by a force $F$ (Fig.a). Calculate the shortening of the assembly and Determine the normal Forces in the cylinder and in the tube..




## Solution: 4 unknowns, $F_S, F_C, \Delta l_S, \Delta l_C$

Denote the compressive forces in the steel cylinder and in the copper tube by $F_S$ and $F_C$, respectively (Fig.b). Equilibrium at the F. B. D. of the plate yields

$$F_S + F_C = F.$$

Since equilibrium furnishes only one equation for the two unknown forces $F_s$ and $F_c$, the problem is statically indeterminate.

obtain a second equation by taking into account the deformation of the system. The shortenings (here counted positive) of the two parts are given according to $\Delta l = (l/EA)F$ , by

$$\Delta l_C = \frac{lF_C}{E_C A_C} \text{ and } \Delta l_S = \frac{lF_S}{E_S A_S}$$

$$F_S + F_C = F.$$

$$\Delta l_C = \Delta l_S$$

$$\Delta l_C = \Delta l_S = \Delta l$$

The plate and the floor are assumed to be rigid. Therefore the geometry of the problem requires that the shortenings of the copper tube and of the steel cylinder coincide. This gives the compatibility condition

Sub. in the two last equations gives $F_S = E_S A_S \Delta l / l$ and $F_C = E_C A_C \Delta l / l$ Sub. these into the equilibrium Eq. gives

$$\Delta l = \frac{Fl}{E_S A_S + E_C A_C} \qquad F_S = \frac{E_S A_S}{E_S A_S + E_C A_C} F \qquad \sigma_S = \frac{E_S}{E_S A_S + E_C A_C} F \qquad F_C = \frac{E_C A_C}{E_S A_S + E_C A_C} F \qquad \sigma_C = \frac{E_C}{E_S A_S + E_C A_C} F$$