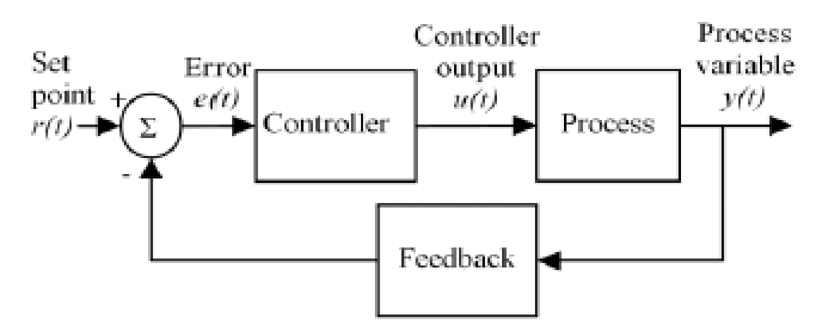


Lecture (5)

Time Domain Analysis of Feedback Control Systems Performance and Characteristics -Part2

Mechatronics Engineering Department
Assistant Professor Isam Asaad

References

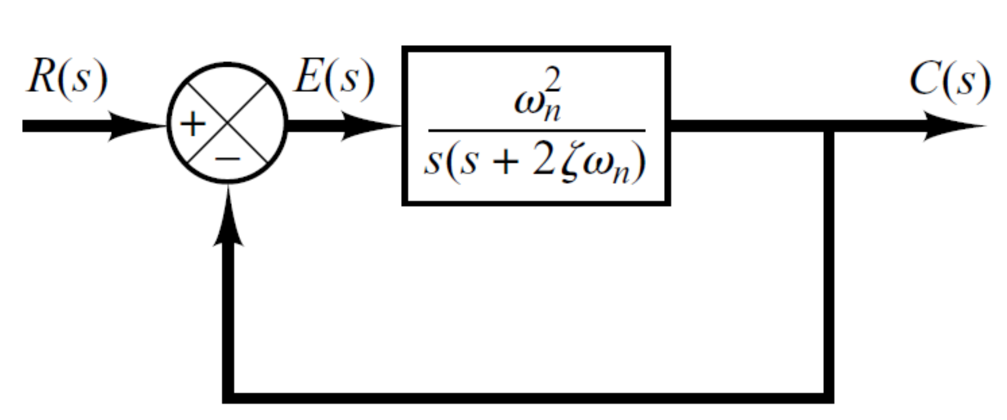
- Control Systems Course, professor Aniket Khandekar, Zeal college of engineering and Research, Pune.
- Gopal, M. - Control Systems_ Principles and Design 3rd edition-Tata McGraw Hill Publishing Co. Ltd. (2008)
- Modern Control Systems, Richard C. Dorf and Robert H. Bishop, Prentice Hall, 12th edition, 2010, ISBN-10: 0-13-602458-0
- Modelling, Dynamics and Control, University of Sheffield, John Anthony Rossiter.
- https://www.tutorialspoint.com/control_systems/

Contents

- Response of Second Order Systems.

Second Order Systems

- A general second-order system (without zeros) is characterized by the following transfer function.



$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$$

$$\frac{C(s)}{R(s)} = \frac{\left(\frac{\omega_n^2}{s(s+2\xi\omega_n)}\right)}{1+\left(\frac{\omega_n^2}{s(s+2\xi\omega_n)}\right)} = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$$

$$G(s) = \frac{\omega_n^2}{s(s+2\zeta\omega_n)}$$

➔ Open-Loop Transfer Function

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

➔ Closed-Loop Transfer Function

Second Order Systems

جَـامعة المَـنارة

$$\frac{C(s)}{R(s)} = \frac{k\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

- ζ ➔ damping ratio of the second order system, which is a measure of the degree of resistance to change in the system output.
- ω_n ➔ un-damped natural frequency of the second order system, which is the frequency of oscillation of the system without damping

Example 1

- Determine the un-damped natural frequency and damping ratio of the following second order system.

$$\frac{C(s)}{R(s)} = \frac{4}{s^2 + 2s + 4}$$

- Compare the numerator and **denominator** of the given transfer function with the general 2nd order transfer function.

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$\omega_n^2 = 4 \quad \Rightarrow \omega_n = 2\ rad/\sec$$

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = s^2 + 2s + 4 \Rightarrow 2\zeta\omega_n s = 2s$$

$$\Rightarrow \zeta\omega_n = 1$$

$$\Rightarrow \zeta = 0.5$$

Example 2

- For the second order system described by the closed loop transfer function T(s), determine ω_n and ξ .

$$T(s)=\frac{C(s)}{R(s)}=\frac{24}{4s^2+12s+256}=\frac{6}{s^2+3s+64}$$

- Compare with the standard equation we have:

$$Since \quad \omega_n^2=64 \quad and \quad 2\zeta\,\omega_n=3 \quad and \quad k\omega_n^2=6$$

$$==> \quad \omega_n=8 \quad , \quad \zeta=\frac{3}{16}=0.8 \quad and \;\; k=0.094$$

Second Order Systems - Poles

- The second order system Transfer function is

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

- The characteristic polynomial of a second order system is:

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = (s - s_1)(s - s_2) = 0$$

- The closed-loop poles of the system are

$$s_1, s_2 = \frac{-2\zeta\omega_n \pm \sqrt{4\zeta^2\omega_n^2 - 4\omega_n^2}}{2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

$$s_1 = -\omega_n\zeta + \omega_n\sqrt{\zeta^2 - 1}$$

$$s_2 = -\omega_n\zeta - \omega_n\sqrt{\zeta^2 - 1}$$

➔ξ<1 ➔ $s_{1,2} = -\omega_n\zeta \pm j\omega_n\sqrt{1-\zeta^2}$

Magnitude
Angle

Classification of second order systems' response according to damping ratio:

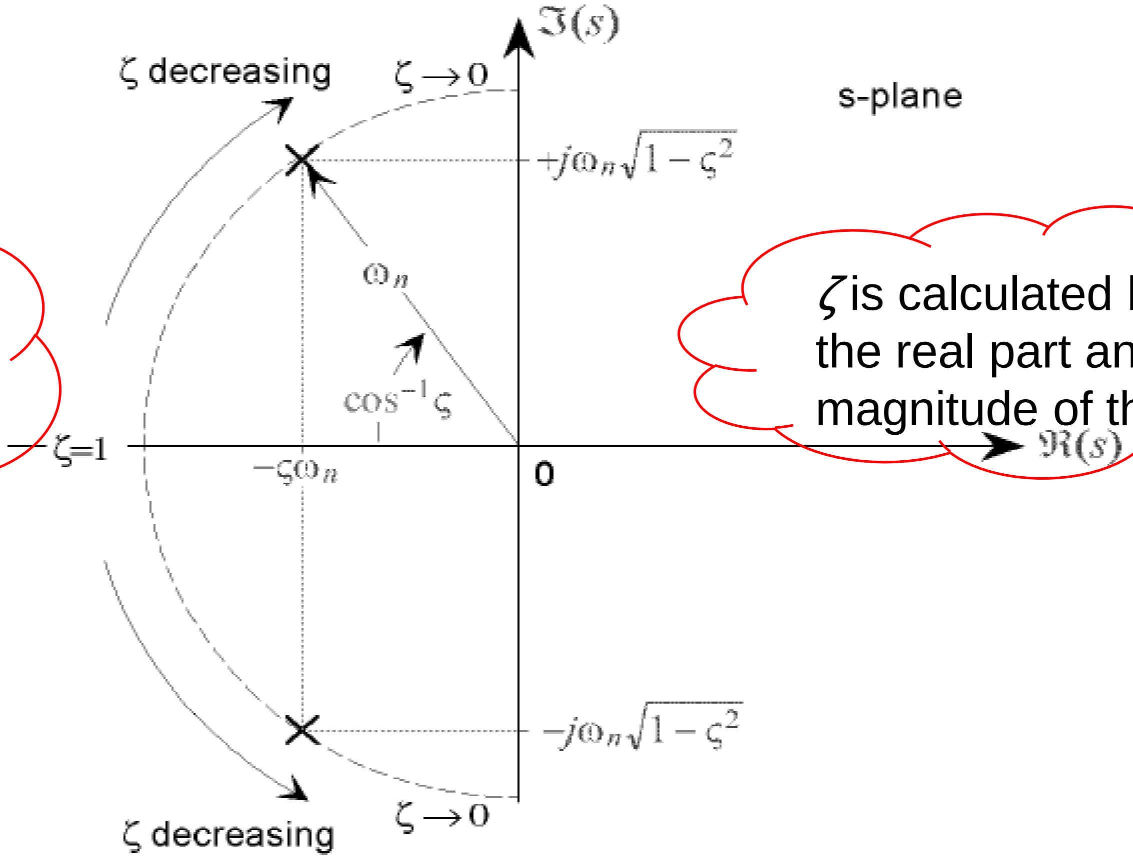
- According the value of ζ , a second-order system's response can be set into one of the four categories:
  - **Case 1: Over damped response (ξ > 1)** [two roots are real but not equal]
  - **Case 2: Critically damped response (ξ = 1)** [two roots are real and equal]
  - **Case 3: Under damped response (0 < ξ <1)** [two roots are complex conjugate]
  - **Case 4: No damped (undamped) response (ξ = 0)** [two roots are imaginary]

$$s_1 = -\omega_n \zeta + \omega_n \sqrt{\zeta^2 - 1}$$

$$s_2 = -\omega_n \zeta - \omega_n \sqrt{\zeta^2 - 1}$$

Classification of second order systems' response according to damping ratio:

If $\zeta \geq 1$, corresponding to an overdamped system, the two poles are real and lie in the left-half plane.



ζ is calculated based on the real part and magnitude of the pole.

Definition of the parameters ω_n and ζ for an underdamped, second-order system from the complex conjugate pole locations.

Unit-step Response of Second Order Systems

$$\frac{Y(s)}{R(s)} = G(s) = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$\tau = \frac{1}{\zeta\omega_n}$$

The unit-step response of the transfer function is given by

$\zeta = 0$ $\quad y(t) = K(1-\cos\omega_n t);\ t \geq 0$

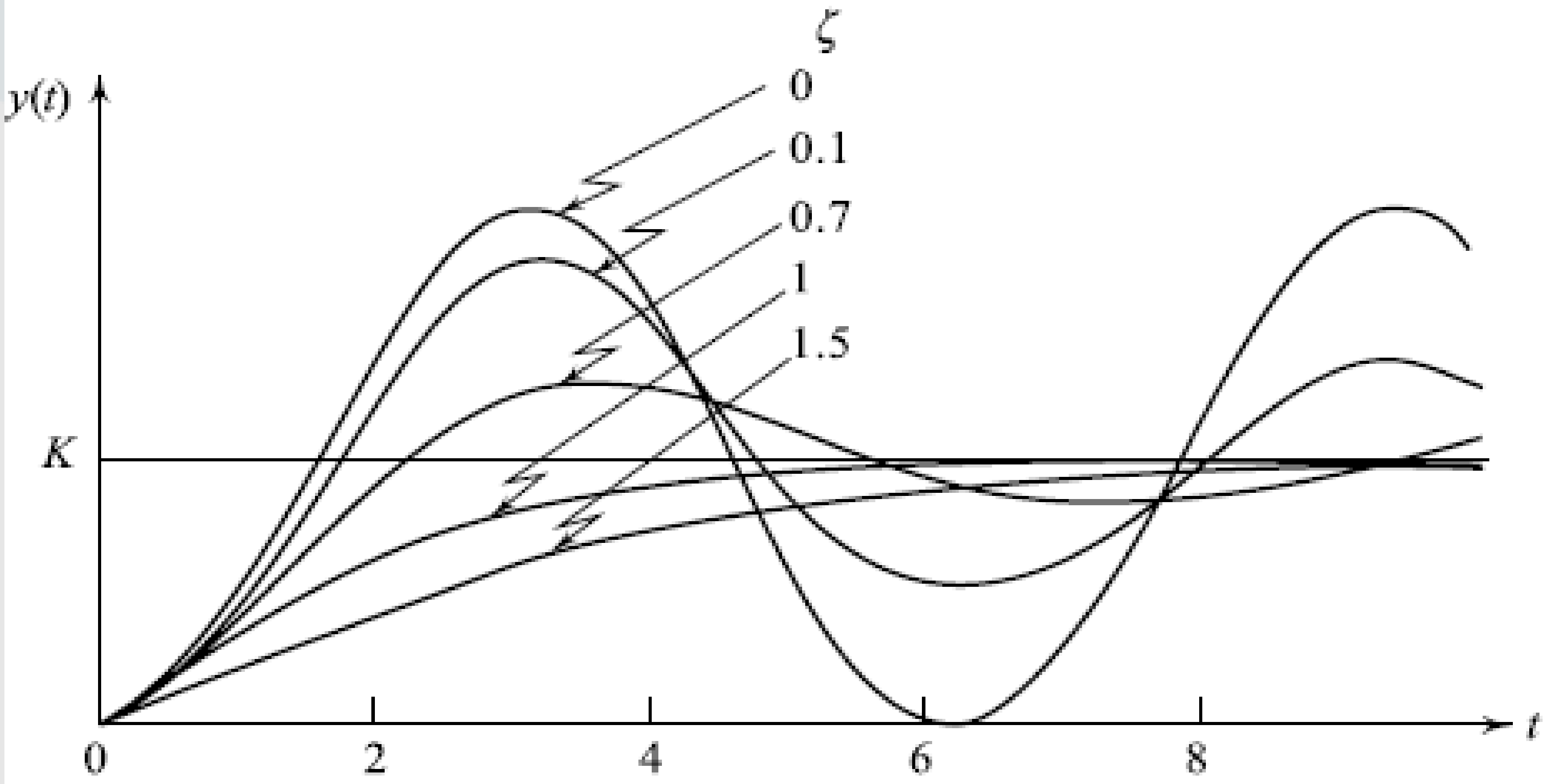
$0 < \zeta < 1$ $\quad y(t) = K\left[1 - e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right)\right] = K\left[1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin\left(\omega_d t + \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right)\right];\ t \geq 0$

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

$\zeta = 1$ $\quad y(t) = K[1 - e^{-\omega_n t} - \omega_n t e^{-\omega_n t}];\ t \geq 0$

$\zeta > 1$ $\quad y(t) = K\left[1 + \frac{1}{2\sqrt{\zeta^2-1}(\zeta+\sqrt{\zeta^2-1})}e^{-(\zeta+\sqrt{\zeta^2-1})\omega_n t} - \frac{1}{2\sqrt{\zeta^2-1}(\zeta-\sqrt{\zeta^2-1})}e^{-(\zeta-\sqrt{\zeta^2-1})\omega_n t}\right];\ t \geq 0$

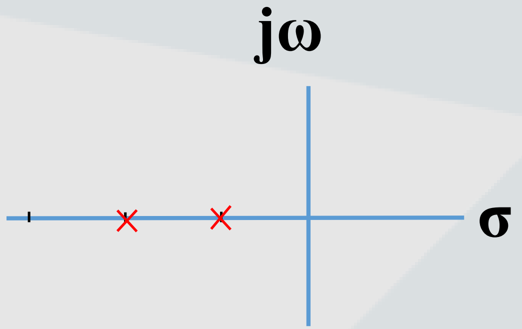
Unit-step Response of Second Order Systems



Step-response curves of a second-order system ($\omega_n = 1$)

Unit-Step Response of Second Order Systems

- **Case 1: Over damped response (ξ > 1)**
  - The two roots of the characteristic equation s1 and s2 are real and distinct.



$$s_1 = -\omega_n \zeta + \omega_n \sqrt{\zeta^2 - 1}$$

$$s_2 = -\omega_n \zeta - \omega_n \sqrt{\zeta^2 - 1}$$

- **Example:** Calculate and plot the output of the system with the following transfer function:
- **Solution:** With unit step input, its response is:

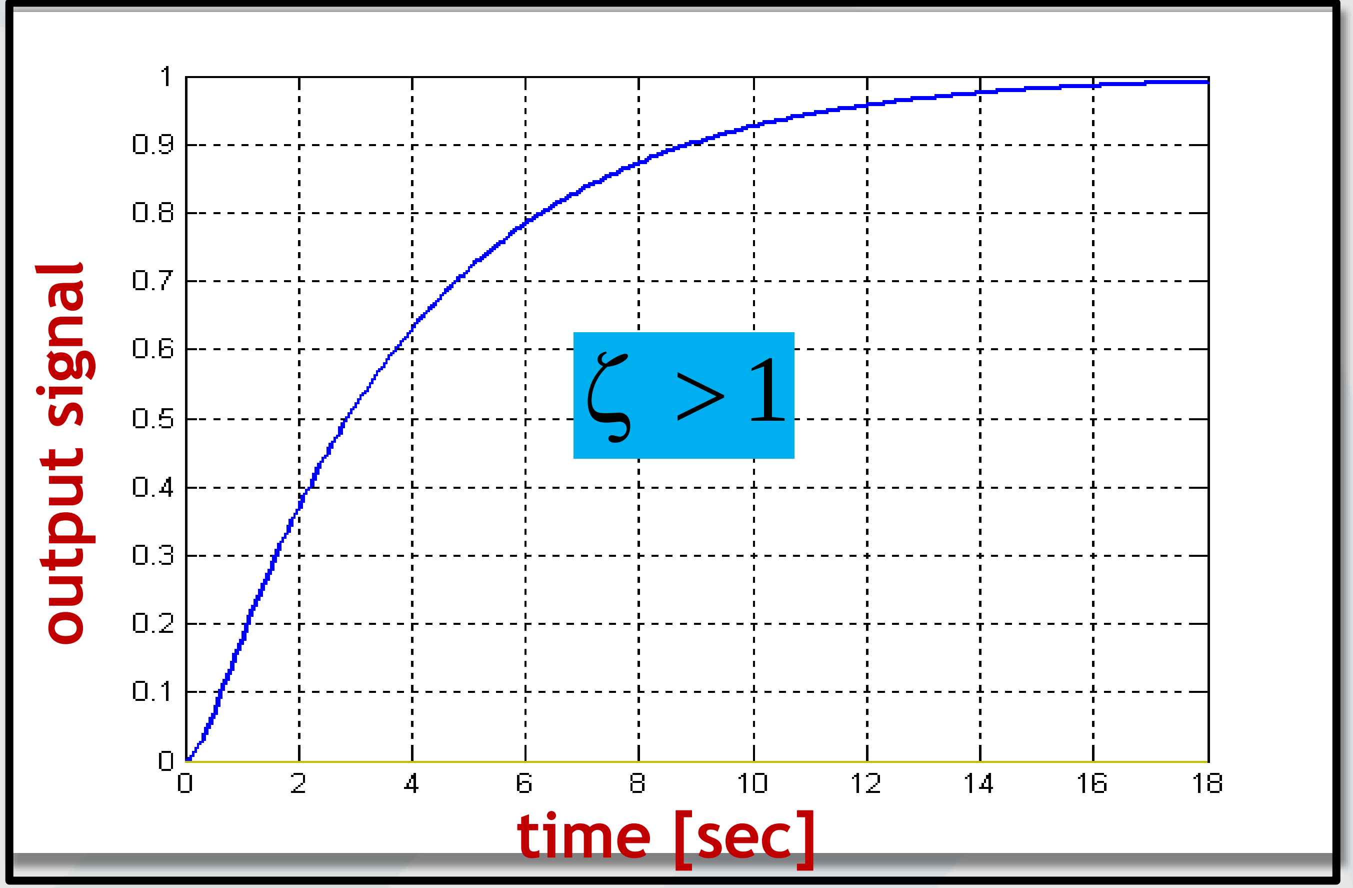
$$T(s) = \frac{2}{s^2 + 3s + 2}$$

$$C(s) = R(s)\,T(s) = \frac{2}{s(s^2 + 3s + 2)} = \frac{1}{s} + \frac{-2}{s+1} + \frac{1}{s+2}$$

Unit-Step Response of Second Order Systems

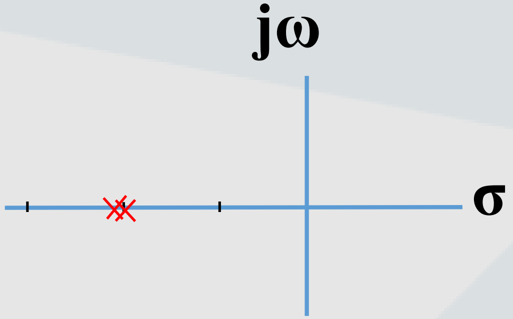
- The corresponding time domain output is given by:

$$c(t) = L^{-1}\{C(s)\} = [1 - 2\, e^{-t} + e^{-2t}]u(t)$$



Unit-Step Response of Second Order Systems

- **Case 2: Critically damped response (ξ = 1)**
  - The two roots of the characteristic equation s1 and s2 are real and equal.



$$s_{1,2} = -\omega_n$$

- **Example:** Calculate and plot the output of the system with the following transfer function:

- **Solution:** With unit step input, its response is:

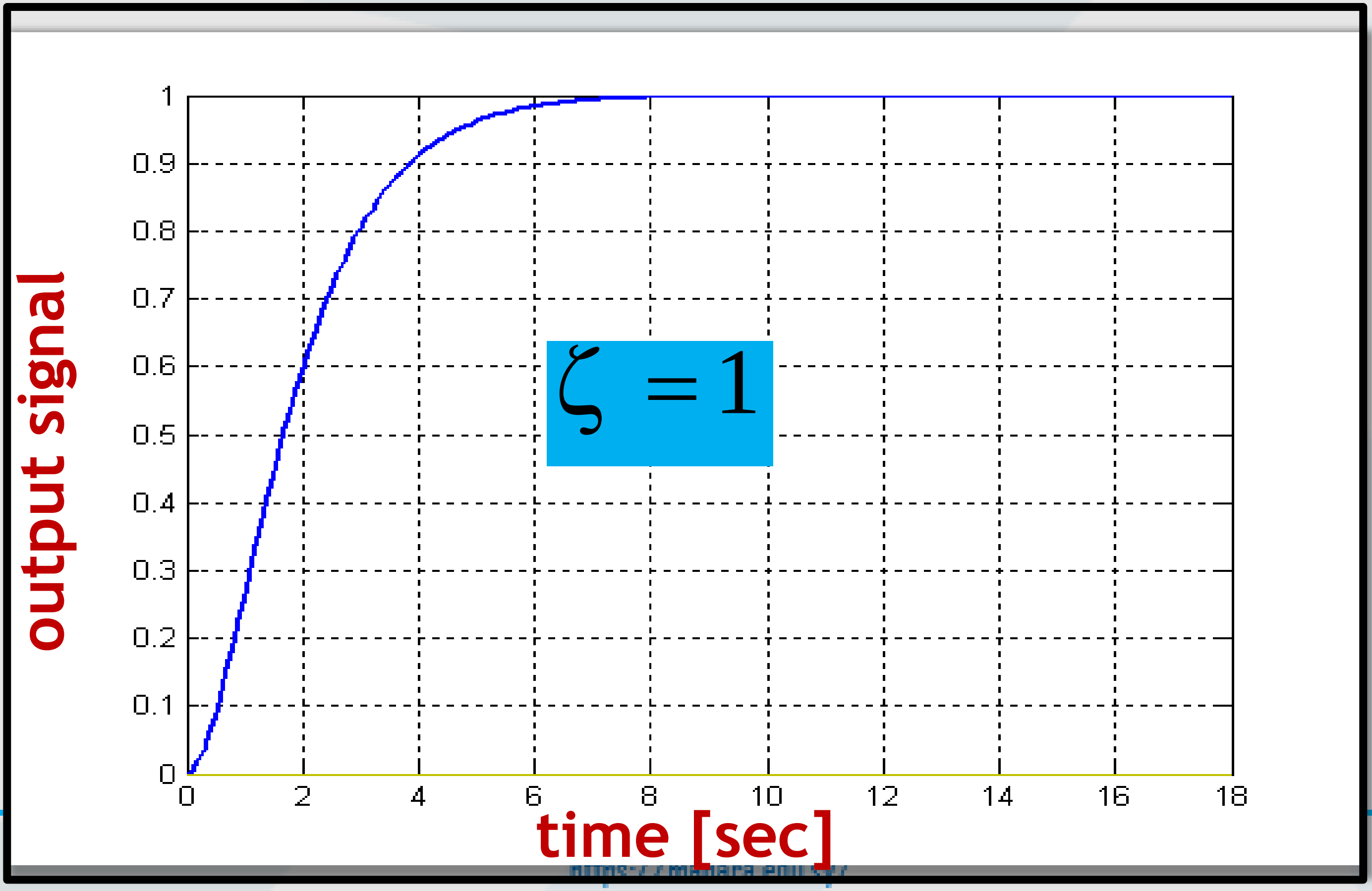
$$T(s) = \frac{5s + 4}{s^2 + 4s + 4}$$

$$C(s) = R(s)T(s) = \frac{5s+4}{s(s^2+4s+4)} = \frac{1}{s} + \frac{-1}{s+2} + \frac{3}{(s+2)^2}$$

Unit-Step Response of Second Order Systems

- The corresponding time domain output is given by:

$$c(t) = L^{-1}\{C(s)\} = [1 - e^{-2t} + 3\ t\ e^{-2t}]u(t)$$



Unit-Step Response of Second Order Systems

- **Case 3: Under damped response ($\xi < 1$)**
  - The two roots of the characteristic equation s1 and s2 are complex conjugates of one another.

$$s_{1,2} = -\omega_n\zeta \pm j\omega_n\sqrt{1-\zeta^2}$$

$$s_{1,2} = -\omega_n\zeta \pm j\omega_d$$

- **Example:** Calculate and plot the output of the system with the following transfer function:
- **Solution:** With unit step input, its response is:

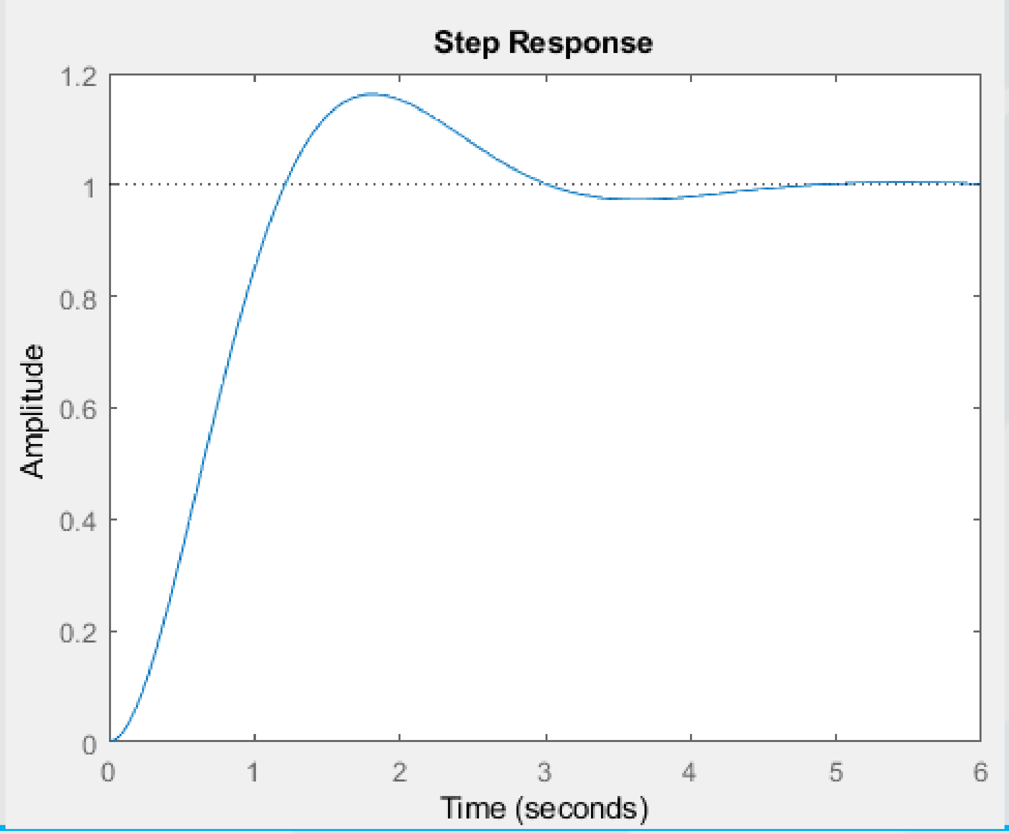
$$T(s) = \frac{4}{s^2 + 2s + 4}$$

$$C(s) = R(s)T(s) = \frac{4}{s(s^2+2s+4)} = \frac{1}{s} + \frac{-s-2}{s^2+2s+4}$$

Unit-Step Response of Second Order Systems

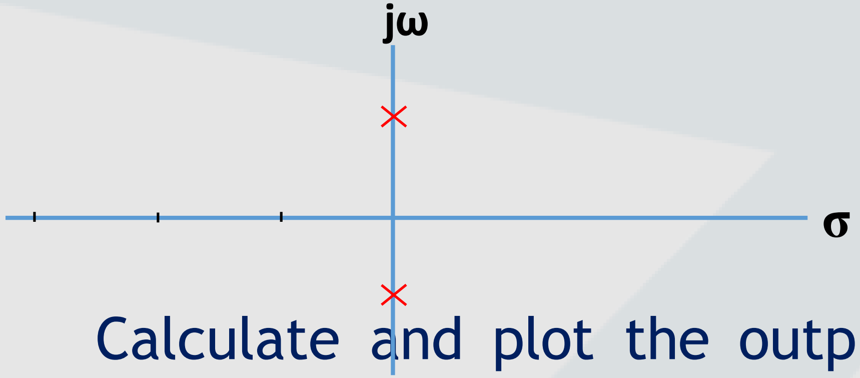
- The corresponding time domain output is given by:

$$c(t) = L^{-1}\{C(s)\} = 1 - \exp^{-t} * (\cos(\sqrt{3} * t) + \frac{1}{\sqrt{3}} * \sin(\sqrt{3} * t))$$



Unit-Step Response of Second Order Systems

- **Case 4: Undamped response ($\xi = 0$)**
  - The two roots of the characteristic equation s1 and s2 are imaginary poles.



$$s_{1,2} = \pm j\omega_n$$

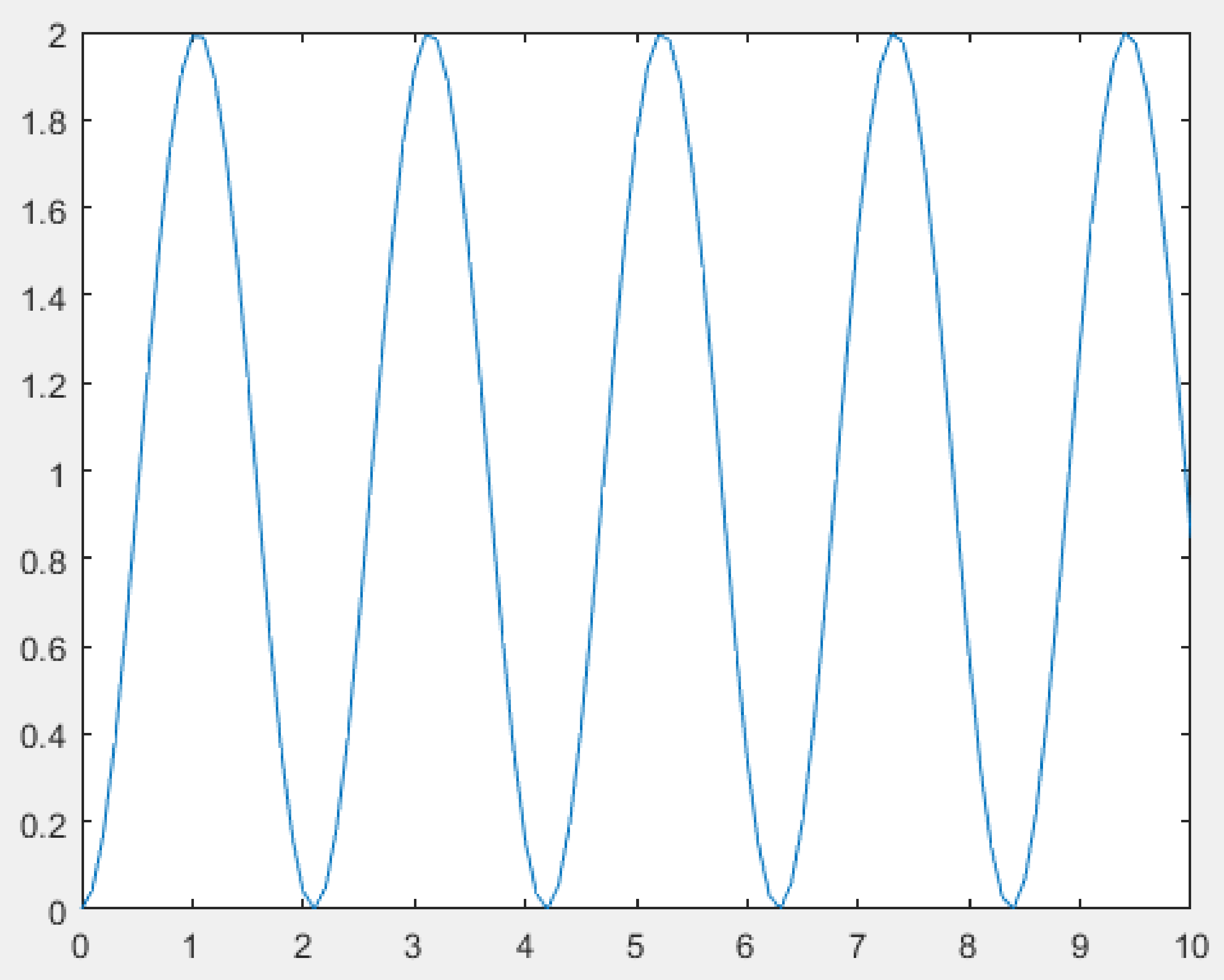
- **Example:** Calculate and plot the output of the system with the following transfer function:
- **Solution:** With unit step input, its response is:

$$T(s) = \frac{4}{s^2 + 4}$$

$$C(s) = R(s)\ T(s) = \frac{4}{s(s^2+4)} = \frac{1}{s} + \frac{-s}{s^2 + 2^2}$$

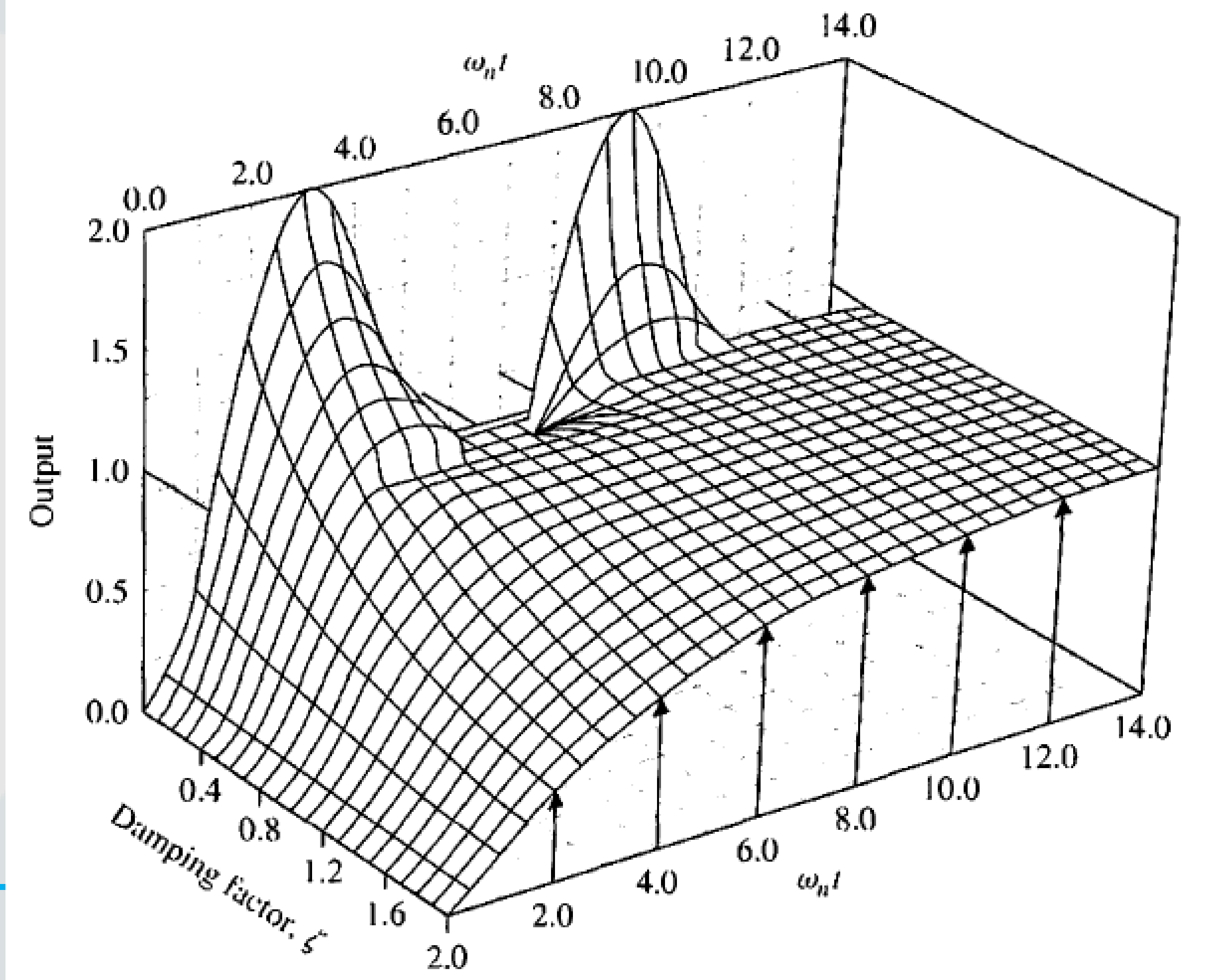
$$c(t) = L^{-1}\{C(S)\} = (1 - \cos 2t)\, u(t)$$

Step Response of underdamped System



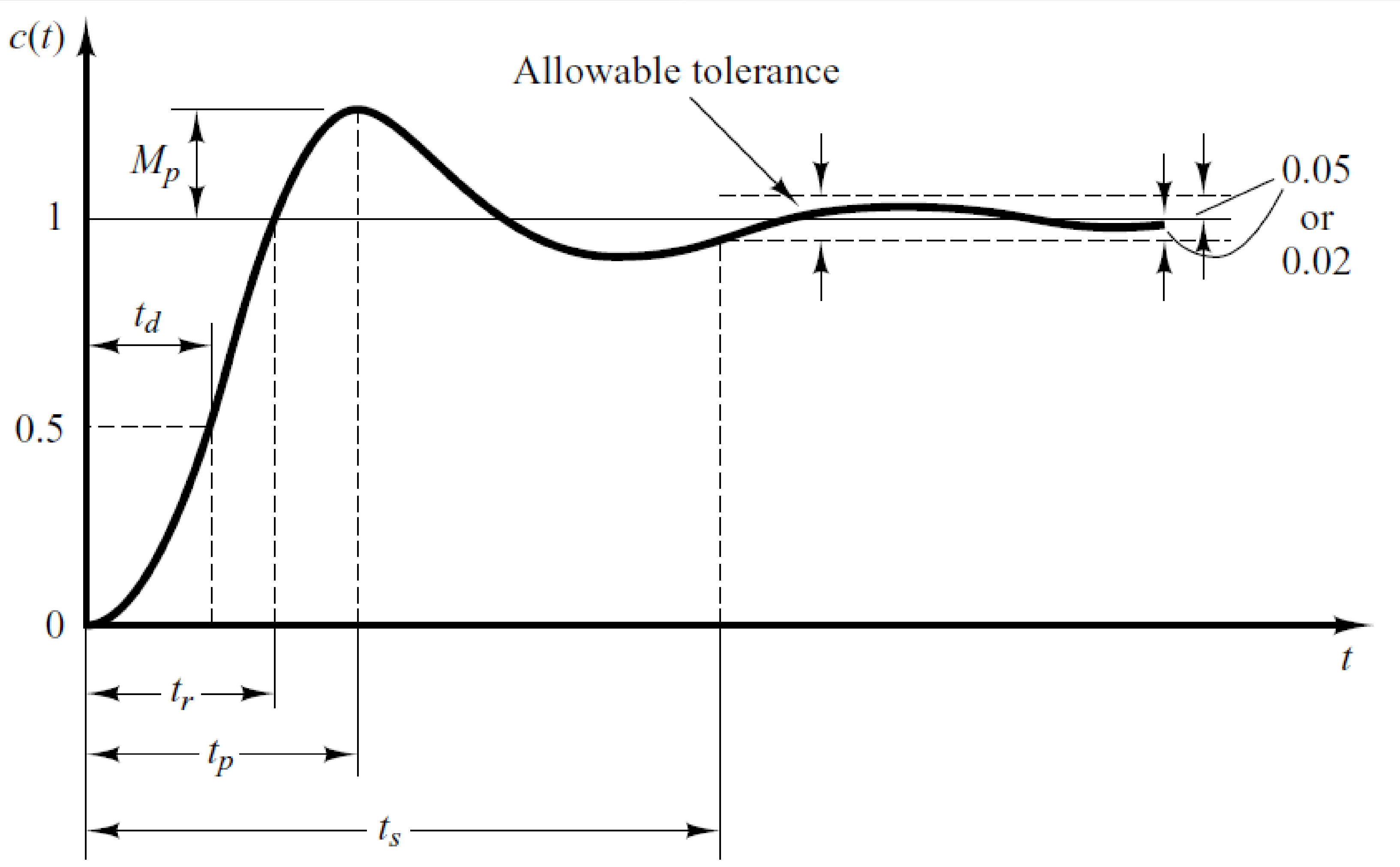
Step Response of underdamped System

$$c(t) = 1 - e^{-\zeta\omega_n t}\left[\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right]$$



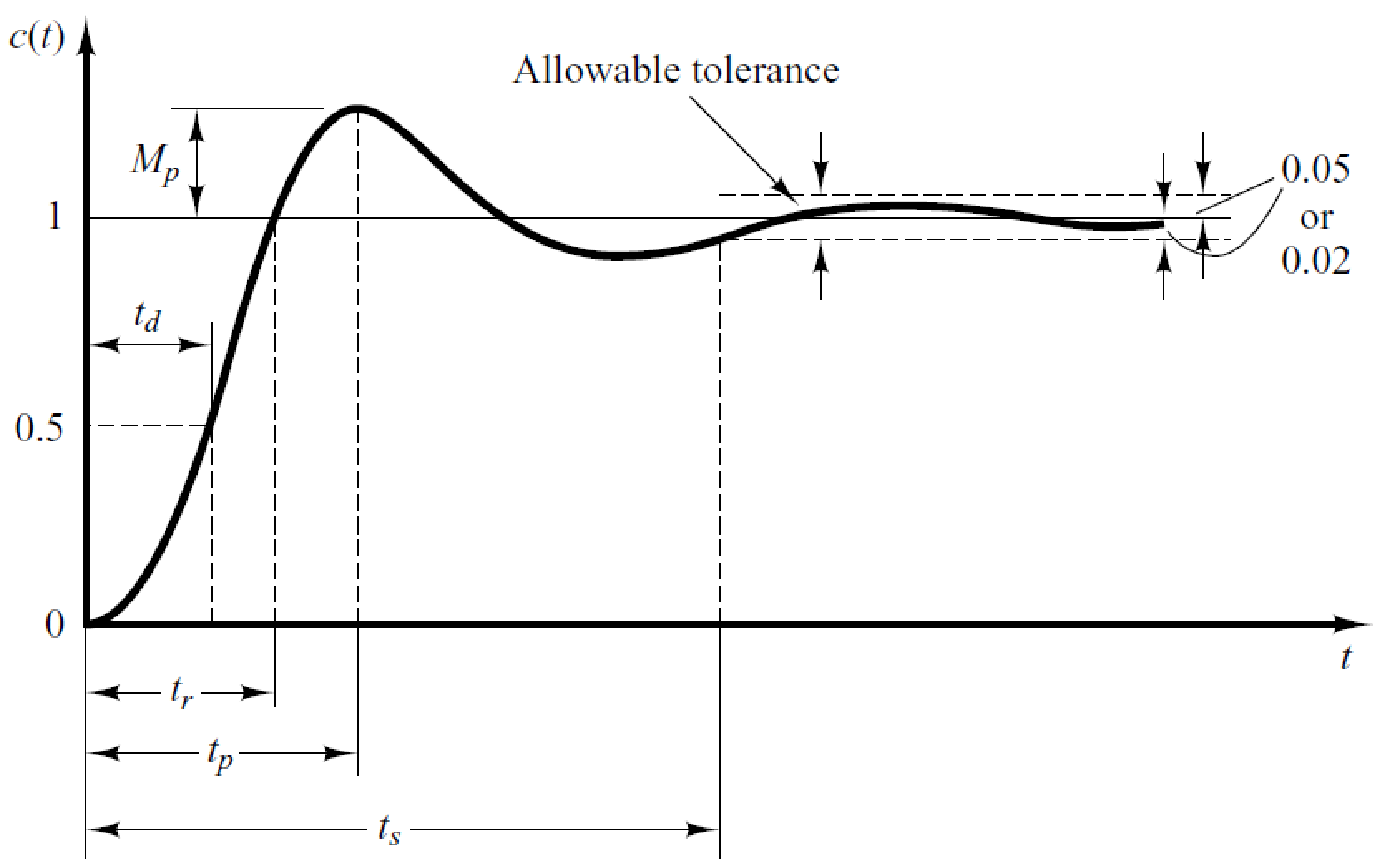
Time-Domain Specification (underdamped systems)

For $0 < \zeta < 1$ and $\omega_n > 0$, the 2nd order system's response due to a unit step input looks like



Time-Domain Specification -Delay Time

- The delay (t_d) time is the time required for the response to reach <u>half</u> the <u>final</u> <u>value</u> the <u>very</u> <u>first</u> <u>time</u>.



Time-Domain Specification -Delay Time

$$c(t) = 1 - \left(\frac{e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}}\right)\sin(\omega_d t + \theta)$$

The final value of the step response is one.

Therefore, at $t = t_d$, the value of the step response will be 0.5. Substitute, these values in the above equation.

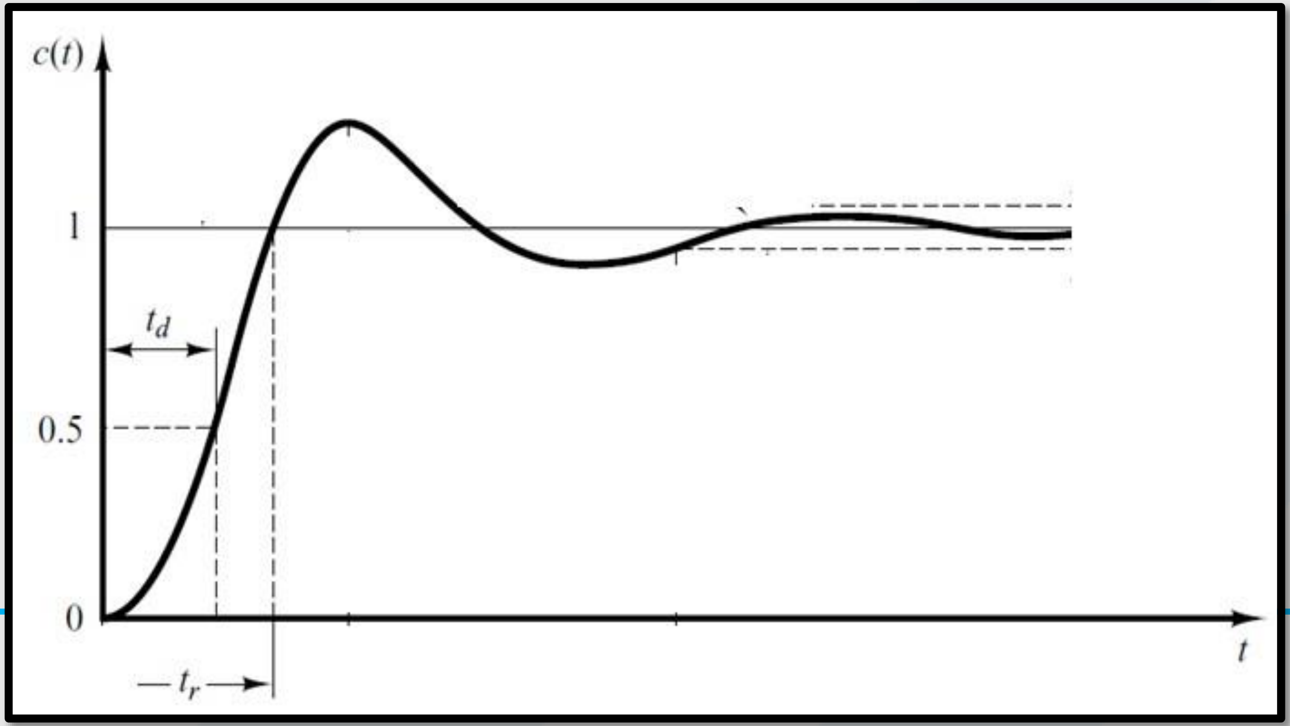
$$c(t_d) = 0.5 = 1 - \left(\frac{e^{-\delta\omega_n t_d}}{\sqrt{1-\delta^2}}\right)\sin(\omega_d t_d + \theta)$$

$$\Rightarrow \left(\frac{e^{-\delta\omega_n t_d}}{\sqrt{1-\delta^2}}\right)\sin(\omega_d t_d + \theta) = 0.5$$

$$t_d = \frac{1 + 0.7\delta}{\omega_n}$$

Time-Domain Specification – Rise Time

- **Rise-Time (T_r):** The rise time is the time required for the response to rise from
  - 10% to 90% of its final value, ➔ over damped systems
  - 5% to 95% of its final value, ➔ Critical damped systems
  - or 0% to 100% of its final value. ➔ under damped systems



$$t_r = \frac{\pi - \theta}{\omega_d}$$

$$\theta = \tan^{-1}\left(\frac{\omega_n\sqrt{1-\zeta^2}}{\zeta\omega_n}\right)$$

Time-Domain Specification – Rise Time

At $t = t_1 = 0$, $c(t_1) = 0$.

$$c(t) = 1 - \left(\frac{e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}}\right)\sin(\omega_d t + \theta)$$

$$c(t_2) = 1 = 1 - \left(\frac{e^{-\delta\omega_n t_2}}{\sqrt{1-\delta^2}}\right)\sin(\omega_d t_2 + \theta)$$

$$\Rightarrow \left(\frac{e^{-\delta\omega_n t_2}}{\sqrt{1-\delta^2}}\right)\sin(\omega_d t_2 + \theta) = 0$$

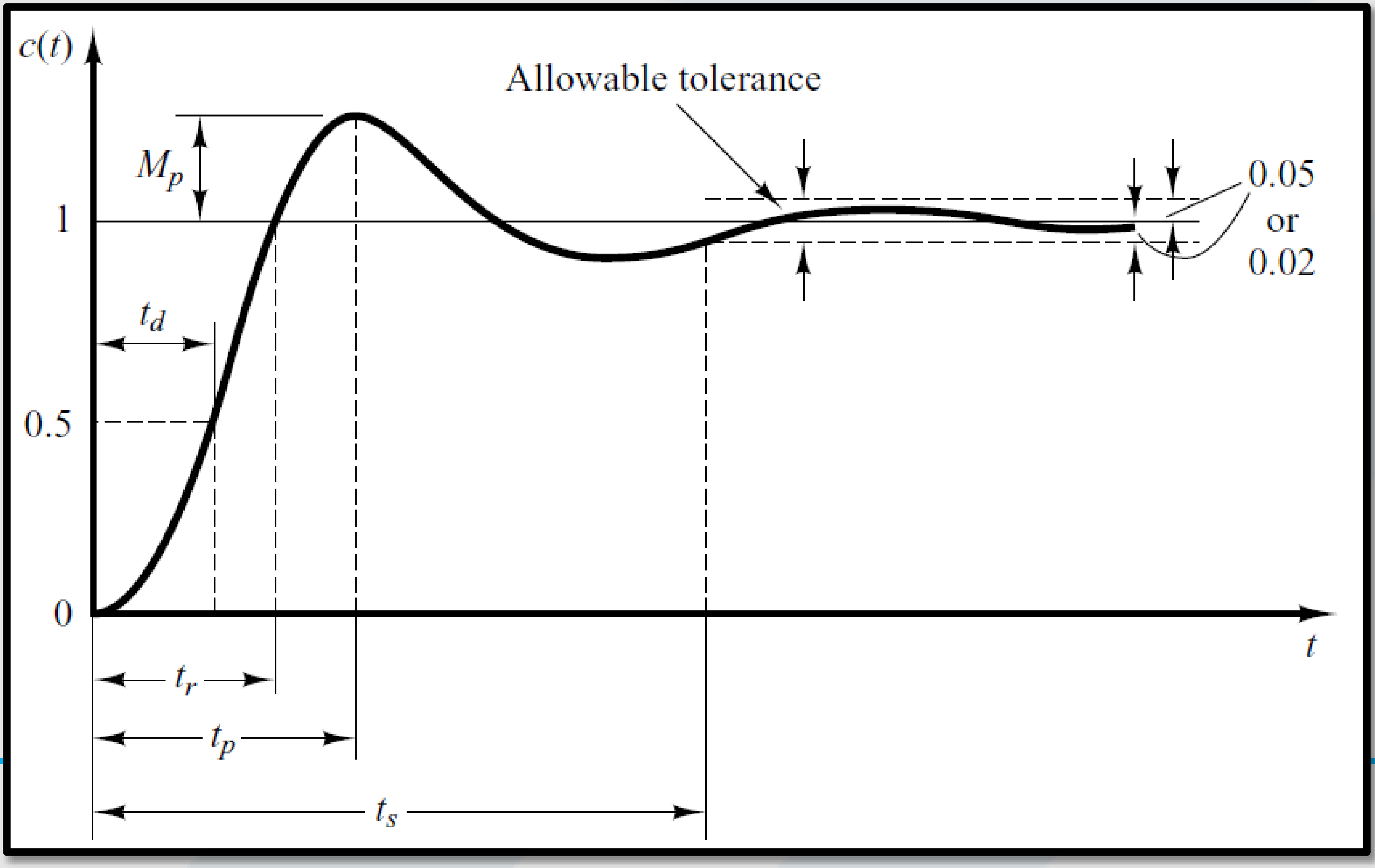
$$\Rightarrow \sin(\omega_d t_2 + \theta) = 0$$

$$\Rightarrow \omega_d t_2 + \theta = n\pi \text{ ;n=1}$$ For rise time

$$\Rightarrow t_2 = \frac{\pi - \theta}{\omega_d}$$

Time-Domain Specification – Settling time

- The settling time (Ts): is the time required for the response curve to reach and stay within a range about the final value of size specified by absolute percentage of the final value (usually 2% or 5%).



Settling Time (2%)

$$t_s = \frac{4}{\zeta\omega_n}$$

Settling Time (5%)

$$t_s = \frac{3}{\zeta\omega_n}$$

Time-Domain Specification – Settling time

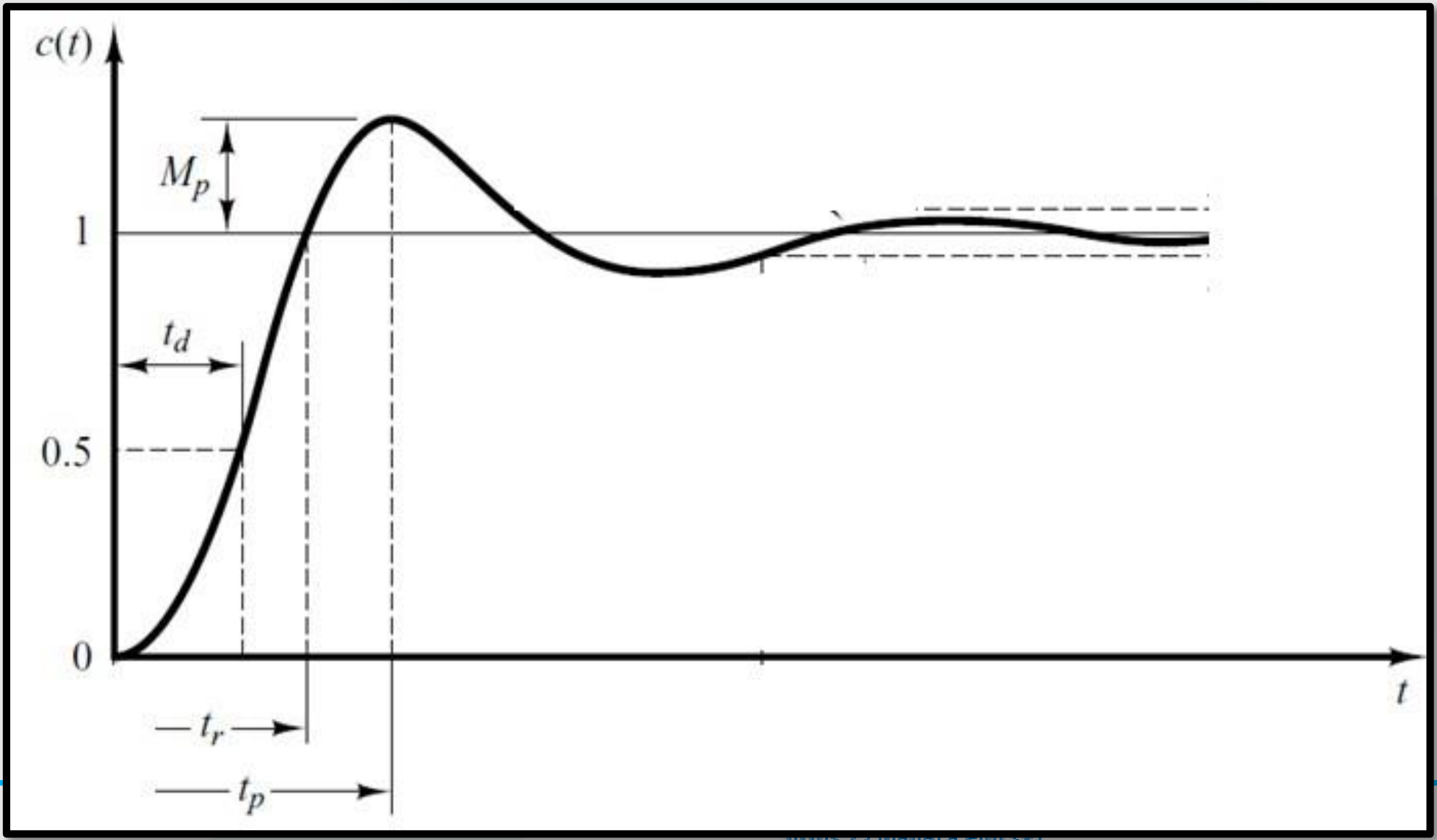
The settling time for 5% tolerance band is: $t_s = \frac{3}{\zeta\,\omega_n} = 3\tau$

The settling time for 2% tolerance band is: $t_s = \frac{4}{\zeta\,\omega_n} = 4\tau$

Where, τ is the time constant and is equal to $\frac{1}{\zeta\omega_n}$

Time-Domain Specification – Peak Time

- **Peak Time (Tp):** The peak time is the time required for the response to reach the first (maximum) peak of the overshoot.



$$t_p = \frac{\pi}{\omega_d}$$

Time-Domain Specification – Peak Time

$$c(t) = 1 - \left(\frac{e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}}\right)\sin(\omega_d t + \theta)$$

Differentiate $c(t)$ with respect to 't'.

$$\frac{dc(t)}{dt} = -\left(\frac{e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}}\right)\omega_d \cos(\omega_d t + \theta) - \left(\frac{-\delta\omega_n e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}}\right)\sin\ (\omega_d t + \theta)$$

Substitute, $t = t_p$ and $\frac{dc(t)}{dt} = 0$ in the above equation.

$$0 = -\left(\frac{e^{-\delta\omega_n t_p}}{\sqrt{1-\delta^2}}\right)\left[\omega_d \cos(\omega_d t_p + \theta) - \delta\omega_n \sin(\omega_d t_p + \theta)\right]$$

$$\Rightarrow \omega_n\sqrt{1-\delta^2}\cos(\omega_d t_p + \theta) - \delta\omega_n \sin(\omega_d t_p + \theta) = 0$$

$$\Rightarrow \sqrt{1-\delta^2}\cos(\omega_d t_p + \theta) - \delta \sin(\omega_d t_p + \theta) = 0$$

$$\Rightarrow \sin(\theta)\cos(\omega_d t_p + \theta) - \cos(\theta)\sin(\omega_d t_p + \theta) = 0$$

$$\Rightarrow \sin(\theta - \omega_d t_p - \theta) = 0$$

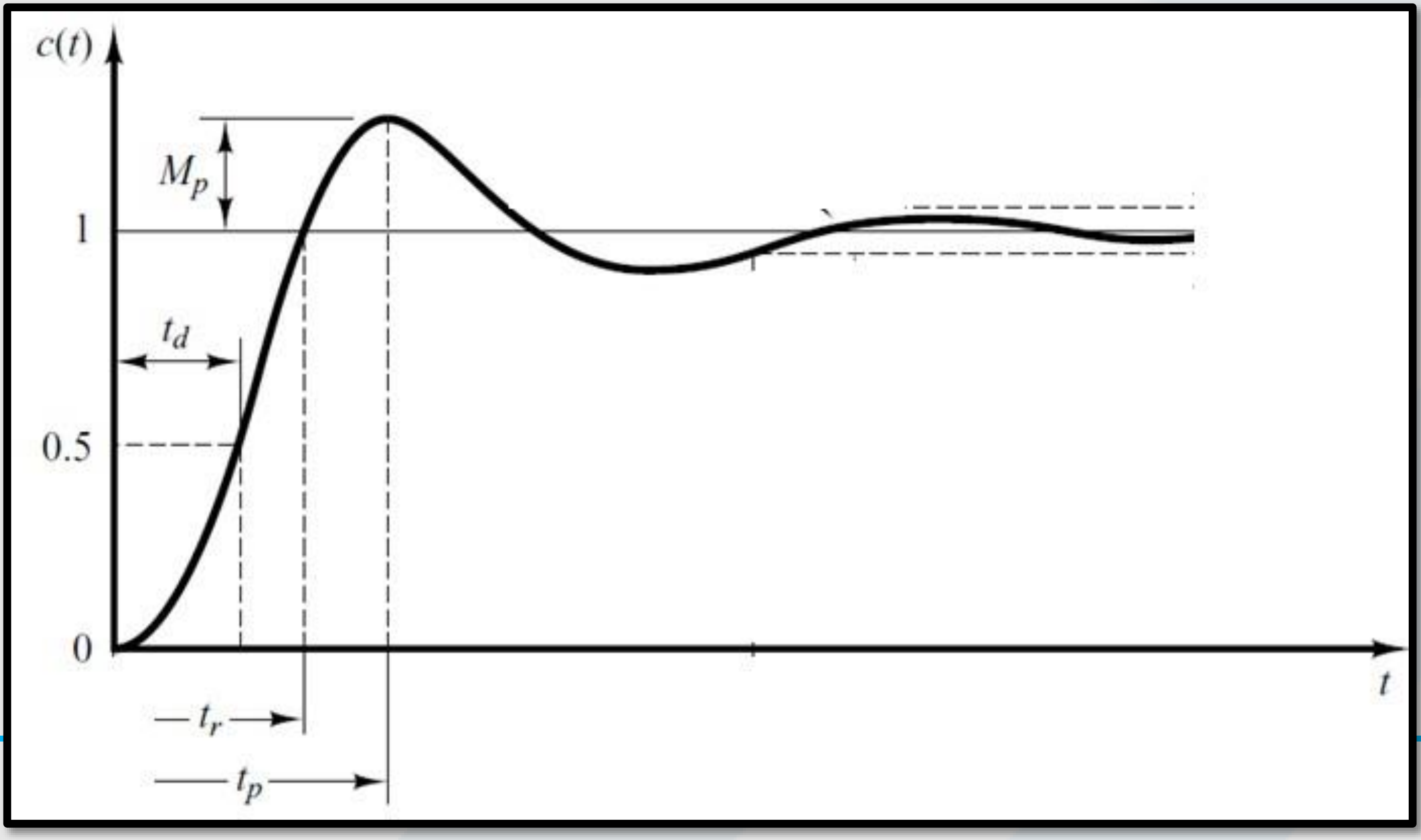
$$\Rightarrow sin(-\omega_d t_p) = 0 \Rightarrow -\sin(\omega_d t_p) = 0 \Rightarrow sin(\omega_d t_p) = 0$$

$$\Rightarrow \omega_d t_p = \pi$$ $\sin(n\pi)=0; n=1$ for the first peak

$$\Rightarrow t_p = \frac{\pi}{\omega_d}$$

Time-Domain Specification – Maximum Overshoot

- **Maximum Overshoot (MP)**: is the maximum peak value of the response curve measured from unity.
- **Maximum percent overshoot (P.O)**: is defined as follows:



$$M_P = e^{-\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right)\pi}$$

$$P.O = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100\%$$

Time-Domain Specification – Maximum Overshoot

The maximum overshoot is the maximum peak value of the response curve measured from unity. If the final steady-state value of the response differs from unity, then it is common to use the maximum percent overshoot. It is defined by

$$\text{Maximum percent overshoot} = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100\%$$

The amount of the maximum (percent) overshoot directly indicates the relative stability of the system.

Time-Domain Specification – Maximum Overshoot

$$c(t_p) = 1 - \left(\frac{e^{-\delta\omega_n t_p}}{\sqrt{1-\delta^2}}\right)\sin(\omega_d t_p + \theta)$$

Substitute, $t_p = \frac{\pi}{\omega_d}$ in the right hand side of the above equation.

$$c(t_P) = 1 - \left(\frac{e^{-\delta\omega_n\left(\frac{\pi}{\omega_d}\right)}}{\sqrt{1-\delta^2}}\right)\sin\left(\omega_d\left(\frac{\pi}{\omega_d}\right) + \theta\right)$$

$$\Rightarrow c(t_p) = 1 - \left(\frac{e^{-\left(\frac{\delta\pi}{\sqrt{1-\delta^2}}\right)}}{\sqrt{1-\delta^2}}\right)(-\sin(\theta))$$

We know that

$$\sin(\theta) = \sqrt{1-\delta^2}$$

So, we will get $c(t_p)$ as

$$c(t_p) = 1 + e^{-\left(\frac{\delta\pi}{\sqrt{1-\delta^2}}\right)}$$

Time-Domain Specification – Maximum Overshoot

$$c(t_p) = 1 + e^{-\left(\frac{\delta\pi}{\sqrt{1-\delta^2}}\right)}$$

Substitute the values of $c(t_p)$ and $c(\infty)$ in the peak overshoot equation

$c(\infty)=1$

$$M_p = 1 + e^{-\left(\frac{\delta\pi}{\sqrt{1-\delta^2}}\right)} - 1$$

$$\text{Maximum percent overshoot} = \frac{c(t_p) - c(\infty)}{c(\infty)}$$

$$\Rightarrow M_p = e^{-\left(\frac{\delta\pi}{\sqrt{1-\delta^2}}\right)}$$

Time-Domain Specification – Maximum Overshoot

Percentage of peak overshoot % M_p can be calculated by using this formula.

$$\%M_p = \frac{M_p}{c(\infty)} \times 100\%$$

By substituting the values of M_p and $c(\infty)$ in above formula, we will get the Percentage of the peak overshoot $\%M_p$ as

$$\%M_p = \left(e^{-\left(\frac{\delta\pi}{\sqrt{1-\delta^2}}\right)} \right) \times 100\%$$

Time Domain Specifications

Rise Time

$$t_r = \frac{\pi - \theta}{\omega_d} = \frac{\pi - \theta}{\omega_n \sqrt{1-\zeta^2}}$$

Peak Time

$$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n \sqrt{1-\zeta^2}}$$

Delay Time

$$t_d = \frac{1+0.7\zeta}{\omega_n}$$

Settling Time (2%)

$$t_s = 4T = \frac{4}{\zeta\omega_n}$$

$$t_s = 3T = \frac{3}{\zeta\omega_n}$$

Settling Time (5%)

Maximum Overshoot

$$M_p = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} \times 100$$

Example 1

Ex-1 consider a second order system with closed loop transfer function $G(s) = \frac{25}{s^2+8s+25}$. Determine time domain specifications and output response y(t) for step input.

<u>soln</u> We have, $G(s) = \frac{25}{s^2+8s+25}$

comparing denominator of G(s) with $s^2+2\xi\omega_n s+\omega_n^2$ we get,

$\omega_n^2 = 25 \Rightarrow \omega_n = 5$ rad/sec

$2\xi\omega_n = 8 \Rightarrow \xi = 0.8$

$\omega_d = \omega_n\sqrt{1-\xi^2} = 5\sqrt{1-(0.8)^2} = 3$ rad/sec

$t_d = \frac{1+0.7\xi}{\omega_n} = \frac{1+0.7(0.8)}{5} = 0.312$ sec

$t_r = \frac{\pi-\theta}{\omega_d} = \frac{\pi-\cos^{-1}\xi}{\omega_d} = \frac{\pi-0.6435}{3} = 0.8327$ sec

$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{3} = 1.0472$ sec

$t_s = \frac{4}{\xi\omega_n} = \frac{4}{4} = 1$ sec (2% band)

$= \frac{3}{\xi\omega_n} = \frac{3}{4} = 0.75$ sec (5% band)

$M_p = e^{-\pi\xi/\sqrt{1-\xi^2}} = 0.0152 = 1.52\%$

Example 1

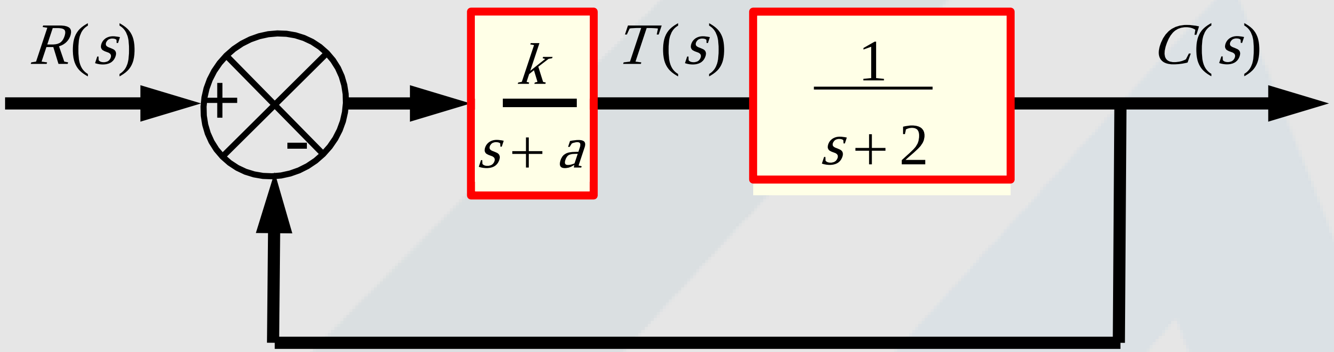
$$y(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin(\omega_d t + \theta)$$

$$= 1 - \frac{e^{-4t}}{0.6} \sin(3t + 0.6435^c)$$

$$= 1 - 1.6667 e^{-4t} \sin(3t + 0.6435^c)$$

Example2

- For the control system shown in Figure, determine k and a that satisfies the following requirements:
  - a) Maximum percentage overshoot P.O =10%.
  - b) The 5% settling time t_s = 1 sec.



- **Solution:** The closed loop transfer function is given by

$$\frac{C(s)}{R(s)} = \frac{\left(\frac{k}{s+a}\right)\left(\frac{1}{s+2}\right)}{1+\left(\frac{k}{s+a}\right)\left(\frac{1}{s+2}\right)} = \frac{k}{(s+a)(s+2)+k} = \frac{k}{s^2+s(a+2)+(2a+k)}$$

Example2

- The maximum percent overshoot (P.O)is given by:

$$P.O = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} = \frac{10}{100} \rightarrow \zeta = 0.6$$

- For 5%, the settling time ts is given by:

$$t_s = 3\tau = \frac{3}{\zeta\omega_n} = 1 \longrightarrow \omega_n = 5$$

$$a + 2 = 2\zeta\omega_n \qquad \& \qquad 2a + k = \omega_n^2$$

- From these two equations we get a + 2 = 6 then a = 4 and k = 17

Example 3

Ex.-2 Consider unity feedback system with open loop transfer function $G(s) = \frac{16}{s(s+4)}$. Determine time domain specifications of closed loop system.

Sol^n We have $G(s) = \frac{16}{s(s+4)}$, $H(s) = 1$

$$G_{CL}(s) = \frac{16}{s^2+4s+16}$$

$$\omega_n^2 = 16 \Rightarrow \omega_n = 4 \text{ rad/sec}$$

$$2\xi\omega_o = 4 \Rightarrow \xi = 0.5$$

$$\omega_d = \omega_n\sqrt{1-\xi^2} = 4\sqrt{1-(0.5)^2} = 3.4641 \text{ rad/sec}$$

$$t_d = \frac{1+0.7\xi}{\omega_n} = \frac{1+0.7(0.5)}{4} = 0.3375 \text{ sec}$$

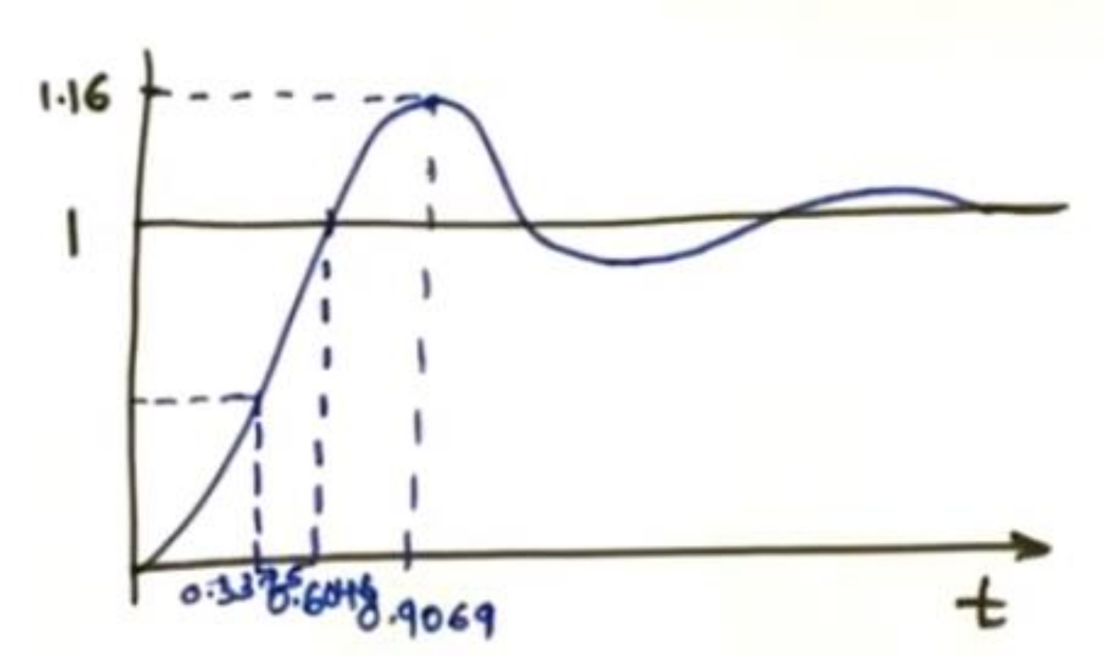
$$t_r = \frac{\pi-\theta}{\omega_d} = \frac{\pi-\cos^{-1}0.5}{3.4641} = 0.6046 \text{ sec}$$

$$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{3.4641} = 0.9069 \text{ sec}$$

$$t_s = \frac{4}{\xi\omega_n} = \frac{4}{2} = 2 \text{ sec}$$

$$M_p = e^{-\pi\xi/\sqrt{1-^2}} = 0.1630 = 16.30\%$$

Example 3



Example 4

Ex-3 Consider unity feed back system with open loop transfer function $G(s) = \frac{12}{(s+1)(s+4)}$. Determine all time domain specifications and step response of closed loop system.

Soln We have $G(s) = \frac{12}{(s+1)(s+4)}$, $H(s) = 1$

$$\therefore G_{CL}(s) = \frac{G(s)}{1+G(s)H(s)} = \frac{12}{s^2+5s+16}$$

$$\omega_n^2 = 16 \Rightarrow \omega_n = 4 \text{ rad/sec}$$

$$2\xi\omega_n = 5 \Rightarrow \xi = 0.625$$

$$\omega_d = \omega_n\sqrt{1-\xi^2} = 4\sqrt{1-(0.625)^2} = 3.1225 \text{ rad/sec}$$

$$t_d = \frac{1+0.7\xi}{\omega_n} = \frac{1+0.7(0.625)}{4} = 0.3594 \text{ sec}$$

$$t_r = \frac{\pi-\theta}{\omega_d} = \frac{\pi-\cos^{-1}\xi}{\omega_d} = 0.7193 \text{ sec}$$

$$t_p = \frac{\pi}{\omega_d} = 1.0061 \text{ sec}$$

$$t_s = \frac{4}{\xi\omega_n} = \frac{4}{2.5} = 1.6 \text{ sec } (2\% \text{ band})$$

$$M_p = e^{-\pi\xi/\sqrt{1-\xi^2}} = 0.0808 = 8.08\%$$

Example 4

k=0.75

$$y(t) = \frac{3}{4}\left[1 - \frac{e^{-\zeta\omega_0 t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t + \theta)\right]$$

$$= \frac{3}{4}\left[1 - \frac{e^{-2.5t}}{0.7806}\sin(3.1225t + 0.8957^c)\right]$$

Example 5

Ex.-4 A closed loop system is represented by the differential equation $\ddot{y}+6\dot{y}+25y=25r$. Determine its time domain specifications.

System input is r

Sol^n We have $\ddot{y}+6\dot{y}+25y=25r$

Taking L.T. on both sides we get

$$s^2Y(s)+6sY(s)+25Y(s)=25R(s)$$

$$\therefore \frac{Y(s)}{R(s)} = \frac{25}{s^2+6s+25}$$

$\omega_n^2=25 \Rightarrow \omega_n = 5$ rad/sec

$2\xi\omega_n = 6 \Rightarrow \xi = 0.6$

$\omega_d = \omega_n\sqrt{1-\xi^2} = 5\sqrt{1-(0.6)^2} = 4$ rad/sec

$t_d = \frac{1+0.7\xi}{\omega_n} = \frac{1+(0.7)(0.6)}{5} = 0.284$ sec

$t_r = \frac{\pi-\theta}{\omega_d} = \frac{\pi-\cos^{-1}0.6}{4} = 0.5536$ sec

$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{4} = 0.7854$ sec

$t_s = \frac{4}{\xi\omega_n} = \frac{4}{3} = 1.3333$ sec (2% band)

$M_p = e^{-\pi\xi/\sqrt{1-\xi^2}} = 0.0948 = 9.48\%$

Example 6

Ex.-5 Consider a system with closed loop transfer function $G(s) = \frac{k_2}{s^2 + k_1 s + k_2}$. Determine k_1 and k_2 such that peak overshoot is 15% and peak time is 2 seconds. Also determine delay time, rise time and settling time.

Soln: We have $M_p = 15\%$, $t_p = 2$ sec

$$M_p = e^{-\pi\xi/\sqrt{1-\xi^2}}$$

$$0.15 = e^{-\pi\xi/\sqrt{1-\xi^2}}$$

$$\frac{-\pi\xi}{\sqrt{1-\xi^2}} = -1.8971$$

$$\frac{\xi}{\sqrt{1-\xi^2}} = 0.6039$$

$$\frac{\xi^2}{1-\xi^2} = 0.3647$$

$$\xi^2 = 0.3647 - 0.3647\xi^2$$

$$\xi^2 = \frac{0.3647}{1.3647} = 0.2672$$

$$\xi = 0.5170$$

$$t_p = \frac{\pi}{\omega_d} = 2$$

Example 6

$$\omega_d = \frac{\pi}{2} = 1.5708$$

$$\therefore \omega_n\sqrt{1-\xi^2} = 1.5708$$

$$\omega_n = \frac{1.5708}{\sqrt{1-0.2672}} = 1.8350 \text{ rad/sec}$$

From given T.F.

$$k_2 = \omega_n^2 = 3.3671$$

$$k_1 = 2\xi\omega_n = 2(0.5170)(1.8350)$$

$$= 1.8974$$

$$t_d = \frac{1+0.7\xi}{\omega_n} = 0.7421 \text{ sec}$$

$$t_r = \frac{\pi-\theta}{\omega_d} = \frac{\pi-\cos^{-1}0.5170}{1.5708} = 1.3459 \text{ sec}$$

$$t_s = \frac{4}{\xi\omega_n} = \frac{4}{(0.5170)(1.8350)} = 4.2163 \text{ sec}$$