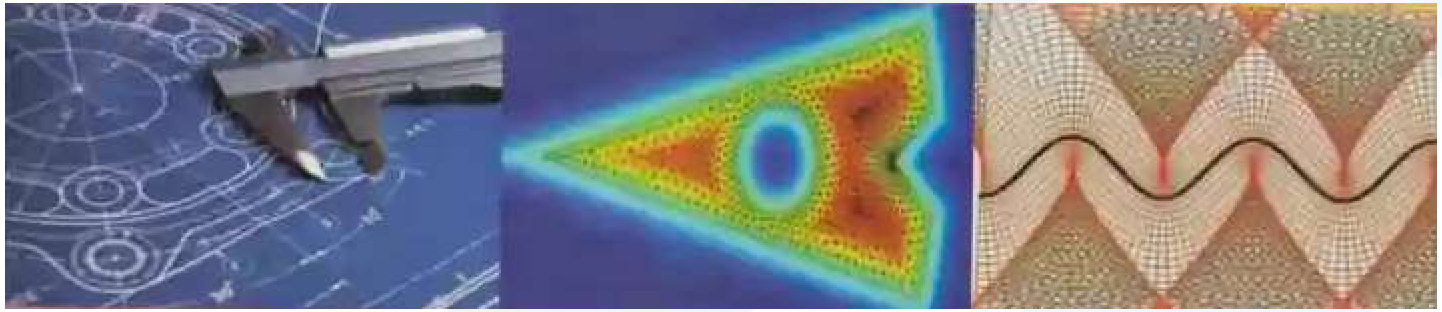
CEDC301: Engineering Mathematics

Lecture Notes 8 & 9: Fourier Analysis



Ramez Koudsieh, Ph.D.

Faculty of Engineering

Department of Robotics and Intelligent Systems

Manara University

Chapter 5

Fourier Analysis

1. Orthogonal Functions

Inner Product

- Let $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$ be vectors in a real vector space V, and let c be any scalar. An inner product on V is a function that associates a real number $<\boldsymbol{u}, \boldsymbol{v}>$ with each pair of vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ and satisfies the following axioms:

(1) $<\boldsymbol{u}, \boldsymbol{v}> = <\boldsymbol{v}, \boldsymbol{u}>$

(2) $<\boldsymbol{u}, \boldsymbol{v} + \boldsymbol{w}> = <\boldsymbol{u}, \boldsymbol{v}> + <\boldsymbol{u}, \boldsymbol{w}>$

(3) $c <\boldsymbol{u}, \boldsymbol{v}> = <c\boldsymbol{u}, \boldsymbol{v}>$

(4) $<\boldsymbol{v}, \boldsymbol{v}> \geq 0$ and $<\boldsymbol{v}, \boldsymbol{v}> = 0$ if and only if $\boldsymbol{v} = \mathbf{0}$

- Note: $<\boldsymbol{u}, \boldsymbol{v}> = \boldsymbol{u}.\boldsymbol{v} = \sum_{i=1}^{n} u_i v_i$ dot product (Euclidean inner product for R^n)

- Definition: The inner product of two piecewise-continuous functions f_1 and f_2 on an interval $[a,\ b]$ is the number:

$$\langle f_1, f_2\rangle = \int_a^b f_1(x)f_2(x)\,dx$$

Orthogonal Functions

- Definition: Two functions f_1 and f_2 are said to be orthogonal on an interval $[a,\ b]$ if:

$$\langle f_1, f_2\rangle = \int_a^b f_1(x)f_2(x)\,dx = 0$$

- Example 1: Orthogonal Functions

  The functions $f_1(x) = x^2$ and $f_2(x) = x^3$ are orthogonal on the interval [–1, 1].

$$\langle f_1, f_2\rangle = \int_{-1}^{1} x^2x^3\,dx = \int_{-1}^{1} x^5\,dx = \frac{1}{6}x^6\Big]_{-1}^{1} = 0$$

- Definition: The norm, or length, of a vector u is given by:

$$\|\boldsymbol{u}\| = \sqrt{<\boldsymbol{u}, \boldsymbol{u}>}$$

Orthogonal Sets

- Definition: A set of real-valued functions $\{\phi_0(x),\ \phi_1(x),\ \phi_2(x),\ \ldots\}$ is said to be orthogonal on an interval $[a,\ b]$ if:

$$\langle \phi_m, \phi_n \rangle = \int_a^b \phi_m(x)\phi_n(x)\,dx = 0, \quad m \neq n$$

Orthonormal Sets

$$\langle \phi_m, \phi_n \rangle = \int_a^b \phi_m(x)\phi_n(x)\,dx = \begin{cases} 1, & m = n \\ 0, & m \neq n \end{cases}$$

- If $\{\phi_n(x)\}$ is an orthogonal set of functions on the interval $[a,\ b]$ with $\|\phi_n(x)\| = 1$ for $n = 0,\ 1,\ 2,\ \ldots,$ then $\{\phi_n(x)\}$ is said to be an orthonormal set on the interval.

- Example 2: Orthogonal Set of Functions

  Show that the set {1, $\cos x$, $\cos 2x$, …} is orthogonal on the interval $[-\pi, \pi]$

$$\langle 1, \cos nx \rangle = \int_{-\pi}^{\pi} \cos nx \, dx = \frac{1}{n} \sin nx \Big]_{-\pi}^{\pi} = \frac{1}{n}\left[\sin n\pi - \sin(-n\pi)\right] = 0$$

$$\langle \cos mx, \cos nx \rangle = \int_{-\pi}^{\pi} \cos mx \cos nx \, dx = \frac{1}{2}\int_{-\pi}^{\pi} \left[\cos(m+n)x + \cos(m-n)x\right] dx$$

$$= \frac{1}{2}\left[\frac{\sin(m+n)x}{m+n} + \frac{\sin(m-n)x}{m-n}\right]_{-\pi}^{\pi} = 0, \quad m \neq n$$

- Example 3: Norms

  Find the norms of each function in the orthogonal set given in Example 2

$$\left\|\phi_0(x)\right\| = \left\|1\right\| = \sqrt{\int_{-\pi}^{\pi} dx} = \sqrt{2\pi}$$

$$\|\phi_n(x)\| = \|\cos nx\| = \sqrt{\int_{-\pi}^{\pi} \cos^2 nx\, dx} = \sqrt{\int_{-\pi}^{\pi} \tfrac{1}{2}[1+\cos 2nx]\, dx} = \sqrt{\pi}, \quad n > 0$$

- Note: Any orthogonal set of nonzero functions $\{\phi_n(x)\}$, $n = 0$, 1, 2, …, can be normalized—that is, made into an orthonormal set—by dividing each function by its norm.

  For example the set $\left\{ \frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{\pi}}, \cdots \right\}$ is orthonormal on $[-\pi, \pi]$.

- Theorem 1 (Coordinates relative to a basis): If $B = \{\phi_n(x)\}$, $n = 0$, 1, 2, …, is an orthogonal basis for an inner product space $V = C[a, b]$, and if f is any vector in V, then:

$$f(x) = \sum_{n=0}^{\infty} c_n \phi_n(x)$$

where

$$c_n = \frac{\langle f, \phi_n \rangle}{\|\phi_n(x)\|^2} = \frac{\int_a^b f(x)\phi_n(x)dx}{\|\phi_n(x)\|^2}$$

$\{\phi_n(x)\}$ is an orthogonal basis: $f(x) = \sum_{n=0}^{\infty} \frac{\langle f, \phi_n \rangle}{\|\phi_n(x)\|^2} \phi_n(x)$

$\{\phi_n(x)\}$ is an orthonormal basis: $f(x) = \sum_{n=0}^{\infty} \langle f, \phi_n \rangle \phi_n(x)$

- Definition: A set of real-valued functions $\{\phi_0(x), \phi_1(x), \phi_2(x), \ldots\}$ is said to be orthogonal with respect to a weight function $w(x)$ on an interval $[a, b]$ if:

$$\langle \phi_m, \phi_n \rangle = \int_a^b w(x)\phi_m(x)\phi_n(x)dx = 0, \quad m \neq n$$

where $w(x)$ is a positive continuous function

- Note: The inner product of two functions f_1 and f_2 on an interval $[a, b]$, used above is:

$$\langle f_1, f_2 \rangle = \int_a^b f_1(x) f_2(x) w(x)\, dx$$

- The set $\{1, \cos x, \cos 2x, \ldots\}$ in Example 2 is orthogonal with respect to the weight function $w(x) = 1$ on the interval $[-\pi, \pi]$.
- The series $f(x) = \sum_{n=0}^{\infty} c_n \phi_n(x)$ is said to be an orthogonal series expansion of f or a generalized Fourier series.

Complete Sets

- To expand f in a series of orthogonal functions, it is certainly necessary that f not be orthogonal to each ϕ_n of the orthogonal set $\{\phi_n(x)\}$.
(If f were orthogonal to every ϕ_n, then $c_n = 0$, $n = 0, 1, 2, \ldots$.)

- To avoid the latter problem we shall assume that an orthogonal set is complete. This means that the only continuous function orthogonal to each member of the set is the zero function.
- Note: Suppose that $\{\phi_0(x), \phi_1(x), \phi_2(x), \ldots\}$ is an infinite set of real-valued functions that are continuous on an interval $[a, b]$. If this set is linearly independent on $[a, b]$, then it can always be made into an orthogonal set using Gram-Schmidt process.

Orthogonal Polynomials

- Let P_∞ be the vector space of all polynomials and define the inner product of two polynomials P and Q, on P_∞ by:

$$\langle P, Q\rangle = \int_a^b P(x)Q(x)w(x)dx$$

- Let $P_0(x)$, $P_1(x)$, ... be a sequence of polynomials with deg $P_n(x) = n$ for each n. If $\langle P_m, P_n \rangle = 0$ whenever $m \neq n$, then $\{P_n(x)\}$ is said to be a sequence of orthogonal polynomials. If $\langle P_m, P_n \rangle = \delta_{mn}$, then $\{P_n(x)\}$ is said to be a sequence of orthonormal polynomials.

Legendre Polynomials

$$\langle P_m, P_n \rangle = \int_{-1}^{1} P_m(x) P_n(x)\,dx \qquad \|P_n(x)\|^2 = \frac{2}{2n+1},\ n = 0, 1, \ldots$$

$P_n(1) = 1$ for each n, then $(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$

$$(1-x^2)P_n''(x) - 2xP_n'(x) + n(n+1)P_n(x) = 0$$

$$P_0(x) = 1 \qquad P_1(x) = x \qquad P_2(x) = \tfrac{1}{2}(3x^2 - 1)$$

$$P_3(x) = \tfrac{1}{2}(5x^3 - 3x) \qquad P_4(x) = \tfrac{1}{8}(35x^4 - 30x^2 + 3)$$

Chebyshev Polynomials

$$\langle T_m, T_n \rangle = \int_{-1}^{1} T_m(x) T_n(x)(1-x)^{-1/2} dx$$

$$\|T_0(x)\|^2 = \pi, \quad \|T_n(x)\|^2 = \frac{\pi}{2}, \; n = 1, 2, \ldots$$

$a_0 = 1, \quad a_k = 2^{k-1} \; (k = 1, 2, \ldots)$

$T_n(\cos\theta) = \cos n\theta$ and $\cos(n+1)\theta = 2\cos\theta\cos n\theta - \cos(n-1)\theta$ gives:

$T_1(x) = xT_0(x), \qquad T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \quad n \geq 1$

$$(1-x^2)T_n''(x) - xT_n'(x) + n^2 T_n(x) = 0$$

$T_0(x) = 1 \qquad T_1(x) = x \qquad T_2(x) = 2x^2 - 1$

$T_3(x) = 4x^3 - 3x \qquad T_4(x) = 8x^4 - 8x^2 + 1$

Hermite Polynomials

$$\langle H_m, H_n \rangle = \int_{-\infty}^{\infty} H_m(x) H_n(x) e^{-x^2} dx \qquad \|H_n(x)\|^2 = \sqrt{\pi}\, 2^n n!,\ n = 0, 1, \ldots$$

$$H_0(x) = 1, \quad H_1(x) = 2x$$

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x), \quad n \geq 1$$

$$H_n''(x) - 2xH_n'(x) + 2nH_n(x) = 0$$

$$H_0(x) = 1 \qquad H_1(x) = 2x \qquad H_2(x) = 4x^2 - 2$$

$$H_3(x) = 8x^3 - 12x \qquad H_4(x) = 16x^4 - 48x^2 + 12$$

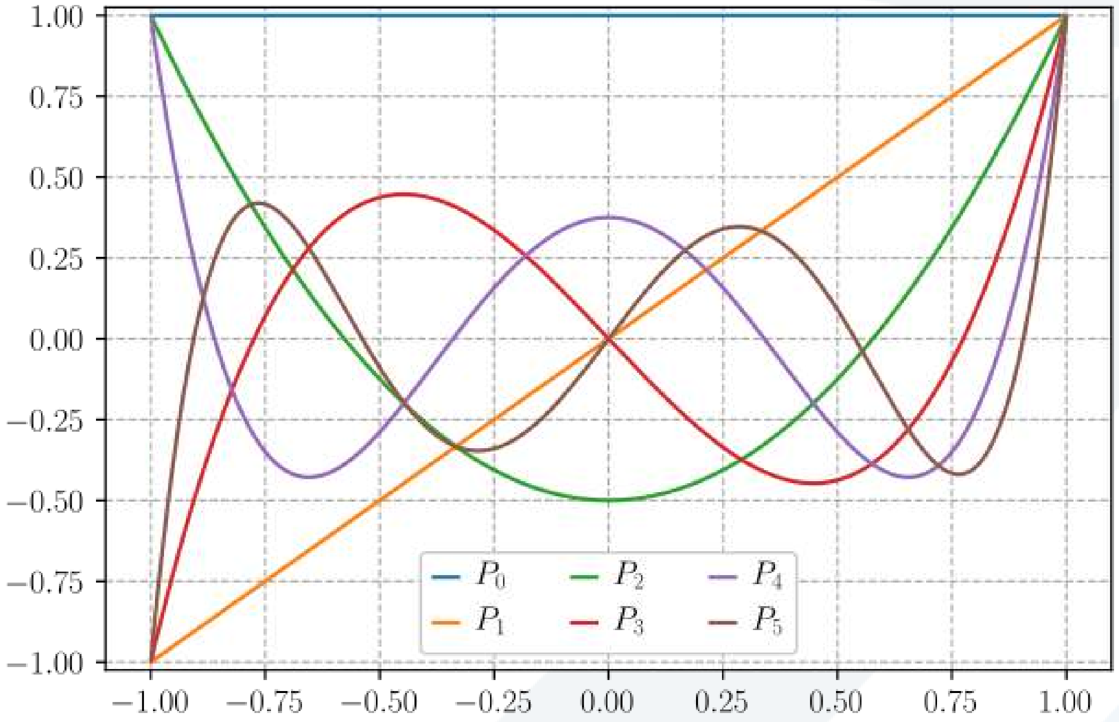
Laguerre Polynomials

$$\langle L_m, L_n \rangle = \int_0^{\infty} L_m(x) L_n(x) e^{-x} dx \qquad \|L_n(x)\|^2 = 1,\ n = 0, 1, \ldots$$

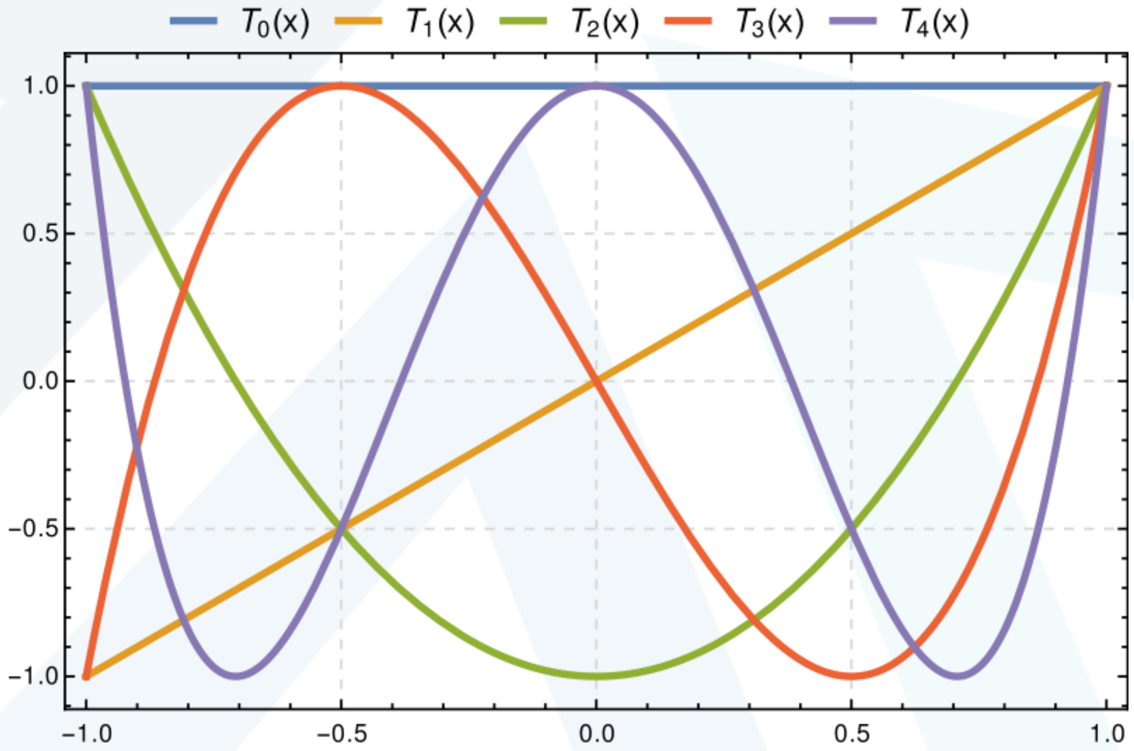
$$(n+1)L_{n+1}(x) = (2n+1-x)L_n(x) - nL_{n-1}(x) \qquad xy'' + (1-x)y' + ny = 0$$

$$L_0(x)=1 \qquad L_1(x)=-x+1 \qquad L_2(x)=\frac{1}{2}(x^2-4x+2)$$

$$L_3(x)=\frac{1}{6}(-x^3+9x^2-18x+6) \qquad L_4(x)=\frac{1}{24}(x^4-16x^3+72x^2-96x+24)$$



Legendre Polynomials



Chebyshev Polynomials

2. Fourier Series

- If $\{\phi_0(x),\ \phi_1(x),\ \phi_2(x),\ \ldots\}$ is a set of real-valued functions that is orthogonal on an interval $[a,\ b]$ and if f is a function defined on the same interval, then we can formally expand f in an orthogonal series $c_0\phi_0(x) + c_1\phi_1(x) + c_2\phi_2(x) + \ldots$.
- In this section we shall expand functions in terms of a special orthogonal set of trigonometric functions.

Trigonometric Series

- The set of trigonometric functions

$$\left\{1,\ \cos\frac{\pi}{L}x,\ \cos\frac{2\pi}{L}x,\ \cos\frac{3\pi}{L}x,\ \cdots,\ \sin\frac{\pi}{L}x,\ \sin\frac{2\pi}{L}x,\ \sin\frac{3\pi}{L}x,\ \cdots\right\}$$

is orthogonal on the interval $[-L,\ L]$.

- Expand a function f defined on $[-L, L]$ in an orthogonal series consisting of the trigonometric functions.

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{L}x + b_n \sin\frac{n\pi}{L}x\right)$$

The coefficients $a_0, a_1, a_2, \ldots, b_1, b_2, \ldots$, can be determined using:

$$\frac{\langle f, \phi_n\rangle}{\|\phi_n(x)\|^2} = \frac{\int_{-L}^{L} f(x)\phi_n(x)dx}{\|\phi_n(x)\|^2}$$

$$\|\phi_0(x)\|^2 = \|1\|^2 = \int_{-L}^{L} dx = 2L, \quad \|\phi_n(x)\|^2 = \left\|\cos\frac{n\pi}{L}x\right\|^2 = \int_{-L}^{L}\cos^2\frac{n\pi}{L}x\,dx = L, \quad n > 0$$

$$\|\phi_n(x)\|^2 = \left\|\sin\frac{n\pi}{L}x\right\|^2 = \int_{-L}^{L}\sin^2\frac{n\pi}{L}x\,dx = L, \quad n > 0$$

$$\frac{\int_{-L}^{L} f(x)\phi_n(x)\,dx}{\left\|\phi_n(x)\right\|^2} = \begin{cases} a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx \\ a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi}{L}x\,dx \\ b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi}{L}x\,dx \end{cases}$$

Fourier coefficients of f

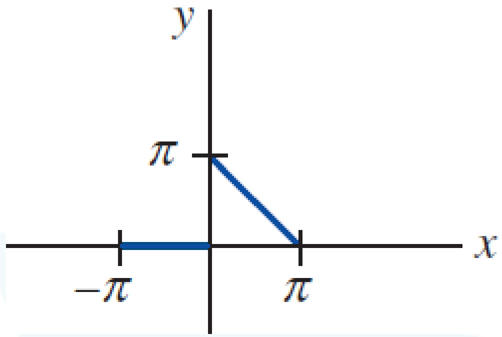
- Definition: The Fourier series (FS) of a function f defined on the interval $(-L, L)$ is given by:

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi}{L}x + b_n\sin\frac{n\pi}{L}x\right)$$

where

$$\frac{1}{2L}\int_{-L}^{L} f(x)\,dx, \quad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi}{L}x\,dx, \quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi}{L}x\,dx$$

- Example 4: Expansion in a Fourier Series



Expand $f(x) = \begin{cases} 0, & -\pi < x < 0 \\ \pi - x, & 0 \le x < \pi \end{cases}$ in a Fourier series

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{2\pi}\int_0^{\pi}(\pi - x)dx = \frac{\pi}{4}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx = \frac{1}{\pi}\int_0^{\pi}(\pi - x)\cos nx dx = \frac{1-\cos n\pi}{n^2\pi} = \frac{1-(-1)^n}{n^2\pi}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx = \frac{1}{\pi}\int_0^{\pi}(\pi - x)\sin nx dx = \frac{1}{n}$$

$$f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty}\left(\frac{1-(-1)^n}{n^2\pi}\cos nx + \frac{1}{n}\sin nx\right)$$

Convergence of a Fourier Series

- Theorem 2 (Conditions for Convergence): Let f and f' be piecewise continuous on the interval $[-L, L]$; that is, let f and f' be continuous except at a finite number of points in the interval and have only finite discontinuities at these points. Then for all x in the interval $(-L, L)$ the FS of f converges to $f(x)$ at a point of continuity. At a point of discontinuity, the FS converges to the average:

$$\frac{f(x^+) + f(x^-)}{2}$$

where $f(x^+)$ and $f(x^-)$ denote the limit of f at x from the right and from the left, respectively.

- The function in Example 4 satisfies the conditions of Theorem 2. Thus for every x in the interval $(-L, L)$, except at $x = 0$, the series will converge to $f(x)$.

At $x = 0$ the function is discontinuous, and so the series will converge to:

$$\frac{f(0^+) + f(0^-)}{2} = \frac{\pi + 0}{2} = \frac{\pi}{2}$$

at $x = \pi/2$ the series converge to $f(\pi/2) = \pi/2$.

$$\frac{\pi}{2} = \frac{\pi}{4} + \sum_{n=1}^{\infty}\left(\frac{1-(-1)^n}{n^2\pi}\cos n\frac{\pi}{2} + \frac{1}{n}\sin n\frac{\pi}{2}\right)$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

at $x = 0$ the series converge to $\pi/2$.

$$\frac{\pi}{2} = \frac{\pi}{4} + \sum_{n=1}^{\infty}\left(\frac{1-(-1)^n}{n^2\pi}\cos 0 + \frac{1}{n}\sin 0\right)$$

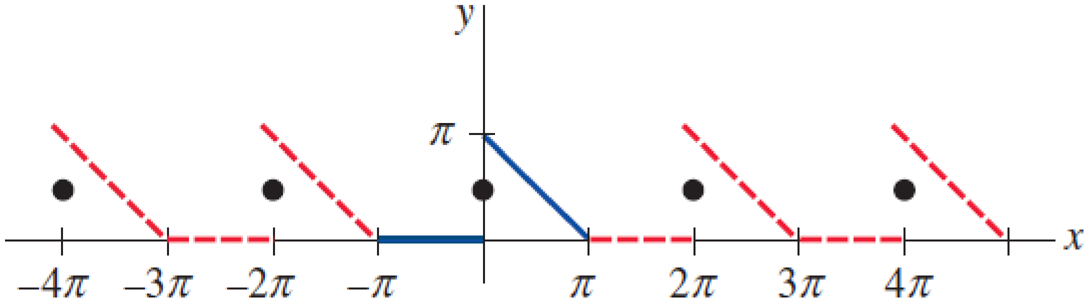
$$\frac{\pi}{4} = \frac{2}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right) \Rightarrow \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

Periodic Extension

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{L} x + b_n \sin \frac{n\pi}{L} x \right)$$

- The right hand side of equation above is $2L$-periodic; indeed, $2L$ is the fundamental period of the sum.
- We conclude that a Fourier series not only represents the function on the interval $(-L, L)$ but also gives the periodic extension of f outside this interval.
- We may assume from the outset that the given function is periodic with period $T = 2L$; that is, $f(x + T) = f(x)$.
- When f is piecewise continuous and the right-and left-hand derivatives exist at $x = -L$ and $x = L$, respectively, then the series converges to $[f(L^-) + f(-L^+)]/2$ at these endpoints and to this value extended periodically to $\pm 3L$, $\pm 5L$, $\pm 7L$, and so on.

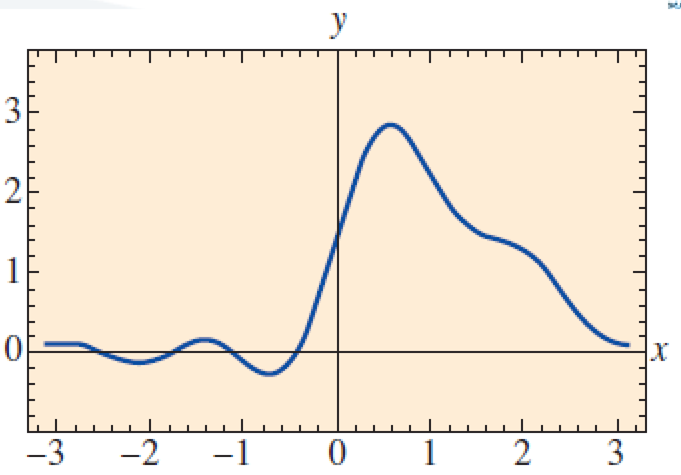
- Fourier series in example 4 converges to the periodic extension of $f(x)$ on the entire x-axis. At 0, $\pm 2\pi$, $\pm 4\pi$, ..., and at $\pm\pi$, $\pm 3\pi$, $\pm 5\pi$, ..., the series converges to the values: $$\frac{f(0^+)+f(0^-)}{2}=\frac{\pi}{2} \quad \text{and} \quad \frac{f(\pi^+)+f(\pi^-)}{2}=0$$ respectively. The solid dots in figure below represent the value $\pi/2$.



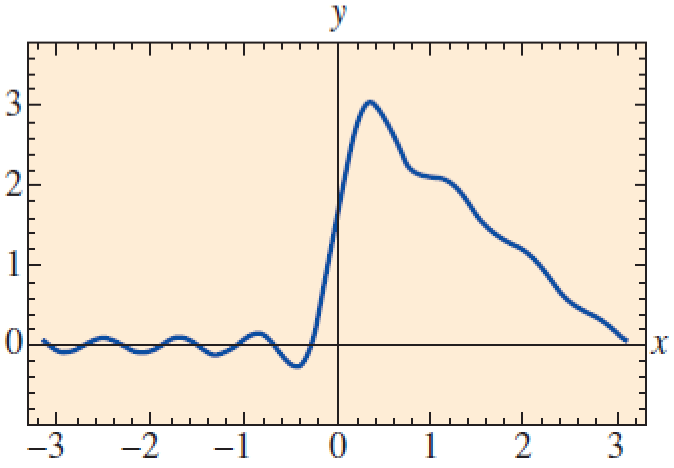
Sequence of Partial Sums

- It is interesting to see how the sequence of partial sums $\{S_N(x)\}$ of a Fourier series approximates a function. In example 4, the first three partial sums are:

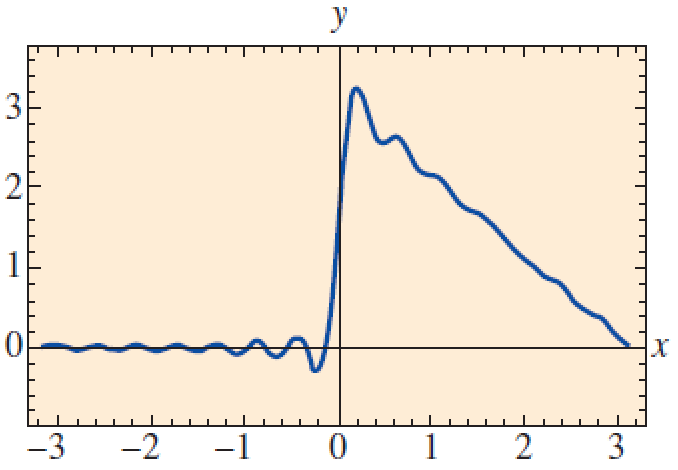
$$S_0(x)=\frac{\pi}{4}, \quad S_1(x)=\frac{\pi}{4}+\frac{2}{\pi}\cos x+\sin x, \quad S_2(x)=\frac{\pi}{4}+\frac{2}{\pi}\cos x+\sin x+\frac{1}{2}\sin 2x$$



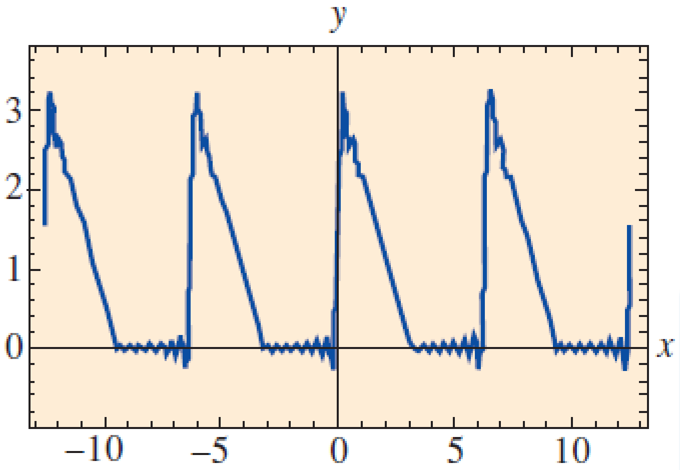
(a) $S_5(x)$ on $(-\pi, \pi)$



(b) $S_8(x)$ on $(-\pi, \pi)$



(c) $S_{15}(x)$ on $(-\pi, \pi)$



(d) $S_{15}(x)$ on $(-4\pi, 4\pi)$

Fourier Cosine and Sine Series

- **Definition:** Fourier Cosine and Sine Series

  (i) The FS of an even function on the interval $(-L, L)$ is the cosine series:

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{L}x$$

where $a_0 = \frac{1}{L}\int_0^L f(x)dx,\qquad a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi}{L}x\,dx$

  (ii) The FS of an odd function on the interval $(-L, L)$ is the sine series:

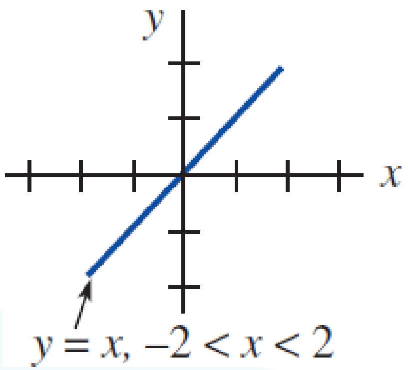
$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{L}x$$

where $b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx$

- **Example 5:** Expansion in a Sine Series

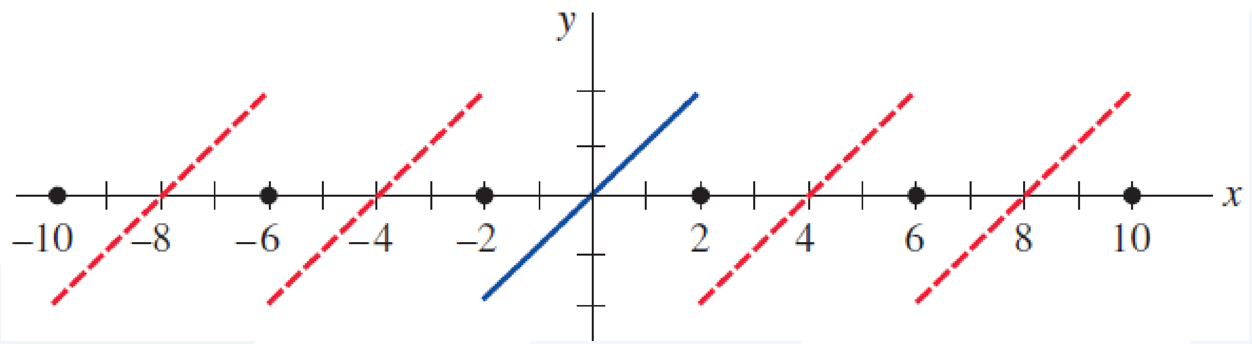
  Expand $f(x) = x$, $-2 < x < 2$, in a Fourier series

The given function is odd on the interval (–2, 2), and so we expand f in a sine series. With the identification $2L = 4$, we have $L = 2$.

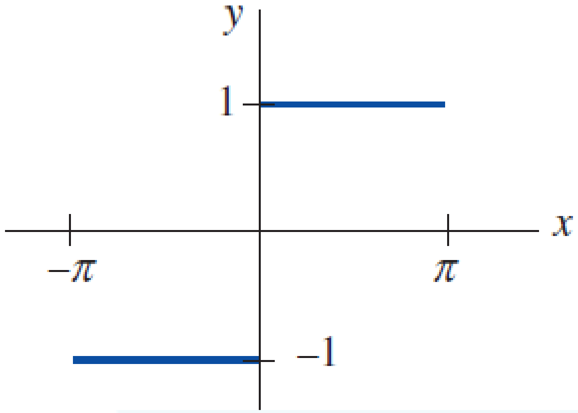


$$b_n = \int_0^2 x \sin\frac{n\pi}{L}x\,dx = \frac{4(-1)^{n+1}}{n\pi} \Rightarrow f(x) = \sum_{n=1}^{\infty}\frac{4(-1)^{n+1}}{n\pi}\sin\frac{n\pi}{2}x$$

The series converges to the function on (–2, 2) and the periodic extension (of period 4).



- Example 6: Expansion in a Sine Series



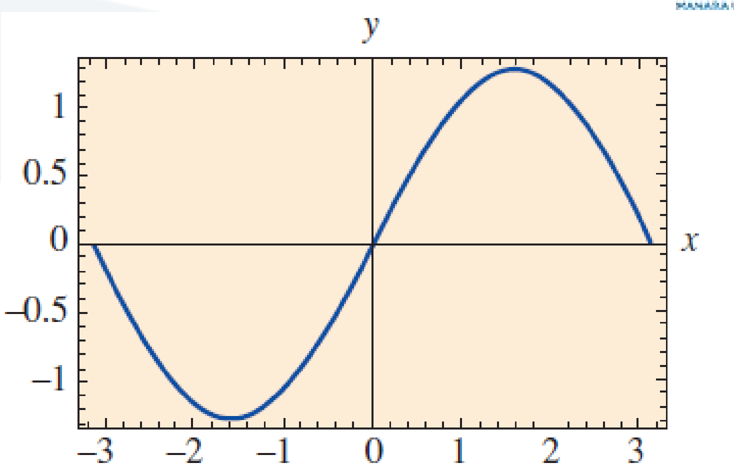
Expand $f(x)=\begin{cases}-1, & -\pi<x<0\\ 1 & 0\le x<\pi\end{cases}$ in a Fourier series

The given function is odd on the interval $(-\pi, \pi)$, and so we expand f in a sine series.

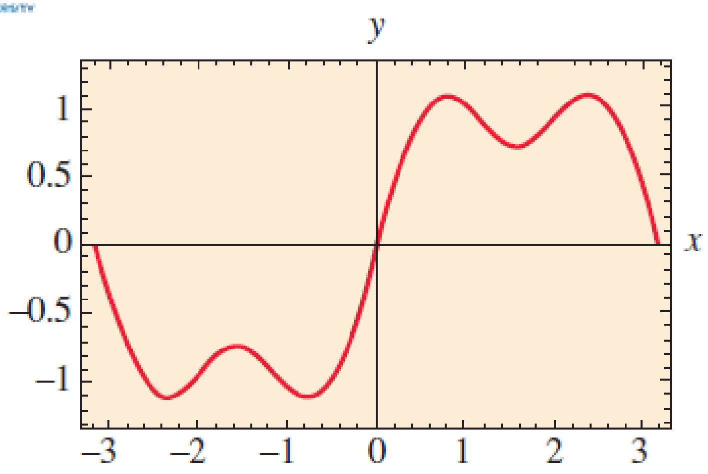
$$b_n=\frac{2}{\pi}\int_0^{\pi}\sin\frac{n\pi}{L}x\,dx=\frac{2}{\pi}\frac{1-(-1)^n}{n}\Rightarrow f(x)=\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n}\sin nx$$

Gibbs Phenomenon

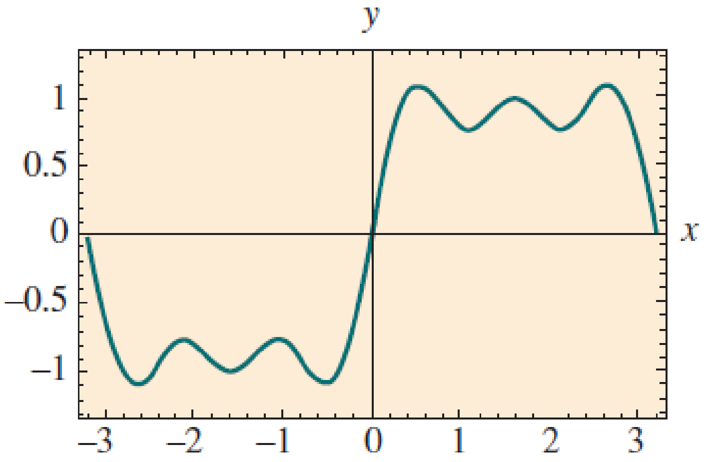
- The partial sums $\{S_N(x)\}$ of a Fourier series shows oscillations (spikes) near the points of discontinuity of $f(x)$. these oscillations don’t disappear as the value of N gets larger. With increasing N, they are shifted closer to the points of discontinuity of $f(x)$. This behavior is known as the Gibbs phenomenon.



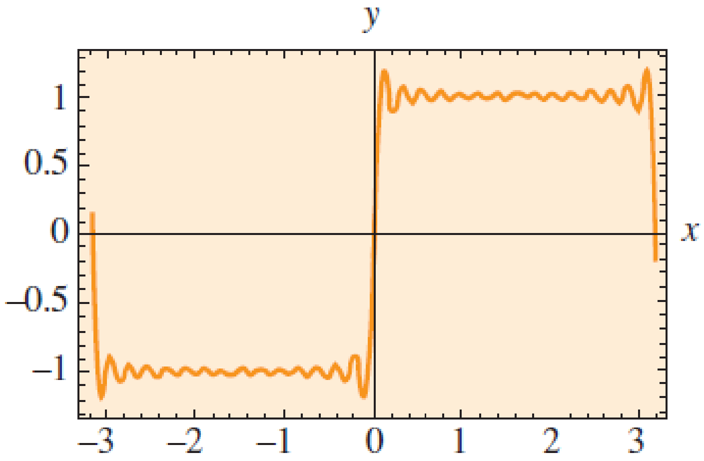
(a) $S_1(x)$



(b) $S_2(x)$



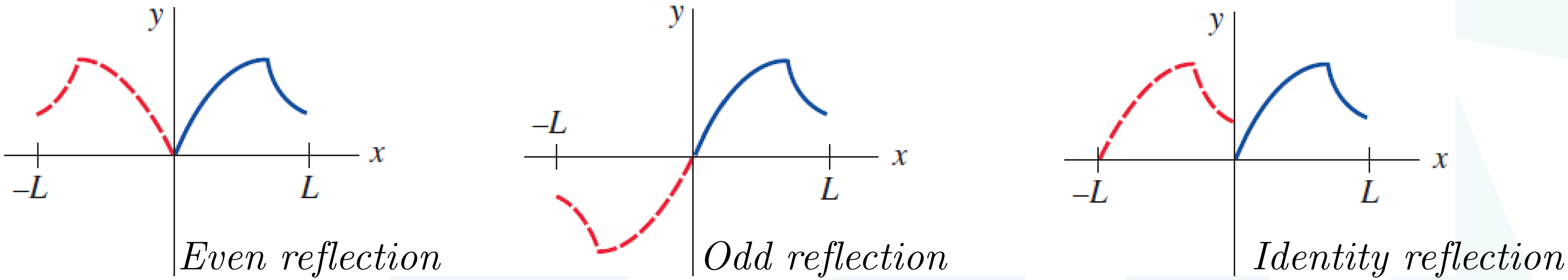
(c) $S_3(x)$



(d) $S_{15}(x)$

Half-Range Expansions

- When we are interested in representing a function that is defined on an interval $(0, L)$ by a trigonometric series. This can be done in many different ways by supplying an arbitrary definition of the function on the interval $(-L, 0)$. Three most important cases:



Even reflection: The function is even on the interval $(-L, L)$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{L}x \qquad a_0 = \frac{1}{L}\int_0^L f(x)dx, \quad a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi}{L}xdx$$

Odd reflection: The function is odd on the interval $(-L, L)$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{L}x \qquad b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx$$

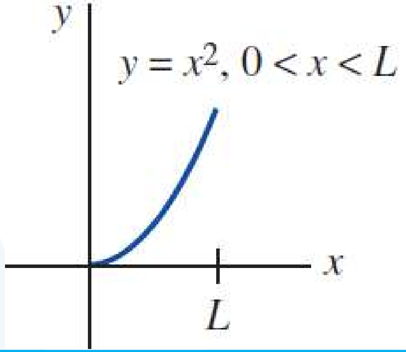
Identity reflection: The function values on the interval $(-L, 0)$ are the same as the values on $(0, L)$. We identify $L \to L/2$ and The resulting Fourier series will give the periodic extension of the function with period L.

- Example 7: Half Range Expansion

Expand $f(x) = x^2$, $0 < x < L$, (a) in a cosine series, (b) in a sine series, (c) in a FS.

(a) $a_0 = \frac{1}{L}\int_0^L x^2 dx = \frac{1}{3}L^2$, $\quad a_n = \frac{2}{L}\int_0^L x^2 \cos\frac{n\pi}{L}x\,dx = \frac{4L^2(-1)^n}{n^2\pi^2}$

$$f(x) = \frac{1}{3}L^2 + \sum_{n=1}^{\infty}\frac{4L^2(-1)^n}{n^2\pi^2}\cos\frac{n\pi}{L}x$$



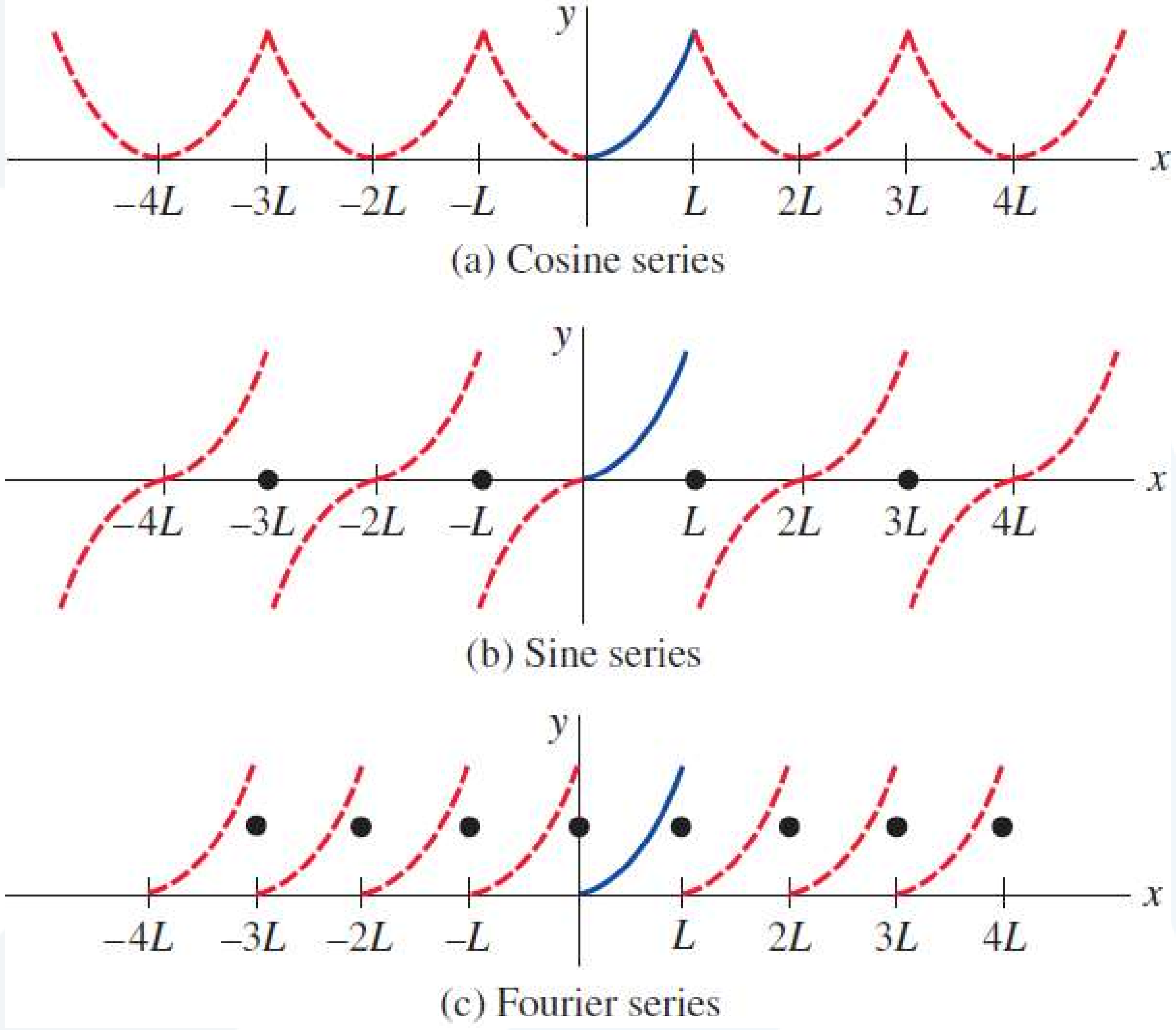
(b) $b_n = \frac{2}{L}\int_0^L x^2 \sin\frac{n\pi}{L}x\,dx = \frac{2L^2(-1)^{n+1}}{n\pi} + \frac{4L^2}{n^3\pi^3}[(-1)^n - 1]$

$$f(x) = \frac{2L^2}{\pi}\sum_{n=1}^{\infty}\left\{\frac{(-1)^{n+1}}{n} + \frac{2}{n^3\pi^3}[(-1)^n - 1]\right\}\sin\frac{n\pi}{L}x$$

(c) $a_0 = \frac{1}{L}\int_0^L x^2\,dx = \frac{1}{3}L^2, \quad a_n = \frac{2}{L}\int_0^L x^2\cos\frac{2n\pi}{L}x\,dx = \frac{L^2}{n^2\pi^2}$

$$b_n = \frac{2}{L}\int_0^L x^2\sin\frac{2n\pi}{L}x\,dx = \frac{-L^2}{n\pi}$$

$$f(x) = \frac{1}{3}L^2 + \frac{L^2}{\pi}\left\{\sum_{n=1}^{\infty}\frac{L^2}{n^2\pi}\cos\frac{2n\pi}{L}x - \frac{1}{n}\sin\frac{2n\pi}{L}x\right\}$$



(a) Cosine series

(b) Sine series

(c) Fourier series

Parseval formula

For a full Fourier Series on $[-L, L]$: $f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{L}x + b_n \sin\frac{n\pi}{L}x\right)$

$$\frac{1}{L}\int_{-L}^{L}[f(x)]^2\,dx = 2a_0^2 + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)$$

- Example 8: Expansion in a Sine Series

  The Fourier series for the function $f(x) = x\ (-\pi < x < \pi)$: $f(x) = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} x^2 dx = \sum_{n=1}^{\infty}\left(\frac{2(-1)^{n+1}}{n}\right)^2 \Rightarrow \frac{2\pi^2}{3} = \sum_{n=1}^{\infty}\frac{4}{n^2}$$

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$$

Differentiation of Fourier Series

- Theorem 3 (Differentiation of FS): Let f be a continuous function on the interval $[-L, L]$ such that $f(-L) = f(L)$, and suppose also that f' is piecewise continuous on the interval $(-L, L)$. Then for any x strictly inside the interval at which $f''(x)$ exists, the derivative of $f(x)$ can be obtained by term-by-term differentiation of the FS representation of f. So, if f has the FS representation:

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{L}x + b_n \sin\frac{n\pi}{L}x\right)$$

Then:

$$f'(x) = \frac{\pi}{L}\sum_{n=1}^{\infty}\left(-na_n \sin\frac{n\pi}{L}x + nb_n \cos\frac{n\pi}{L}x\right) \qquad \text{for } -L < x < L$$

except for points at where $f'(x)$ and $f''(x)$ are not defined.

- Note: Not all Fourier series are differentiable.

- Example 9: a Series is not differentiable

 The Fourier series for the function $f(x) = x$ $(-\pi < x < \pi)$ converges to $f(x)$ at each point in the interval $-\pi < x < \pi$:

$$f(x) = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx$$

 But the differentiated series $2\sum_{n=1}^{\infty}(-1)^{n+1}\cos nx$ does not converge since its nth term fails to approach zero as n tends to infinity.

- Example 10: a Series is differentiable

 The Fourier series for the function $f(x) = \cosh ax$ $(-\pi \le x \le \pi)$ $a \ne 0$

$$\cosh ax = \frac{\sinh a\pi}{a\pi}\left[1 + 2a^2\sum_{n=1}^{\infty} \frac{(-1)^n}{a^2 + n^2} \cos nx\right]$$

This series converges to $\cosh ax$ on the interval $-\pi \leq x \leq \pi$. The hypothesis of the theorem is satisfied when, it follows that:

$$\sinh ax = \frac{2\sinh a\pi}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{a^2+n^2}\sin nx \qquad -\pi < x < \pi$$

- Note: The equation above is valid when the condition $a = 0$ is dropped.

Integration of Fourier Series

- Theorem 4 (Integration of FS): A Fourier series of a piecewise smooth function f can always be integrated term by term and the result is a convergent infinite series that always converges to the integral of f on $[-L, L]$:

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi}{L}x + b_n\sin\frac{n\pi}{L}x\right)$$

The equation

$$\int_{-L}^{x} f(u)du = a_0(x+L) + \frac{L}{\pi}\sum_{n=1}^{\infty}\left[\frac{a_n}{n}\sin\frac{n\pi}{L}x - \frac{b_n}{n}\left(\cos\frac{n\pi}{L}x + (-1)^{n+1}\right)\right]$$

is valid when $-L \le x \le L$.

- Example 11: Integration of Fourier Series

Use the FS representation of the function $f(x) = \begin{cases} -1, & -\pi < x < 0 \\ 1, & 0 < x < \pi \end{cases}$ to find a FS find a FS representation of $F(x) = \int_{-\pi}^{x} f(t)dt$ in the interval $-\pi < x < \pi$

$$f(x) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin(2n-1)x$$

$$F(x) = \frac{4}{\pi}\sum_{n=1}^{\infty}\int_{-\pi}^{x}\frac{\sin(2n-1)t}{2n-1}dt = -\frac{4}{\pi}\left[\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2} - \sum_{n=1}^{\infty}\frac{\cos(2n-1)\pi}{(2n-1)^2}\right]$$

$$F(x) = -\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$$

applying the Parseval formula to the Fourier series representation of $f(x)$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} 1\cdot dx = \sum_{n=1}^{\infty}\left(\frac{4}{\pi}\frac{1}{2n-1}\right)^2 \Rightarrow 2 = \frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} \Rightarrow \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$$

$$F(x) = -\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2} - \frac{4}{\pi}\frac{\pi^2}{8} = -\frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2}$$

$$F(x) = \int_{-\pi}^{x} f(t)dt = \begin{cases} \int_{-\pi}^{x} -1\,dt = -(x+\pi), & -\pi < x < 0 \\ \int_{-\pi}^{0} -1\,dt + \int_{0}^{x} 1\,dt = x - \pi, & 0 < x < \pi \end{cases}$$

$$F(x)=\int_{-\pi}^{x} f(t)dt = |x|-\pi = -\frac{\pi}{2}-\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2}$$

$$|x| = \frac{\pi}{2}-\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2} \qquad -\pi \le x \le \pi$$

3. Complex Fourier Series

- In certain applications, for example, the analysis of periodic signals in electrical engineering, it is actually more convenient to represent a function f in an infinite series of complex-valued functions of a real variable x such as the exponential functions e^{inx}, $n=0, 1, 2, \ldots$, and where i is the imaginary unit.

Complex Fourier Series

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{L}x + b_n \sin\frac{n\pi}{L}x\right)$$

$$\cos\frac{n\pi}{L}x=\frac{e^{in\pi x/L}+e^{-in\pi x/L}}{2},\qquad \sin\frac{n\pi}{L}x=\frac{e^{in\pi x/L}-e^{-in\pi x/L}}{2i}$$

$$f(x)=a_0+\sum_{n=1}^{\infty}\left(a_n\frac{e^{in\pi x/L}+e^{-in\pi x/L}}{2}+b_k\frac{e^{in\pi x/L}-e^{-in\pi x/L}}{2i}\right)$$

$$f(x)=a_0+\sum_{n=1}^{\infty}\left(\frac{1}{2}(a_n-ib_n)e^{in\pi x/L}+\frac{1}{2}(a_n+ib_n)e^{-in\pi x/L}\right)$$

$$f(x)=c_0+\sum_{n=1}^{\infty}c_ne^{in\pi x/L}+\sum_{n=1}^{\infty}c_{-n}e^{-in\pi x/L}$$

where

$$c_0=a_0=\frac{1}{2L}\int_{-L}^{L}f(x)dx$$

$$c_n=\tfrac{1}{2}(a_n-ib_n)=\tfrac{1}{2L}\int_{-L}^{L}f(x)e^{-in\pi x/L}dx,\qquad c_{-n}=\tfrac{1}{2}(a_n+ib_n)=\tfrac{1}{2L}\int_{-L}^{L}f(x)e^{in\pi x/L}dx$$

- Definition: The complex Fourier series (CFS) of a function f defined on the interval $(-L, L)$ is given by:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L}$$

where

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x) e^{-in\pi x/L} dx, \quad n = 0, \pm 1, \pm 2, \ldots$$

- Note: When the function f is real, c_n and c_{-n} are complex conjugates: $c_{-n} = \overline{c_n}$
- Note: The functions $e^{im\pi x/L}$ and $e^{-in\pi x/L}$ are orthogonal over the interval $[-L, L]$.

$$\int_{-L}^{L} e^{im\pi x/L} e^{-in\pi x/L} dx = \begin{cases} 0, & m \neq n \\ 2L, & m = n \end{cases}$$

- If f satisfies the hypotheses of Theorem 2, a CFS converges to $f(x)$ at a point of continuity and to the average $\frac{1}{2}[f(x^+) + f(x^-)]$ at a point of discontinuity.

- Example 12: Complex Fourier Series

Expand $f(x) = e^{-x}$, $-\pi < x < \pi$, (a) in a complex Fourier series.

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-x}e^{-inx}dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-(in+1)x}dx = \frac{1}{2\pi(in+1)}\left[e^{-(in+1)\pi} - e^{(in+1)\pi}\right]$$

$$c_n = (-1)^n \frac{(e^{\pi} - e^{-\pi})}{2(in+1)\pi} = (-1)^n \frac{\sinh \pi}{\pi}\frac{1-ni}{n^2+1}$$

$$f(x) = \frac{\sinh \pi}{\pi}\sum_{n=-\infty}^{\infty}(-1)^n \frac{1-ni}{n^2+1}e^{inx}$$

The series converges to the 2π-periodic extension of f.

Fundamental Frequency

- The Fourier series define a periodic function and the fundamental period of that function (that is, the periodic extension of f) is $T = 2L$.

$$a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega x + b_k \sin n\omega x) \quad \text{and} \quad \sum_{n=-\infty}^{\infty} c_n e^{in\omega x}$$

where number $\omega = 2\pi/T$ is called the fundamental angular frequency.

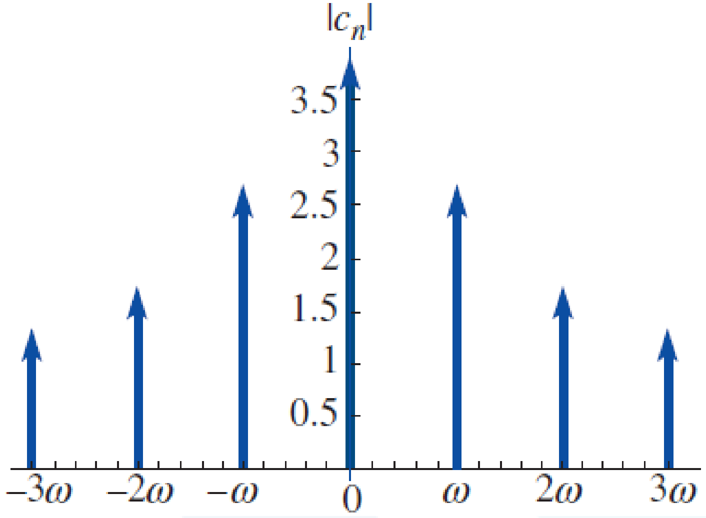
Frequency Spectrum

- If f is periodic and has fundamental period T, the plot of the points $(n\omega, |c_n|)$, where ω is the fundamental angular frequency and the c_n are the Fourier coefficients, is called the frequency spectrum of f.

- Example 13: Frequency Spectrum

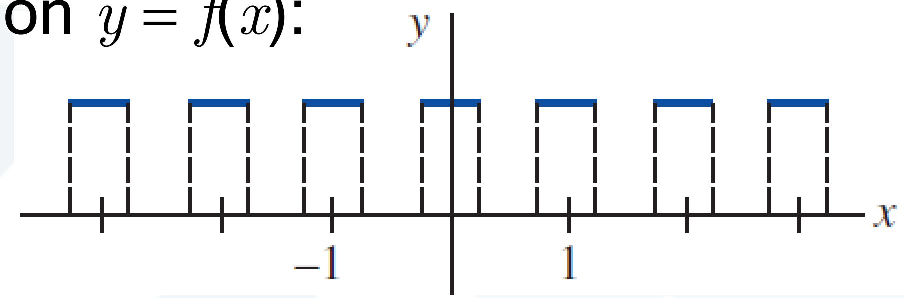
  From example 12:

$$|c_n| = \frac{\sinh \pi}{\pi} \frac{1}{\sqrt{n^2+1}}$$



- Example 14: Frequency Spectrum

Find the frequency spectrum of the periodic square wave or periodic pulse. The wave is the periodic extension of the function $y = f(x)$:

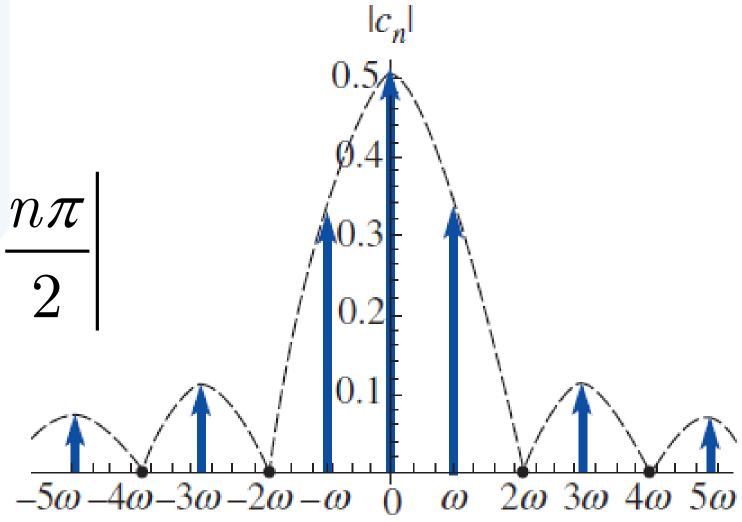


$$f(x) = \begin{cases} 0, & -\frac{1}{2} < x < -\frac{1}{4} \\ 1, & -\frac{1}{4} < x < \frac{1}{4} \\ 0, & \frac{1}{4} < x < \frac{1}{2} \end{cases}$$

$$c_n = \int_{-1/4}^{1/4} 1 \cdot e^{-2in\pi x} dx = -\frac{e^{-2in\pi x}}{2in\pi}\Bigg]_{-1/4}^{1/4}$$

$$= \frac{1}{n\pi}\frac{e^{in\pi/2} - e^{-in\pi/2}}{2i} = \frac{1}{n\pi}\sin\frac{n\pi}{2}$$

$$|c_n| = \frac{1}{n\pi}\left|\sin\frac{n\pi}{2}\right|$$



4. Sturm - Liouville Problem

Linear equations	General solutions
$y' + \alpha y = 0,$	$y = c_1 e^{-\alpha x}$
$y'' + \alpha^2 y = 0, \quad \alpha > 0$	$y = c_1 \cos \alpha x + c_2 \sin \alpha x$
$y'' - \alpha^2 y = 0, \quad \alpha > 0$	$\begin{cases} y = c_1 e^{-\alpha x} + c_2 e^{\alpha x}, & \text{or} \\ y = c_1 \cosh \alpha x + c_2 \sinh \alpha x \end{cases}$

Cauchy-Euler equation	General solutions, $x > 0$
$x^2 y'' + xy' - \alpha^2 y = 0, \quad \alpha \geq 0$	$\begin{cases} y = c_1 e^{-\alpha} + c_2 e^{\alpha}, & \alpha \neq 0 \\ y = c_1 + c_2 \ln x, & \alpha \neq 0 \end{cases}$

Parametric Bessel equation ($\nu > 0$)	General solutions, $x > 0$
$x^2 y'' + y' - \alpha^2 xy = 0$	$y = c_1 J_0(\alpha x) + c_2 Y_0(\alpha x)$

- Can we replace the trigonometric system by other orthogonal systems? The answer is “yes” and will lead to generalized Fourier series, including the Fourier-Legendre series and the Fourier-Bessel series. To prepare for this generalization, we first have to introduce the Sturm - Liouville Problem.

- Let p, q, r, and r' be real-valued functions continuous on an interval $[a, b]$, and let $r(x) > 0$ and $p(x) > 0$ for every x in the interval. Then

$$\textit{Solve:} \qquad [p(x)y']' + [q(x) + \lambda r(x)]y = 0 \qquad (1)$$

$$\textit{Subject to:} \qquad k_1 y + k_2 y_r = 0 \quad \text{at } x = a \qquad (2.a)$$

$$l_1 y + l_2 y_r = 0 \quad \text{at } x = b \qquad (2.b)$$

λ is a parameter, and k_1, k_2, l_1, l_2, are real constants and independent of λ. Furthermore, at least one of each constant in each condition must be different from zero.

- Equation (1) is known as a Sturm-Liouville equation. Together with conditions (2) it is know as the Sturm-Liouville problem. It is an example of a boundary value problem (BVP).

Eigenvalues, Eigenfunctions

- Orthogonal functions arise in the solution of DE. More to the point, an orthogonal set of functions can be generated by solving a two-point boundary-value problem involving a linear second-order DE containing a parameter λ.
- Clearly, $y = 0$ is a solution–the "trivial solution"–of the problem (1), (2) for any λ because (1) is homogeneous. We want to find eigenfunctions $y(x)$, that is, solutions of (1) satisfying (2) without being identically zero.
- We call a number λ for which an eigenfunction exists an eigenvalue of the Sturm-Liouville problem (1), (2).

- Theorem 5 (Properties of the Problem Sturm-Liouville Problems):
  (a) There exist an infinite number of real eigenvalues that can be arranged in increasing order $\lambda_1 < \lambda_2 < \lambda_3 < \ldots < \lambda_n$ … such that $\lambda_n \to \infty$ as $n \to \infty$.
  (b) For each eigenvalue there is only one eigenfunction (except for nonzero constant multiples).
  (c) Eigenfunctions corresponding to different eigenvalues are linearly independent.
  (d) The set of eigenfunctions corresponding to the set of eigenvalues is orthogonal with respect to the weight function $p(x)$ on the interval $[a, b]$.

- Example 15: Nontrivial Solutions of a BVP: Vibrating String
  Solve the homogeneous boundary-value problem: $y'' + \lambda y = 0,\ y(0) = y(L) = 0.$

Case I: For $\lambda = 0$, the solution of the DE $y'' = 0$ is $y = c_1x + c_2$. $y(0) = y(L) = 0$ $\Rightarrow c_1 = c_2 = 0$. Hence, the only solution of the BVP is the trivial solution $y = 0$.

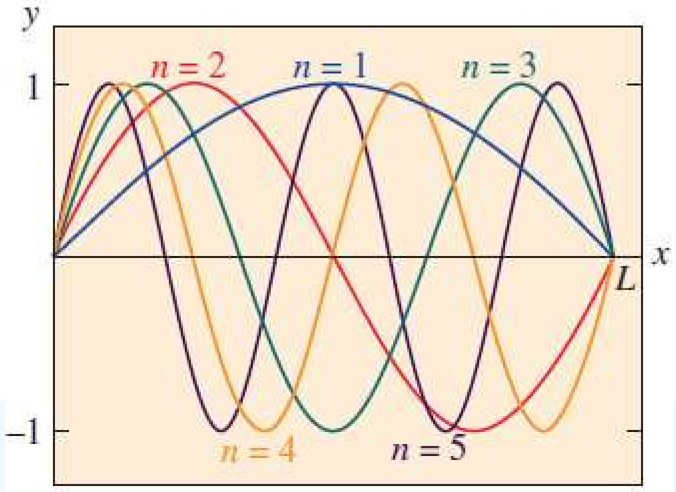
Case II: For $\lambda < 0$, it is convenient to write $\lambda = -\alpha^2$, where $\alpha > 0$. The general solution of $y'' - \alpha^2 y = 0$ is $y = c_1 \cosh \alpha x + c_2 \sinh \alpha x$. $y(0) = y(L) = 0 \Rightarrow$ $c_1 = c_2 = 0$. Once again the only solution of the BVP is the trivial solution $y = 0$.

Case III: For $\lambda > 0$, it is convenient to write $\lambda = \alpha^2$, where $\alpha > 0$. The general solution of $y'' + \alpha^2 y = 0$ is $y = c_1 \cos \alpha x + c_2 \sin \alpha x$. $y(0) = 0 \Rightarrow c_1 = 0$ and so $y = c_2 \sin \alpha x$. $y(L) = 0 \Rightarrow c_2 \sin \alpha L = 0$. If $c_2 = 0$, then necessarily $y = 0$. But this time we can require $c_2 \neq 0$, $\sin \alpha L = 0 \Rightarrow \alpha L = n\pi$ or $\alpha = n\pi / L$ or:

$$\lambda_n = \alpha_n^2 = \left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, 3, \cdots$$

Therefore for any real nonzero c_2, $y_n = c_2 \sin(n\pi x / L)$ is a nontrivial solution of the original problem.

- The numbers $\lambda_n = n^2\pi^2/L^2$, $n = 1, 2, \ldots$ for which the BVP possesses nontrivial solutions are the eigenvalues. The nontrivial solutions that depend on these values of λ_n, $y_n = c_2\sin(n\pi x/L)$ are the eigenfunctions.



- Example 16: Sturm - Liouville Problem

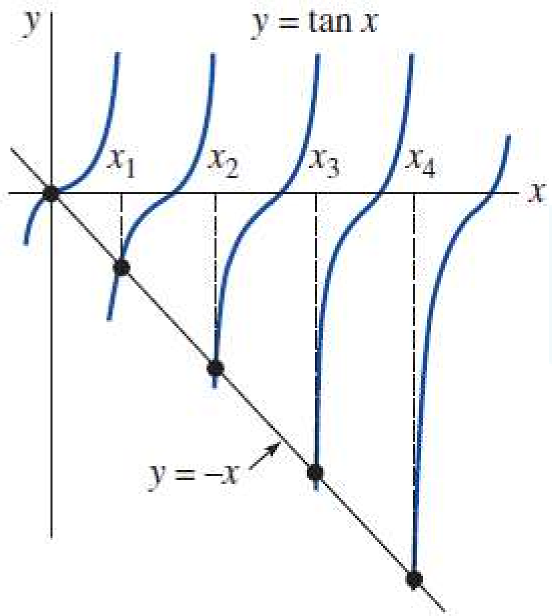
  Solve the homogeneous BVP: $y'' + \lambda y = 0$, $y(0) = 0$, $y(1) + y'(1) = 0$.

  For $\lambda = 0$ and for $\lambda = -\alpha^2 < 0$, where $\alpha > 0$, the BVP possesses only the trivial solution $y = 0$. For $\lambda = \alpha^2 > 0$, where $\alpha > 0$, the general solution of $y'' + \alpha^2 y = 0$ is $y = c_1\cos\alpha x + c_2\sin\alpha x$. $y(0) = 0 \Rightarrow c_1 = 0$ and so $y = c_2\sin\alpha x$. $y(1) + y'(1) = 0 \Rightarrow c_2\sin\alpha + c_2\alpha\cos\alpha = 0$. Choosing $c_2 \neq 0$, the eigenvalues are then $\lambda_n = \alpha_n^2$, $n = 1, 2, 3, \ldots$, are the consecutive positive roots $\alpha_1, \alpha_2, \ldots$ of $\tan\alpha = -\alpha$.

$y(1) + y'(1) = 0 \Rightarrow c_2 \sin\alpha + c_2\alpha\cos\alpha = 0$. Choosing $c_2 \neq 0$, the eigenvalues are then $\lambda_n = \alpha_n^2$, $n = 1, 2, 3, \ldots$, are the consecutive positive roots $\alpha_1, \alpha_2, \ldots$ of $\tan\alpha = -\alpha$.

The eigenfunctions are $\{\sin\alpha_n x\}$, $n = 1, 2, 3, \ldots$.

Thus $\{\sin\alpha_n x\}$, $n = 1, 2, 3, \ldots$ is an orthogonal set with respect to the weight function $r(x) = 1$ on the interval $[0, 1]$.



$$\langle y_m, y_n \rangle = \int_a^b r(x) y_m(x) y_n(x)\,dx = 0, \quad m \neq n$$

- If $p(a) = 0$, then (2a) can be dropped from the problem. If $p(b) = 0$, then (2b) can be dropped. It is then required that y and y' remain bounded at such a point, and the problem is called singular.

If $p(a) = p(b)$, then (2) can be replaced by the "periodic boundary conditions"

$$y(a) = y(b),\ y'(a) = y'(b) \qquad (3)$$

The BVP consisting of the Sturm-Liouville equation (1) and the periodic boundary conditions (3) is called a periodic Sturm-Liouville problem.

Orthogonal Series - Generalized Fourier Series

- Let y_0, y_1, y_2, … be orthogonal with respect to a weight function $r(x)$ on an interval $a \le x \le b$, and let $f(x)$ be a function that can be represented by a convergent series

$$f(x) = \sum_{n=0}^{\infty} = a_n y_n(x) a_0 y_0(x) + a_1 y_1(x) + \cdots$$

$$a_n = \frac{\langle f, y_n \rangle}{\|y_n\|^2} = \frac{1}{\|y_n\|^2} \int_a^b r(x) f(x) y_n(x)\, dx$$

This is called an orthogonal series, orthogonal expansion, or generalized Fourier series.

- Example 16: Fourier-Legendre expansion of the function: $f(x) = \begin{cases} 0, & -1 < x < 0 \\ 1, & 0 < x < 1 \end{cases}$

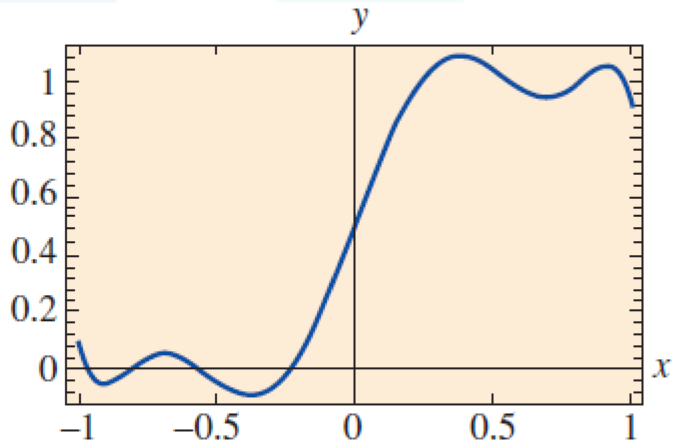
For Legendre's equation $a_n = \frac{2n+1}{2}\int_{-1}^{1} f(x)P_n(x)dx$

$$f(x) = \sum_{n=0}^{\infty} a_n P_n(x) = a_0 + a_1 P_1(x) + \cdots$$

$$a_0 = \frac{1}{2}, \quad a_1 = \frac{3}{4}, \quad a_2 = 0, \quad a_3 = -\frac{7}{16}, \cdots$$

$$f(x) = \frac{1}{2} + \frac{3}{4}x - \frac{7}{16}(5x^3 - 3x) + \cdots$$

$$S_5(x) = \frac{1}{2} + \frac{3}{4}x - \frac{7}{16}(5x^3 - 3x) + \frac{11}{256}(63x^5 - 70x^3 + 15x) - \frac{65}{4096}(429x^7 - 693x^5 + 315x^3 - 35x)$$



5. Boundary-Value Problems in Rectangular Coordinates

Separable Partial Differential Equations PDEs

- A PDE is an equation that contains one or more partial derivatives of an unknown function, call it u, that depends on at least two variables. Usually one of these deals with time t and the remaining with space (spatial variable(s)).
- The most important PDEs are the wave equations that can model the vibrating string and the vibrating membrane, the heat equation for temperature in a bar or wire, and the Laplace equation for electrostatic potentials.
- PDEs are very important in dynamics, elasticity, heat transfer, electromagnetic theory, and quantum mechanics.
- PDEs, like ordinary differential equations (ODEs), are classified as either linear or nonlinear.

- The dependent variable u and its partial derivatives in a linear PDE are only to the first power. We shall be interested in linear second-order PDEs.
- Example 17: Important Second-Order PDEs

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad \textit{One-dimensional wave equation}$$

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad \textit{One-dimensional heat equation}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \textit{Two-dimensional Laplace equation}$$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \qquad \textit{Two-dimensional wave equation}$$

- the general form of a linear second-order PDE is given by:

$$A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial x\partial y}+C\frac{\partial^2 u}{\partial y^2}+D\frac{\partial u}{\partial x}+E\frac{\partial u}{\partial y}+Fu=G$$

where the coefficients A, B, C, ..., G are functions of x and y. When $G(x, y)=0$, the equation is said to be homogeneous; otherwise, it is nonhomogeneous.

- A solution of a linear PDE is a function $u(x, y)$ of two independent variables that possesses all partial derivatives occurring in the equation and that satisfies the equation in some region of the xy-plane.

- It is often difficult to obtain a general solution of a linear second-order PDE. In general, the totality of solutions of a PDE is very large. For example, the functions: $u=x^2-y^2$, $u=e^x\cos y$, $u=\sin x\cosh y$, $u=\ln(x^2+y^2)$ which are entirely different from each other, are solutions of 2D Laplace equation.

- There are several methods that can be tried to find particular solutions of a linear PDE, the one we are interested is called the method of separation of variables. In this method we seek a particular solution of the form of a product of a function of x and a function of y: $u(x,y) = X(x)Y(y)$

$$\frac{\partial u}{\partial x} = X'Y, \quad \frac{\partial u}{\partial y} = XY', \quad \frac{\partial^2 u}{\partial x^2} = X''Y, \quad \frac{\partial^2 u}{\partial y^2} = XY''$$

- Example 18: Separation of Variables

Find product solutions of $\frac{\partial^2 u}{\partial x^2} = 4\frac{\partial u}{\partial y}$

Substituting $u(x,y) = X(x)Y(y)$ into the partial differential equation

$$X''Y = 4XY' \Rightarrow \frac{X''}{4X} = \frac{Y'}{Y}$$

Since the left-hand side of the last equation is independent of y and is equal to the right-hand side, which is independent of x, we conclude that both sides of the equation are independent of x and y.

$$\frac{X''}{4X} = \frac{Y'}{Y} = -\lambda \qquad X'' + 4\lambda X = 0 \quad \text{and} \quad Y' + \lambda Y = 0$$

Case I If $\lambda = 0$, then the two ODEs are: $X'' = 0$ and $Y' = 0$

$$X = c_1 + c_2 x \text{ and } Y = c_1 \Rightarrow u = XY = A_1 + B_1 x$$

Case II If $\lambda = -\alpha^2 < 0$, then the two ODEs are:

$$X'' - 4\alpha^2 X = 0 \text{ and } Y' - \alpha^2 Y = 0$$

$$X = c_4 \cosh 2\alpha x + c_5 \sinh 2\alpha x \text{ and } Y = c_6 e^{\alpha^2 y}$$

$$\Rightarrow u = XY = A_2 e^{\alpha^2 y} \cosh 2\alpha x + B_2 e^{\alpha^2 y} \sinh 2\alpha x$$

Case III If $\lambda = \alpha^2 > 0$, then the two ODEs are:

$$X'' + 4\alpha^2 X = 0 \text{ and } Y' + \alpha^2 Y = 0$$

$$X = c_7 \cos 2\alpha x + c_8 \sin 2\alpha x \text{ and } Y = c_9 e^{-\alpha^2 y}$$

$$\Rightarrow u = XY = A_3 e^{-\alpha^2 y} \cos 2\alpha x + B_2 e^{-\alpha^2 y} \sin 2\alpha x$$

- **Theorem 6 (Superposition principle):** If u_1, u_2, ..., u_k are solutions of a homogeneous linear PDE, then the linear combination $u = c_1 u_1 + c_2 u_2 + \dots + c_k u_k$, where the c_i, $i = 1, 2, \dots, k$, are constants, is also a solution.

- **Definition:** classification of equations

  The linear second-order partial differential equation

$$A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} + D \frac{\partial u}{\partial x} + E \frac{\partial u}{\partial y} + Fu = G$$

where the coefficients A, B, C, ..., G are real constants, is said to be

hyperbolic if $B^2 - 4AC > 0$,

parabolic if $B^2 - 4AC = 0$,

elliptic if $B^2 - 4AC < 0$.

Heat Equation: Solution by Fourier Series

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, \quad t > 0$$

$u(0, t) = 0, \quad u(L, t) = 0, \quad t > 0$ boundary conditions

$u(x, 0) = f(x), \quad 0 < x < L$ initial condition

$$u(x,t) = X(x)T(t) \Rightarrow \frac{X''}{X} = \frac{T'}{c^2 T} = -\lambda$$

$$X'' + \lambda X = 0 \quad \text{and} \quad T' + c^2\lambda T = 0$$

$$u(0,t) = X(0)T(t) = 0 \quad \text{and} \quad u(L,t) = X(L)T(t) = 0$$

$T(t) \neq 0$ for all $t \Rightarrow X(0) = 0$ and $X(L) = 0$

$$X'' + \lambda X = 0, \quad X(0) = 0, \quad X(L) = 0$$

$$X(x) = c_1 + c_2 x, \qquad \lambda = 0$$

$$X(x) = c_1 \cosh \alpha x + c_2 \sinh \alpha x, \quad \lambda = -\alpha^2 < 0$$

$$X(x) = c_1 \cos \alpha x + c_2 \sin \alpha x, \quad \lambda = \alpha^2 > 0$$

- When the boundary conditions $X(0) = 0$ and $X(L) = 0$ are applied to the first and 2nd equations, these solutions yield only $X(x) = 0$, so $u = 0$.
- But when $X(0) = 0$ is applied to the 3rd equation, we find that $c_1 = 0$ and $X(x) = c_2 \sin \alpha x$. The second boundary condition then implies that $X(L) = c_2 \sin \alpha L = 0$.

- To obtain a nontrivial solution, we must have $c_2 \neq 0$ and $\sin \alpha L = 0$. So $\alpha L = n\pi$ or $\alpha = n\pi/L$.
- Hence $X'' + \lambda X = 0$ possesses nontrivial solutions when:

$$\lambda_n = \alpha_n^2 = n^2\pi^2/L^2, \quad n = 1, 2, 3, \ldots$$

- These values of λ are the eigenvalues of the problem; the eigenfunctions are:

$$X_n(x) = c_2 \sin \frac{n\pi}{L} x$$

$$T' + c^2\lambda T = 0 \Rightarrow T_n(t) = c_3 e^{-c^2(n^2\pi^2/L^2)t}$$

$$u_n(x,t) = X_n(x)T_n(t) = c_2 \sin \frac{n\pi}{L} x \, c_3 e^{-c^2(n^2\pi^2/L^2)t} = A_n e^{-c^2(n^2\pi^2/L^2)t} \sin \frac{n\pi}{L} x$$

- Each of the product functions $u_n(x,t)$ is a particular solution of the PDE, and each $u_n(x,t)$ satisfies both boundary conditions as well.

- The solution of the entire problem: by the superposition principle

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} A_n e^{-c^2(n^2\pi^2/L^2)t} \sin\frac{n\pi}{L}x$$

- To satisfy the initial condition, we would have to choose the coefficient A_n in such a manner that:

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi}{L}x$$

- Hence A_n must be the coefficients of the Fourier sine series (half-range expansion of f in a sine series), thus

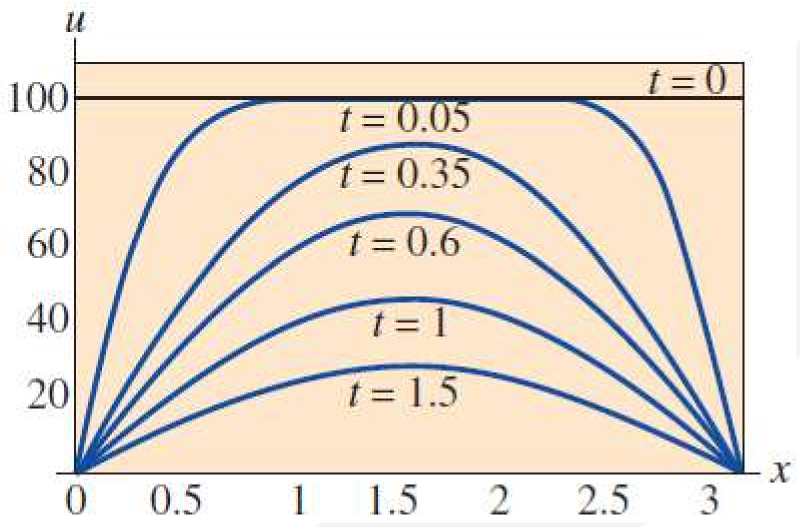
$$A_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx$$

$$u(x,t) = \frac{2}{L}\sum_{n=1}^{\infty}\left(\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx\right)e^{-c^2(n^2\pi^2/L^2)t}\sin\frac{n\pi}{L}x$$

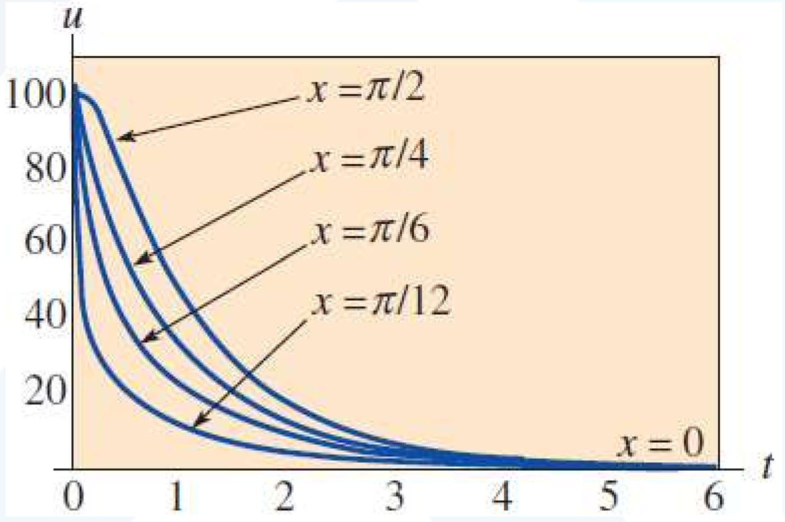
In the special case when the initial temperature is $u(x,0)=100$, $L=\pi$, and $c^2=1$,

$$A_n = \frac{200}{\pi}\left[\frac{1-(-1)^n}{n}\right]$$

$$u(x,t) = \frac{200}{\pi}\sum_{n=1}^{\infty}\left[\frac{1-(-1)^n}{n}\right]e^{-n^2 t}\sin nx$$



u(x, t) graphed as a function of x



u(x, t) graphed as a function of t

6. Fourier transform (FT)

- Fourier series are powerful tools for problems involving functions that are periodic or defined over a finite interval $-L \le x \le L$.
- If a nonperiodic function $f(x)$ is to be represented over an arbitrarily large interval, some generalization of a Fourier series is required.
- Letting $L \to \infty$ in a FS leads to the introduction of a diff. type of representation called a Fourier integral representation, where the function f is defined for all x and need not be periodic. This representation forms the basis of so called FT.

Fourier Transform and Its Inverse

$$F(\omega) = \mathcal{F}\{f(x)\} = \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx$$ Fourier transform

$$f(x) = \mathcal{F}^{-1}\{F(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega) e^{i\omega x} d\omega$$ Inverse Fourier transform

- Theorem 7 (Existence of the Fourier Transform): If $f(x)$ is absolutely integrable on the x-axis and piecewise continuous on every finite interval, then the Fourier transform $F(\omega)$ exists.

- Example 19: Fourier Transform

  Find the Fourier transform of $f(x) = 1$ if $|x| < 1$ and $f(x) = 0$ otherwise.

$$F(\omega) = \int_{-1}^{1} 1 \cdot e^{-i\omega x} dx = \left.\frac{e^{-i\omega x}}{-i\omega}\right|_{-1}^{1} = \frac{1}{-i\omega}(e^{-i\omega} - e^{i\omega}) = 2\frac{\sin\omega}{\omega} = 2\mathrm{sinc}\left(\frac{\omega}{\pi}\right)$$

- Example 20: Fourier Transform

  Find the Fourier transform of $f(x) = e^{-ax}$ if $x > 0$ and $f(x) = 0$ if $x < 0$, $a > 0$

$$F(\omega) = \int_{0}^{\infty} e^{-ax} \cdot e^{-i\omega x} dx = \left.\frac{e^{-(a+i\omega)x}}{-(a+i\omega)}\right|_{0}^{\infty} = \frac{1}{a+i\omega}$$

- Example 21: Fourier Transform for the Delta Dirac Function

$$\delta(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ \text{undefined} & \text{if } x = 0 \end{cases} \quad \text{and} \quad \int_{-\infty}^{\infty} \delta(x)\,dx = 1$$

$$f(x)\delta(x-a) = f(a)\delta(x-a) \qquad \int_{-\infty}^{\infty} f(x)\delta(x-a)\,dx = f(a)$$

$$\mathcal{F}\{\delta(x)\} = \int_{-\infty}^{\infty} \delta(x)\,e^{-i\omega x}dx = e^{-i\omega x}\Big|_{x=0} = 1$$

$$\mathcal{F}\{\delta(x-a)\} = \int_{-\infty}^{\infty} \delta(x-a)\,e^{-i\omega x}dx = e^{-i\omega x}\Big|_{x=a} = e^{-i\omega a}$$

properties of the Fourier transform

- Theorem 8 (Linearity of the Fourier Transform): The FT is a linear operation; that is, for any functions $f(x)$ and $g(x)$ whose FT exist and any constants a and b, the FT of $af + bg$ exists, and $\mathcal{F}\{af(x) + bg(x)\} = a\mathcal{F}\{f(x)\} + b\mathcal{F}\{g(x)\}$

- Theorem 9 (Differentiation in the time domain): Let $f(x)$ be continuous on the x-axis and $f(x) \to 0$ as $|x| \to \infty$. Furthermore, let $f'(x)$ be absolutely integrable on the x-axis. Then

$$\mathcal{F}\{f'(x)\} = i\omega\mathcal{F}\{f(x)\}$$

$$\mathcal{F}\{f^{(n)}(x)\} = (i\omega)^n\mathcal{F}\{f(x)\}$$

- Theorem 10 (Differentiation in the frequency domain): Let $f(x)$ be a continuous and differentiable function with an n times differentiable Fourier transform $F(\omega)$. Then

$$\mathcal{F}\{xf(x)\} = i\frac{d}{d\omega}[F(\omega)]$$

$$\mathcal{F}\{x^n f(x)\} = i^n\frac{d^n}{d\omega^n}[F(\omega)]$$

for all n such that $F^{(n)}(\omega) \to 0$ as $|\omega| \to \infty$

- Example 22: Fourier Transform

Find the Fourier transform of $f(x) = e^{-a^2x^2}$, $a > 0$

The function $f(x)$ is continuous and differentiable for all x and

$$\int_{-\infty}^{\infty} \left|e^{-a^2x^2}\right| dx = \int_{-\infty}^{\infty} e^{-a^2x^2} dx = \frac{1}{a}\int_{-\infty}^{\infty} e^{-u^2} du = \frac{\sqrt{\pi}}{a}$$

absolutely integrable over the interval $(-\infty, \infty)$. $f(x)$ satisfies the differential equation: $f' + 2a^2xf = 0$.

$$\mathcal{F}\{f'(x)\} + 2a^2\mathcal{F}\{xf(x)\} = 0 \Rightarrow 2a^2F'(\omega) + \omega F(\omega) = 0$$

$$\int \frac{F'}{F} d\omega = -\frac{1}{2a^2}\int \omega d\omega \Rightarrow \ln F(\omega) = -\frac{\omega^2}{4a^2} + \ln A \Rightarrow F(\omega) = Ae^{-\frac{\omega^2}{4a^2}}$$

$$F(0) = A = \int_{-\infty}^{\infty} f(x)dx = \frac{\sqrt{\pi}}{a} \Rightarrow F(\omega) = \mathcal{F}\{f(x)\} = \frac{\sqrt{\pi}}{a} e^{-\frac{\omega^2}{4a^2}}$$

Fourier transform of a Gaussian

$f(x) = e^{-\pi x^2}$ Normalized Gaussian function $\int_{-\infty}^{\infty} e^{-\pi x^2} dx = 1$

$$F(\omega) = \mathcal{F}\{f(x)\} = e^{-\frac{\omega^2}{4\pi}}$$

$\omega = 2\pi f \Rightarrow F(f) = e^{-\pi f^2}$ The Gaussian $f(x) = e^{-\pi x^2}$ is its own FT

Convolution property

$$(f * g)(x) = \int_{-\infty}^{\infty} f(t)g(x-t)dt = \int_{-\infty}^{\infty} f(x-t)g(t)dt$$

- Theorem 11 (The convolution theorem for Fourier transforms): Let the functions $f(x)$ and $g(x)$ be piecewise continuous, bounded, and absolutely integrable over $(-\infty, \infty)$ with the respective FT $F(\omega)$ and $G(\omega)$. Then

$$\mathcal{F}\{(f * g)(x)\} = \mathcal{F}\{f(x)\}\mathcal{F}\{g(x)\} = F(\omega)G(\omega)$$

and, conversely,

$$(f * g)(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)G(\omega)e^{i\omega x}d\omega$$

- Example 23: Fourier Transform

  It was shown in Example 19 that the function $f(x) = 1$ if $|x| < 1$ and $f(x) = 0$ otherwise, has the Fourier transform $F(\omega) = 2\text{sinc}(\omega/\pi)$, so by the convolution theorem it follows that $\mathcal{F}\{(f * f)(x)\} = F(\omega)F(\omega) = 4\text{sinc}^2(\omega/\pi)$.
  Confirm this result by calculating $(f * f)(x)$ and finding its Fourier transform.

$$f(t)f(x-t) = \begin{cases} 1, & -1 < t < x+1, \ (-2 < x < 0) \\ 0, & \text{otherwise} \end{cases}$$

$$f(t)f(x-t) = \begin{cases} 1, & x-1 < t < 1, \ (0 < x < 2) \\ 0, & \text{otherwise} \end{cases}$$

$$(f*f)(x)=\begin{cases}\int_{-1}^{x+1} dt = 2+x, & (-2<x<0)\\ \int_{x-1}^{1} dt = 2-x, & (0<x<2)\end{cases} \quad \text{and } (f*f)(x)=0 \text{ otherwise}$$

$$\mathcal{F}\{(f*f)(x)\} = \int_{-2}^{0}(2+x)e^{-i\omega x}dx + \int_{0}^{2}(2-x)e^{-i\omega x}dx$$

$$= 2\frac{1-\cos 2\omega}{\omega^2} = 4\frac{\sin^2\omega}{\omega^2} = 4\operatorname{sinc}^2\left(\frac{\pi}{\omega}\right)$$

Parseval formula

- **Theorem 12 (The Parseval relation for the Fourier transforms):** If $f(x)$ has the Fourier transforms $F(\omega)$, Then

$$\int_{-\infty}^{\infty}|f(x)|^2\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty}|F(\omega)|^2\,d\omega$$

- Example 24: Using the result of Example 19 and the Parseval relation, show that

$$\int_{-\infty}^{\infty} \frac{\sin^2 \omega}{\omega^2} d\omega = \pi$$

The FT of $f(x) = 1$ if $|x| < 1$ and $f(x) = 0$ otherwise is $F(\omega) = 2\frac{\sin\omega}{\omega}$

$$\int_{-1}^{1} 1^2 dx = 2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} 4\frac{\sin^2\omega}{\omega^2} d\omega \Rightarrow \int_{-\infty}^{\infty} \frac{\sin^2\omega}{\omega^2} d\omega = \pi$$

- Theorem 13 (Fourier transforms involving scaling x by a, shifting x by a, and shifting ω by ω_0): If $f(x)$ has the Fourier transforms $F(\omega)$, Then:

$$\mathcal{F}\{f(ax)\} = \frac{1}{a}F(\omega/a),\ a > 0$$

$$\mathcal{F}\{f(x-a)\} = e^{-i\omega a}F(\omega)$$

$$\mathcal{F}\{e^{i\omega_0 x}f(x)\} = F(\omega - \omega_0)$$

Duality property

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega x}d\omega \Rightarrow 2\pi f(x) = \int_{-\infty}^{\infty} F(\omega)e^{i\omega x}d\omega = \int_{-\infty}^{\infty} F(\lambda)e^{i\lambda x}d\lambda$$

$$2\pi f(-\omega) = \int_{-\infty}^{\infty} F(\lambda)e^{-i\omega\lambda}d\lambda = \int_{-\infty}^{\infty} F(x)e^{-i\omega x}dx = \mathcal{F}\{F(x)\}$$

$$\mathcal{F}\{F(x)\} = 2\pi f(-\omega)$$

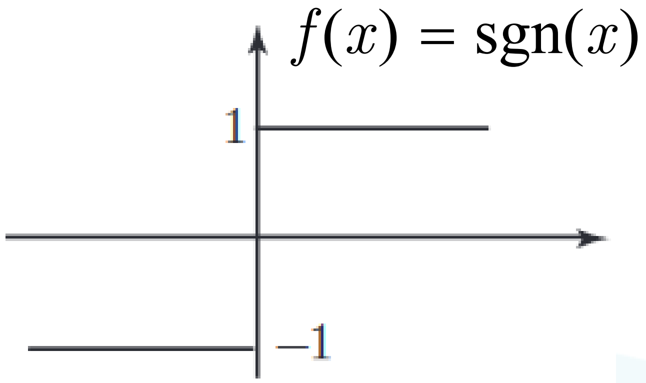
- Example 25: Find the Fourier transform of $f(x) = 1$

$F(\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx = \int_{-\infty}^{\infty} e^{-i\omega x}dx$ could not be evaluated

The function $f(x) = 1$ does not satisfy the existence conditions; it is neither absolute integrable nor square integrable. Its FT does not converge.

$$\mathcal{F}\{\delta(t)\} = 1 \quad \Rightarrow \quad \mathcal{F}\{1\} = 2\pi\delta(-\omega) = 2\pi\delta(\omega)$$

- Example 26: Fourier transform of the signum function

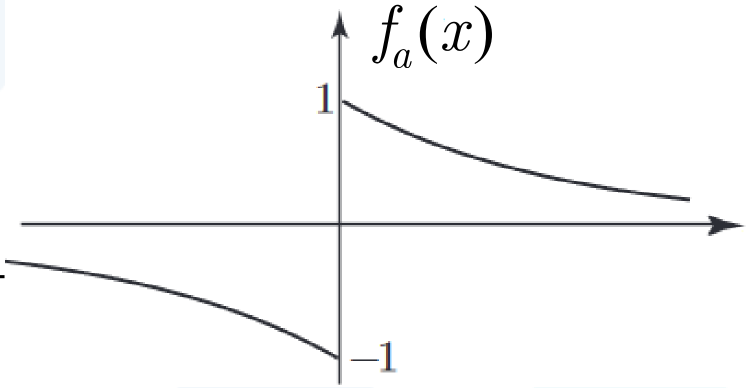


$$f(x) = \mathrm{sgn}(x) = \begin{cases} -1, & x < 0 \\ 1, & x > 0 \end{cases}$$

$$F(\omega) = \int_{-\infty}^{0} (-1)\, e^{-i\omega x} dx + \int_{0}^{\infty} (1)\, e^{-i\omega x} dx$$

The two integrals cannot be evaluated. Instead, we will define an intermediate signal $f_a(x)$ as:

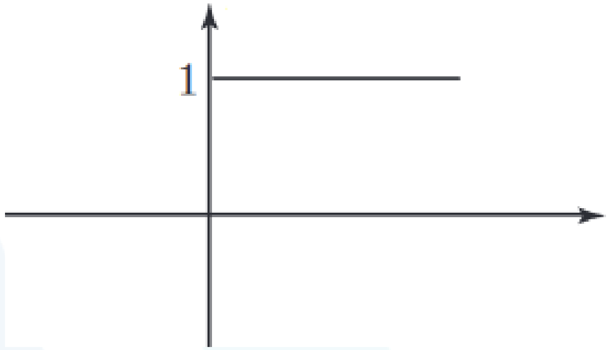
$$f_a(x) = \begin{cases} -e^{ax}, & x < 0 \\ e^{-ax}, & x > 0 \end{cases}, \text{ where } a \geq 0$$



$$F_a(\omega) = \int_{-\infty}^{0} (-e^{at})\, e^{-i\omega t} dt + \int_{0}^{\infty} (e^{-at})\, e^{-i\omega t} dt = -\frac{i2\omega}{a^2 + \omega^2}$$

$$F(\omega) = \mathcal{F}\{\text{sgn}(x)\} = \lim_{a\to 0}\left[-\frac{i2\omega}{a^2+\omega^2}\right] = \frac{2}{i\omega}$$

- Example 27: Fourier transform of the unit step function



$$f(x) = H(x) = \begin{cases} 0, & x < 0 \\ 1, & x > 0 \end{cases}$$

$$\mathcal{F}\{H(x)\} = \int_{-\infty}^{\infty} H(x)\, e^{-i\omega x} dx = \int_0^{\infty} e^{-i\omega x} dx$$ could not be evaluated

$$H(x) = \tfrac{1}{2} + \tfrac{1}{2}\,\text{sgn}(x) \Rightarrow \mathcal{F}\{H(x)\} = \mathcal{F}\{\tfrac{1}{2} + \tfrac{1}{2}\,\text{sgn}(x)\}$$

$$\mathcal{F}\{H(x)\} = \pi\delta(\omega) + \frac{1}{i\omega}$$