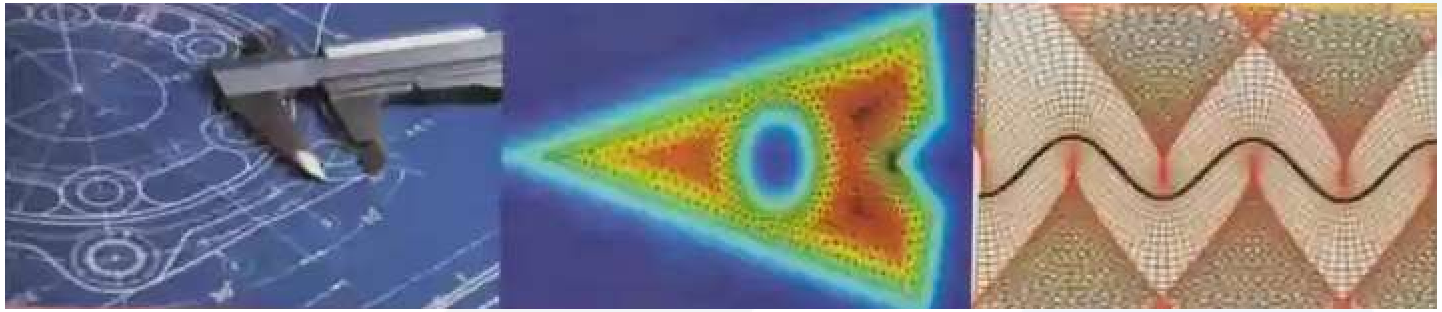
CEDC301: Engineering Mathematics

Lecture Notes 10: Z-Transform



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Chapter 6

Z-Transform

1. Definition of the Z-Transform
2. Properties of the Z-Transform
3. Inverse Z-Transform
4. Solution of Difference Equations
5. Summation of Infinite Series

1. Definition of the Z-Transform

- The z-transform plays the same role in the discrete analysis as Laplace and Fourier transforms in a continuous system.
- Difference equations are formed in the discrete system, and their solution and analysis are carried out by z-transform, similar to the method of Laplace transformation in connection with differential equations.
- Definition: The z-transform of a sequence $f(n)$, is defined as:

$$Z\{f(n)\} = F(z) = \sum_{n=0}^{\infty} f(n) z^{-n}$$ provided that the power series converges

where z, the independent variable of the transform is a complex number.

- Example 1: A simple z-transform example

$$f(n) = \{1.5, 1.2, \ -1.5, 3.6, 5.1\}$$

$$F(z)=\sum_{n=0}^{\infty} f(n)z^{-n} = 1.5+1.2z^{-1}-1.5z^{-2}+3.6z^{-3}+5.1z^{-4}$$

The transform converges at all points in the complex z-plane except of $z=0$.

- Example 2: Z-Transform of an exponential sequence

$$f(n)=a^n,\ n\geq 0$$

$$F(z)=Z\{a^n\}=\sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty}(az^{-1})^n = \frac{1}{1-az^{-1}} = \frac{z}{z-a}$$

converge if: $|az^{-1}|<1 \Rightarrow |z|>|a|$

when $a=1$, we obtain $Z\{1\}=\sum_{n=0}^{\infty} z^{-n} = \frac{z}{z-1}, \quad |z|>1$

- Example 3: Z-Transform of the unit-impulse

$$\delta(n)=\begin{cases}1, & n=0\\ 0, & n\neq 0\end{cases}$$

$$F(z) = \mathcal{Z}\{\delta(n)\} = \sum_{n=0}^{\infty} f(n) z^{-n} = f(0) z^{0} = 1$$

It converges at every point in the z-plane

- Example 4: Z-Transform of the unit-step

$$u(n) = \begin{cases} 0, & n < 0 \\ 1, & n \geq 0 \end{cases}$$

$$F(z) = \mathcal{Z}\{u(n)\} = \sum_{n=0}^{\infty} z^{-n} = \frac{1}{1 - z^{-1}} = \frac{z}{z - 1}$$

converge if: $|z^{-1}| < 1 \Rightarrow |z| > 1$

- Example 5: Z-Transform of a discrete-time pulse

$$f(n) = \begin{cases} 1, & 0 \leq n < N \\ 0, & \text{otherwise} \end{cases}$$

$$F(z) = \sum_{n=0}^{N-1} (1) z^{-n} = \frac{1 - z^{-N}}{1 - z^{-1}}, \quad |z| > 0$$

$$F(z) = \frac{z^N - 1}{z^{N-1}(z-1)}$$ It seems as though $F(z)$ might have a pole at $z = 1$

Zeros: $z_k = e^{i2\pi k/N}, \quad k = 0, \cdots, N-1$

Poles: $z = 1$ and $p_k = 0, \quad k = 1, \cdots, N-1$

The factors $(z - 1)$ in numerator and denominator polynomials cancel each other, therefore there is neither a zero nor a pole at $z = 1$.

- Example 6: Z-Transform of complex exponential $f(n) = e^{inx}$

$$F(z) = \sum_{n=0}^{\infty} e^{inx} z^{-n} = \sum_{n=0}^{\infty} (e^{ix} z^{-1})^n = \frac{1}{1 - e^{ix} z^{-1}} = \frac{z}{z - e^{ix}}, \quad \left| e^{ix} z^{-1} \right| < 1 \Rightarrow |z| > 1$$

2. Properties of the Z-Transform

Linearity

- **Theorem 1 (Linearity of the Z-Transform):** The z-transform is a linear operation; that is, for any sequences $f(n)$ and $g(n)$ whose z-transforms exist and any constants a and b, the z-transform of $af + bg$ exists, and

$$Z\{af(n)+bg(n)\} = aZ\{f(n)\} + bZ\{g(n)\}$$

- **Example 7:** Z-Transform of a cosine and sine

$$\cos(nx) = \tfrac{1}{2}e^{inx}u + \tfrac{1}{2}e^{-inx} \Rightarrow Z\{\cos(nx)\} = \tfrac{1}{2}Z\{e^{inx}\} + \tfrac{1}{2}Z\{e^{-inx}\}$$

$$Z\{\cos(nx)\} = \frac{1/2}{1-e^{ix}z^{-1}} + \frac{1/2}{1-e^{-ix}z^{-1}} = \frac{1-\cos x\, z^{-1}}{1-2\cos x\, z^{-1}+z^{-2}} = \frac{z(z-\cos x)}{z^2-2\cos x\, z+1}$$

$$|z| > 1$$

$$\sin(nx) = \tfrac{1}{2i}e^{inx}u - \tfrac{1}{2i}e^{-inx} \Rightarrow Z\{\cos(nx)\} = \tfrac{1}{2i}Z\{e^{inx}\} - \tfrac{1}{2i}Z\{e^{-inx}\}$$

$$\mathcal{Z}\{\sin(nx)\} = \frac{1/2i}{1-e^{ix}z^{-1}} - \frac{1/2i}{1-e^{-ix}z^{-1}} = \frac{\sin x\, z^{-1}}{1-2\cos x\, z^{-1}+z^{-2}} = \frac{z\sin x}{z^2-2\cos x\, z+1}$$

$$|z|>1$$

- Example 8: Z-Transform of a cosine

  Find the Z-transform of $\cos\left(\frac{n\pi}{2}+\frac{\pi}{4}\right)$

$$\mathcal{Z}\left\{\cos\left(\frac{n\pi}{2}+\frac{\pi}{4}\right)\right\} = \mathcal{Z}\left\{\cos\left(\frac{n\pi}{2}\right)\cos\left(\frac{\pi}{4}\right) - \sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{\pi}{4}\right)\right\}$$

$$= \frac{1}{\sqrt{2}}\left[\frac{z(z-\cos\frac{\pi}{2})}{z^2-2\cos\frac{\pi}{2}z+1} - \frac{z\sin\frac{\pi}{2}}{z^2-2\cos\frac{\pi}{2}z+1}\right]$$

$$= \frac{1}{\sqrt{2}}\left[\frac{z^2}{z^2+1} - \frac{z}{z^2+1}\right] = \frac{z(z-1)}{\sqrt{2}(z^2+1)}$$

Shifting

- The effect of shifting depends upon whether it is to the right or to the left. For the sequence $f(n-2)$, no values from the sequence $f(n)$ are lost; thus, we anticipate that the z-transform of $f(n-2)$ only involves $F(z)$.
- For the sequence $f(n+2)$, the first two values of $f(n)$ are lost, and we anticipate that the z-transform of $f(n+2)$ cannot be expressed solely in terms of $F(z)$ but must include those two lost pieces of information.

n	$f(n)$	$f(n-2)$	$f(n+2)$
0	1	0	4
1	2	0	8
2	4	1	16
3	8	2	64
$\vdots$	$\vdots$	$\vdots$	$\vdots$

- **Theorem 2 (Time Shifting):** If $Z\{f(n)\} = F(z)$ and $m \geq 0$, then

$$Z\{f(n-m)\} = z^{-m}F(z), \qquad Z\{f(n+m)\} = z^{m}\left[F(z) - \sum_{r=0}^{m-1} f(r)z^{-r}\right]$$

$$Z\{f(n+1)\} = z\{F(z) - f(0)\},$$
$$Z\{f(n+2)\} = z^2\{F(z) - f(0)\} - zf(1),$$
$$Z\{f(n+3)\} = z^3\{F(z) - f(0)\} - z^2 f(1) - zf(2)$$

- **Example 9:** Shifting to left

  Find $Z\{4^{n+3}\}$

$$Z\{4^n\} = \frac{z}{z-4} \Rightarrow Z\{4^{n+3}\} = z^3\{\frac{z}{z-4} - 4^0\} - z^2 4^1 - z4^2$$

$$Z\{4^{n+3}\} = z^3\{\frac{z}{z-4} - 4^0\} - z^2 4^1 - z4^2 = \frac{z^4}{z-4} - z^3 - 4z^2 - 16z = \frac{64z}{z-4}, \quad |z| > 4$$

Multiplication by an exponential

- **Theorem 3 (Multiplication by an exponential):** If $Z\{f(n)\} = F(z)$, then

$$Z\{a^n f(n)\} = F(a^{-1}z), \quad |z| > |a|$$

- Example 10: Multiplication by an exponential

  Find $\mathcal{Z}\{a^n\}$, $n \geq 0$

$$F(z) = \mathcal{Z}\{u(n)\} = \frac{z}{z-1}$$

$$\mathcal{Z}\{a^n u(n)\} = F(a^{-1}z) = \frac{a^{-1}z}{a^{-1}z-1} = \frac{z}{z-a}, \quad |z| > |a|$$

Derivative of z-transforms

- Theorem 4 (Derivative of z-transforms): If $\mathcal{Z}\{f(n)\} = F(z)$, then

$$\mathcal{Z}\{nf(n)\} = -z\frac{d}{dz}F(z)$$

  More generally,

$$\mathcal{Z}\{n^k f(n)\} = -z\frac{d}{dz}\mathcal{Z}\{n^{k-1}f(n)\}, \quad k = 1, 2, \ldots$$

$$\mathcal{Z}\{n^k\} = -z\frac{d}{dz}\mathcal{Z}\{n^{k-1}\}$$

- Example 11: Derivative of z-transforms. Find $\mathcal{Z}\{n\}$, $\mathcal{Z}\{n^2\}$, and $\mathcal{Z}\{n^3\}$

$$\mathcal{Z}\{n\} = -z\frac{d}{dz}\mathcal{Z}\{n^0\} = -z\frac{d}{dz}\frac{z}{z-1} = \frac{z}{(z-1)^2}$$

$$\mathcal{Z}\{n^2\} = -z\frac{d}{dz}\mathcal{Z}\{n^1\} = -z\frac{d}{dz}\frac{z}{(z-1)^2} = \frac{z(z+1)}{(z-1)^3}$$

$$\mathcal{Z}\{n^3\} = -z\frac{d}{dz}\mathcal{Z}\{n^2\} = -z\frac{d}{dz}\frac{z(z+1)}{(z-1)^3} = \frac{z(z^2+4z+1)}{(z-1)^4}$$

Convolution

- Theorem 5 (Convolution property): If $\mathcal{Z}\{f(n)\} = F(z)$ and $\mathcal{Z}\{g(n)\} = G(z)$, then the z-transform of the convolution $f(n) * g(n)$ is given by:

$$\mathcal{Z}\{f(n) * g(n)\} = \mathcal{Z}\{f(n)\}\mathcal{Z}\{g(n)\}$$

where $f(n) * g(n) = \sum_{m=0}^{\infty} f(n-m)g(m) = \sum_{m=0}^{\infty} f(m)g(n-m)$

- Example 12: Determine $h(n) = f(n) * g(n)$ using z-transform techniques

$$f(n) = \{4, 3, 2, 1\}, \quad g(n) = \{3, 7, 4\}$$

$$F(z) = 4 + 3z^{-1} + 2z^{-2} + z^{-3}, \quad G(z) = 3 + 7z^{-1} + 4z^{-2}$$

$$H(z) = F(z)G(z) = 12 + 37z^{-1} + 43z^{-2} + 29z^{-3} + 15z^{-4} + 4z^{-5}$$

$$h(n) = \{12, 37, 43, 29, 15, 4\}$$

Initial Value Theorem

- Theorem 6 (Initial Value Theorem): If $\mathcal{Z}\{f(n)\} = F(z)$, then $f(0) = \lim_{z\to\infty} F(z)$

  Also, if $f(0) = 0$, then $f(1) = \lim_{z\to\infty} zF(z)$ provided the limits exist

- Example 13: Verify the initial value theorem for $f(n) = a^n$

$$f(0) = a^0 = 1, \quad \lim_{z\to\infty} F(z) = \lim_{z\to\infty} \frac{z}{z-a} = 1$$

Final Value Theorem

- **Theorem 7 (Final Value Theorem):** If $\mathcal{Z}\{f(n)\} = F(z)$, then

$$\lim_{n\to\infty} f(n) = \lim_{z\to 1}\{(z-1)F(z)\}$$ provided the limits exist

- **Example 14:** Initial value theorem for the sequence with the z-transform:

$$F(z) = \frac{10z^2 + 2z}{(z-1)(5z-1)^2}$$

$$\lim_{z\to 1}\left\{(z-1)\frac{10z^2+2z}{(z-1)(5z-1)^2}\right\} = \lim_{z\to 1}\left\{\frac{10z^2+2z}{(5z-1)^2}\right\} = \frac{3}{4}$$

Verification:

$$F(z) = \frac{10z^2+2z}{(z-1)(5z-1)^2} = \frac{3}{4}\frac{z}{z-1} - \frac{3}{4}\frac{z}{z-1/5} - \frac{1}{5}\frac{z}{(z-1/5)^2}$$

$$f(n)=\frac{3}{4}-\frac{3}{4}\left(\frac{1}{5}\right)^n-n\left(\frac{1}{5}\right)^n \Rightarrow \lim_{n\to\infty} f(n)=\frac{3}{4}$$

- **Note:**

$$F(z)=Z\{f(n)\}=\sum_{n=0}^{\infty} f(n)z^{-n}=f(0)+\frac{f(1)}{z}+\frac{f(2)}{z^2}+\frac{f(3)}{z^3}+\cdots$$

$$f(0)=\lim_{z\to\infty} F(z)$$

$$f(1)=\lim_{z\to\infty} z\left[F(z)-f(0)\right]$$

$$f(2)=\lim_{z\to\infty} z^2\left[F(z)-f(0)-\frac{f(1)}{z}\right]$$

$$f(3)=\lim_{z\to\infty} z^3\left[F(z)-f(0)-\frac{f(1)}{z}-\frac{f(2)}{z^2}\right]$$

$$\vdots$$

- Example 15: Initial Value Theorem

If $F(z)=\dfrac{2z^2+5z+14}{(z-1)^4}$, evaluate $f(2)$, $f(3)$

$$f(0)=\lim_{z\to\infty}F(z)=\lim_{z\to\infty}\frac{2z^2+5z+14}{(z-1)^4}=0$$

$$f(1)=\lim_{z\to\infty}z\left[F(z)-f(0)\right]=\lim_{z\to\infty}z\,\frac{2z^2+5z+14}{(z-1)^4}=0$$

$$f(2)=\lim_{z\to\infty}z^2\left[F(z)-f(0)-\frac{f(1)}{z}\right]=\lim_{z\to\infty}z^2\,\frac{2z^2+5z+14}{(z-1)^4}=2$$

$$f(3)=\lim_{z\to\infty}z^3\left[F(z)-f(0)-\frac{f(1)}{z}-\frac{f(2)}{z^2}\right]=\lim_{z\to\infty}\left\{z^3\,\frac{2z^2+5z+14}{(z-1)^4}-2z\right\}=13$$

Z-Transforms of Some Commonly Used Sequences

	$f(n),\ n \ge 0$	$F(z)$	
1	$\delta(n)$	$1,$	$\forall z$
2	$u(n)$	$\dfrac{z}{z-1},$	$\lvert z\rvert > 1$
3	a^n	$\dfrac{z}{z-a},$	$\lvert z\rvert > a$
4	n	$\dfrac{z}{(z-1)^2},$	$\lvert z\rvert > 1$
5	$\dfrac{a^n}{n!}$	$e^{a/z},$	$\lvert z\rvert > 0$

	$f(n),\ n \ge 0$	$F(z)$	
6	$\cos nx$	$\dfrac{z(z-\cos x)}{z^2-2z\cos x+1},$	$\lvert z\rvert > 1$
7	$\sin nx$	$\dfrac{z\sin x}{z^2-2z\cos x+1},$	$\lvert z\rvert > 1$
8	$\cosh nx$	$\dfrac{z(z-\cosh x)}{z^2-2z\cosh x+1},$	$\lvert z\rvert > e^{\lvert x\rvert}$
9	$\sinh nx$	$\dfrac{z\sinh x}{z^2-2z\cosh x+1},$	$\lvert z\rvert > e^{\lvert x\rvert}$
10	$(\ln a)^n/n!$	$a^{1/z},$	$\lvert z\rvert > 0$

3. Inverse Z-Transform

- There are four methods for evaluating the inverse z-transform:

  (1) Power series, (2) Recursion, (3) Partial fractions, and (4) The residue method. The first three apply only to those functions $F(z)$ that are rational functions while the residue method is more general.

Inverse Z-Transform by Long Division (Power series)

- By means of the long-division process, we can always rewrite $F(z)$ as the Laurent expansion: $F(z) = a_0 + a_1 z^{-1} + a_2 z^{-2} + \cdots$

  From the definition of the z-transform,

$$F(z) = \sum_{n=0}^{\infty} f(n) z^{-n} = f(0) + f(1)z^{-1} + f(2)z^{-2} + \cdots$$

  the desired sequence $f(n)$ is given by a_n

- Example 16: Find the inverse of the z-transform of $F(z) = \dfrac{2z^2 - 1.5z}{z^2 - 1.5z + 0.5}$ using Long Division

$$
\begin{array}{r|llllllll}
 & 2 & + & 1.5z^{-1} & + & 1.25z^{-2} & + & 1.125z^{-3} & + \cdots \\
\hline
z^2 - 1.5z + 0.5 & 2z^2 & - & 1.5z & & & & & \\
 & 2z^2 & - & 3z & + & 1 & & & \\
\cline{2-6}
 & & & 1.5z & - & 1 & & & \\
 & & & 1.5z & - & 2.25 & + & 0.750z^{-1} & \\
\cline{4-8}
 & & & & & 1.25 & - & 0.750z^{-1} & \\
 & & & & & 1.25 & - & 1.870z^{-1} & + \cdots \\
\cline{6-9}
 & & & & & & & 1.125z^{-1} & + \cdots
\end{array}
$$

Thus, $f(0) = 2$, $f(1) = 1.5$, $f(2) = 1.25$, $f(3) = 1.125$, …, or $f(n) = 1 + (1/2)^n$. In general, this technique only produces numerical values for some of the elements of the sequence.

Inverse Z-Transform by Recursive method

- This method obtains the inverse recursively. We begin by assuming that the z-transform is of the form:

$$F(z) = \frac{a_0 z^m + a_1 z^{m-1} + a_2 z^{m-2} + \cdots + a_{m-1} z + a_m}{b_0 z^m + b_1 z^{m-1} + b_2 z^{m-2} + \cdots + b_{m-1} z + b_m}$$

where some of the coefficients a_i and b_i may be zero and $b_0 \neq 0$. Applying the initial-value theorem, $f(0) = \lim_{z \to \infty} F(z) = a_0 / b_0$

Next, we apply the initial-value theorem to $z[F(z) - f(0)]$ and find that

$$f(1) = \lim_{z \to \infty} z[F(z) - f(0)]$$

$$= \lim_{z \to \infty} z \frac{(a_0 - b_0 f(0)) z^m + (a_1 - b_1 f(0)) z^{m-1} + \cdots + (a_m - b_m f(0))}{b_0 z^m + b_1 z^{m-1} + b_2 z^{m-2} + \cdots + b_{m-1} z + b_m} = (a_1 - b_1 f(0))/b_0$$

Note that the coefficient $a_0 - b_0 f(0) = 0$

$$f(2) = \lim_{z\to\infty} z[zF(z) - zf(0) - f(1)]$$

$$= \lim_{z\to\infty} z \frac{\cdots + (a_2 - b_2 f(0) - b_1 f(1))z^{m-1} + \cdots - b_m f(1)}{b_0 z^m + b_1 z^{m-1} + b_2 z^{m-2} + \cdots + b_{m-1} z + b_m}$$

$$= (a_2 - b_2 f(0) - b_1 f(1))/b_0$$

$$a_0 - b_0 f(0) = 0$$

$$a_1 - b_1 f(0) - b_0 f(1) = 0$$

Continuing this process, we finally have that

$$f(n) = (a_n - b_n f(0) - b_{n-1} f(1) - \cdots - b_1 f(n-1))/b_0,$$

where $a_n = b_n = 0$ for $n > m$

- Example 17: Recursive method

Let us redo Example 16 using the recursive method $F(z) = \dfrac{2z^2 - 1.5z}{z^2 - 1.5z + 0.5}$

$a_0 = 2,\ a_1 = -1.5,\ a_2 = 0,\ b_0 = 1,\ b_1 = -1.5,\ b_2 = 0.5$, and $a_n = b_n = 0$ if $n \geq 3$

$f(0) = a_0/b_0 = 2/1 = 2,$

$f(1) = (a_1 - b_1 f(0))/b_0 = [-1.5 - (-1.5)(2)]/1 = 1.5,$

$f(2) = (a_2 - b_2 f(0) - b_1 f(1))/b_0 = [0 - (0.5)(2) - (-1.5)(1.5)]/1 = 1.25,$

$f(3) = (a_3 - b_3 f(0) - b_2 f(1) - b_1 f(2))/b_0 = [0 - 0(2) - 0.5(1.5) - (-1.5)(1.25)]/1 = 1.125$

Inverse Z-Transform by Partial fraction expansion

- In this case, however, we work with $F(z)/z$ instead of $F(s)$. Why $F(z)/z$ and not $F(z)$?
- Examining our standard forms we note that functions of z occur in the form:

$$\frac{z}{z-1},\ \frac{z}{(z-1)^2},\ \frac{z}{z-e^{-a}}$$

and not in the normal partial fraction form $\frac{1}{z-1}, \frac{1}{(z-1)^2}, \frac{1}{z-e^{-a}}$

- It consists of decomposing $F(z)/z$ into partial fractions, multiplying the resulting expansion by z and then inverting the same.

- Example 18: Partial fraction expansion

  Find inverse z-transforms of $F(z) = z/(z^2 - 1)$

$$\frac{F(z)}{z} = \frac{1}{(z-1)(z+1)} = \frac{1/2}{z-1} - \frac{1/2}{z+1}$$

$$F(z) = \frac{1}{2}\left(\frac{z}{z-1} - \frac{z}{z+1}\right) \Rightarrow f(n) = \frac{1}{2}\left(Z^{-1}\left\{\frac{z}{z-1}\right\} - Z^{-1}\left\{\frac{z}{z+1}\right\}\right)$$

$$f(n) = \tfrac{1}{2}[1-(-1)^n],\ n \geq 0$$

- Example 19: Find inverse z-transforms of $F(z) = 2z^2/[(z+2)\,(z+1)^2]$

$$\frac{F(z)}{z} = \frac{2z}{(z+2)(z+1)^2} = \frac{4}{z+2} - \frac{4}{z+1} - \frac{2}{(z+1)^2}$$

$$F(z) = \frac{4z}{z+2} - \frac{4z}{z+1} - \frac{2z}{(z+1)^2} \Rightarrow f(n) = 4(-1)^n - 4(-2)^n + 2n(-1)^n,\ n \geq 0$$

- Example 20: Find inverse z-transforms of $\dfrac{2z}{(z-1)(z^2+1)}$

$$\frac{F(z)}{z} = \frac{2}{(z-1)(z^2+1)} = \frac{1}{z-1} - \frac{z+1}{z^2+1}$$

$$F(z) = \frac{z}{z-1} - \frac{z^2}{z^2+1} - \frac{z}{z^2+1} \Rightarrow f(n) = 1 - \sin\frac{n\pi}{2} - \cos\frac{n\pi}{2},\ n \geq 0$$

Inverse Z-Transform by Residue method

- The inverse z-transform can be evaluated by using the Cauchy residue theorem.

$$f(n) = \frac{1}{2\pi i} \oint_C F(z) z^{n-1} dz,\ n \geq 0$$

where C is a counterclockwise simple closed circular contour centered at the origin and with radius R such that C is in the ROC of F.

- Example 21: Residue method

Find inverse z-transforms of $F(z) = \dfrac{1}{(z-1)(z-2)}$

$$f(n) = \frac{1}{2\pi i} \oint_C \frac{z^{n-1}}{(z-1)(z-2)} dz$$

The integral has simple poles at $z = 1$ and $z = 2$. However, when $n = 0$ we also have a simple pole at $z = 0$.

Case 1: $n = 0$

$$f(0) = Res\left[\frac{1}{z(z-1)(z-2)}, 0\right] + Res\left[\frac{1}{z(z-1)(z-2)}, 1\right] + Res\left[\frac{1}{z(z-1)(z-2)}, 2\right]$$

$$Res\left[\frac{1}{z(z-1)(z-2)}, 0\right] = \left.\frac{1}{(z-1)(z-2)}\right|_{z=0} = \frac{1}{2}$$

$$Res\left[\frac{1}{z(z-1)(z-2)}, 1\right] = \left.\frac{1}{z(z-2)}\right|_{z=1} = -1$$

$$Res\left[\frac{1}{z(z-1)(z-2)}, 2\right] = \left.\frac{1}{z(z-1)}\right|_{z=2} = \frac{1}{2}$$

$$\Rightarrow f(0) = \tfrac{1}{2} - 1 + \tfrac{1}{2} = 0$$

Case 1: $n > 0$

$$f(n) = Res\left[\frac{z^{n-1}}{(z-1)(z-2)}, 1\right] + Res\left[\frac{z^{n-1}}{(z-1)(z-2)}, 2\right]$$

$$Res\left[\frac{z^{n-1}}{(z-1)(z-2)}, 1\right] = \left.\frac{z^{n-1}}{z-2}\right|_{z=1} = -1$$

$$Res\left[\frac{z^{n-1}}{(z-1)(z-2)}, 2\right] = \left.\frac{z^{n-1}}{z-1}\right|_{z=2} = 2^{n-1},\ n > 0$$

$$f(n) = 2^{n-1} - 1,\ n > 0$$

$$f(n) = \begin{cases} 0, & n = 0 \\ 2^{n-1} - 1, & n > 0 \end{cases}$$

- Example 22: Residue method

Find inverse z-transforms of $F(z) = \dfrac{z^2 + 2z}{(z-1)^2}$

$$f(n) = \frac{1}{2\pi i}\oint_C \frac{z^{n+1} + 2z^n}{(z-1)^2} dz = Res\left[\frac{z^{n+1} + 2z^n}{(z-1)^2}, 1\right]$$

$$Res\left[\frac{z^{n+1} + 2z^n}{(z-1)^2}, 1\right] = \frac{d}{dz}\left(z^{n+1} + 2z^n\right)\Big|_{z=1} = 3n + 1$$

$$f(n) = 3n + 1,\ n \geq 0$$

- Example 23: Using the Convolution to find the inverse

Use the Convolution theorem to find the inverse of $\dfrac{z^2}{(z-a)(z-b)}$

$$F(z)=\frac{z}{z-a}, \quad G(z)=\frac{z}{z-b} \Rightarrow f(n)=a^n, \quad g(z)=b^n$$

$$Z^{-1}\{F(z)G(z)\}=f(n)*g(n)=\sum_{m=0}^{n} f(n-m)*g(m)=\sum_{m=0}^{n} a^{n-m}b^m=a^n\sum_{m=0}^{n}\left(\frac{b}{a}\right)^m$$

$$Z^{-1}\left\{\frac{z^2}{(z-a)(z-b)}\right\}=a^n\sum_{m=0}^{n}\left(\frac{b}{a}\right)^m=a^n\left\{\frac{1-\left(\frac{b}{a}\right)^{n+1}}{1-\frac{b}{a}}\right\}=\frac{a^{n+1}}{a-b}\left\{1-\left(\frac{b}{a}\right)^{n+1}\right\}$$

- Note: The existence of the inverse imposes restrictions on $f(n)$ for uniqueness. We require that $f(n)=0$ for $n<0$.
- Example 24: Reconstruction of a Sequence from its Z-Transform

  Suppose $F(z)=\frac{z}{z-1}, \quad |z|>1$ and $G(z)=\frac{z}{z-1}, \quad |z|<1$

$$\mathcal{Z}^{-1}\{F(z)\} = f(n) = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases} \quad \text{and} \quad \mathcal{Z}^{-1}\{G(z)\} = g(n) = \begin{cases} 1, & n < 0 \\ 0, & n \geq 0 \end{cases}$$

- The inverse z-transform of $z(z-1)^{-1}$ is not unique. In general, the inverse z-transform is not unique, unless its region of convergence is specified.

4. Solution of Difference Equations

- We use the z-transform to solve difference equations. The procedure parallels that of solving ordinary differential equations by Laplace transforms.
- Essentially we reduce the difference equation to an algebraic problem. We then find the solution by inverting $F(z)$.
- Example 25: First-Order Difference Equation
  Solve the equation: $f(n+1) + 2f(n) = n$, $f(0) = 1$

$$z\{F(z)-f(0)\}+2F(z)=\frac{z}{(z-1)^2} \Rightarrow F(z)=\frac{z}{z+2}+\frac{z}{(z+2)(z-1)^2}$$

$$F(z)=\frac{z}{z+2}+\frac{1}{9}\frac{z}{z+2}-\frac{1}{9}\frac{z}{z-1}+\frac{3}{9}\frac{z}{(z-1)^2}$$

$$F(z)=\frac{10}{9}\frac{z}{z+2}-\frac{1}{9}\frac{z}{z-1}+\frac{3}{9}\frac{z}{(z-1)^2}$$

$$f(n)=\tfrac{1}{9}[10(-2)^n+3n-1]$$

- Example 26: Second-Order Difference Equation (The Fibonacci Sequence)

  Solve the equation: $f(n+1)=f(n)+f(n-1)$, $f(0)=f(1)=1$

$$F(z)=\frac{z^2}{z^2-z-1}=\frac{z^2}{(z-a)(z-b)}, \quad a=\frac{1+\sqrt{5}}{2} \text{ and } b=\frac{1-\sqrt{5}}{2}$$

$$f(n) = \frac{a^{n+1} - b^{n+1}}{a-b}, \quad n = 0, 1, 2, \ldots \qquad f(n) = 1, 1, 2, 3, 5, \ldots$$

- Example 27: Chebyshev Polynomials

  Solve the equation: $f(n+2) - f(n+1) + f(n) = 0$, $f(0) = 1$ and $f(1) = 2$

$$\{z^2F(z) - z^2 - 2z\} - \{zF(z) - z\} + F(z) = 0$$

$$F(z) = \frac{z^2+z}{z^2-z+1} = \frac{z^2 - \frac{1}{2}z}{z^2-z+1} + \frac{\sqrt{3}\left(\frac{\sqrt{3}}{2}z\right)}{z^2-z+1} \Rightarrow f(n) = \cos\left(\frac{n\pi}{3}\right) + \sqrt{3}\sin\left(\frac{n\pi}{3}\right)$$

- Example 28: Periodic Solution

  Solve the equation: $f(n+2) - 2xf(n+1) + f(n) = 0$, $|x| \le 1$, $f(0) = u_0$ and $f(1) = u_1$

$$\{z^2F(z) - z^2u_0 - zu_1\} - 2x\{zF(z) - zu_0\} + F(z) = 0$$

$$F(z) = u_0\left[\frac{z^2 - zx}{z^2 - 2xz + 1}\right] + (u_1 - xu_0)\left[\frac{z}{z^2 - 2xz + 1}\right]$$

$$F(z) = u_0\left[\frac{z^2 - zx}{z^2 - 2xz + 1}\right] + \frac{(u_1 - xu_0)}{\sqrt{1-x^2}}\left[\frac{z\sqrt{1-x^2}}{z^2 - 2xz + 1}\right]$$

$$F(z) = u_0\left[\frac{z^2 - zx}{z^2 - 2xz + 1}\right] + v_0\left[\frac{z\sqrt{1-x^2}}{z^2 - 2xz + 1}\right]$$

where $v_0 = (u_1 - xu_0)(1 - x^2)^{-1/2}$ is independent of z

Since $|x| \leq 1$, we may write $x = \cos t$

$$f(n) = u_0\cos nt + v_0\sin nt = u_0\cos(n\cos^{-1}x) + v_0\sin(n\cos^{-1}x)$$

$T_n(x) = \cos(n\cos^{-1}x)$ is called Chebyshev polynomial of first kind of degree n.

- Example 29: Solve the following system of difference equations:

$x(n+1) = 4x(n) + 2y(n)$, and $y(n+1) = 3x(n) + 3y(n)$, $x(0) = 0$ and $y(0) = 5$

$$zX(z) - x(0)z = 4X(z) + 2Y(z), \quad zY(z) - y(0)z = 3X(z) + 3Y(z)$$

$$(z-4)X(z) - 2Y(z) = 0, \quad 3X(z) - (z-3)Y(z) = -5z$$

$$X(z) = \frac{10z}{(z-6)(z-1)} = \frac{2z}{z-6} - \frac{2z}{z-1}, \quad Y(z) = \frac{5z(z-4)}{(z-6)(z-1)} = \frac{2z}{z-6} + \frac{3z}{z-1}$$

$$x(n) = -2 + 2(6^n), \quad y(n) = 3 + 2(6^n)$$

5. Summation of Infinite Series

- Theorem 8 (Summation of Infinite Series): If $\mathcal{Z}\{f(n)\} = F(z)$, then

$$(i)\ \sum_{k=0}^{n} f(k) = \mathcal{Z}^{-1}\left\{\frac{z}{z-1}F(z)\right\} \qquad (ii)\ \sum_{k=0}^{\infty} f(k) = \lim_{z\to 1} F(z) = F(1)$$

- Example 30: Summation of Infinite Series

Use the z-transform to show that: $\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$

$$Z\{x^n f(n)\} = F(\frac{z}{x})$$

$$f(n) = \frac{1}{n!} \Rightarrow F(z) = e^{1/z}$$

$$Z\left\{\frac{x^n}{n!}\right\} = e^{x/z} = F\left(\frac{z}{x}\right)$$

$$\sum_{k=1}^{\infty} \frac{x^n}{n!} = \lim_{z \to 1} e^{x/z} = e^x$$

- Example 31: Summation of Infinite Series

Find the sum of the series: $\sum_{n=0}^{\infty} a^n \sin nx$

$$Z\{f(n)\} = Z\{\sin nx\} = \frac{z\sin x}{z^2 - 2z\cos x + 1}$$

$$Z\{a^n \sin nx\} = F\left(\frac{z}{a}\right) = \frac{az\sin x}{a^2 - 2az\cos x + z^2}$$

$$\sum_{n=0}^{\infty} a^n \sin nx = \lim_{z\to 1} F\left(\frac{z}{a}\right) = \frac{a\sin x}{a^2 - 2a\cos x + 1}$$