Tension and Compression in Bars

1. Stress الإجهاد
2. Strain التشوه (الانفعال)
3. Constitutive Law قانون السلوك
4. Single Bar under Tension or Compression قضيب مفرد: شد أو ضغط
5. Systems of Bars جمل القضبان
6. Supplementary Examples أمثلة إضافية

Objectives: *Mechanics of Materials* investigates the stressing and the deformations of structures subjected to applied loads, starting by the simplest structural members, namely, bars in tension or compression.

يدرس ميكانيك المواد إجهادات وتشوهات الجمل الإنشائية (الهياكل الحاملة) الناتجة عن الحمولات الخارجية، مبتدأً بالعناصر الأبسط أي القضبان (العناصر الطولية) المشدودة أو المضغوطة.

In order to treat such problems, the kinematic relations and a constitutive law are needed to complement the equilibrium conditions which are known from Engineering Mechanics (Statics).

تقوم هذه الدراسة على:

(1) معادلات التوازن التي دُرست في الميكانيك الهندسي (علم السكون)

(2) العلاقات الكينماتيكية التي ستدرس وهي تصف التشوهات كمياً أي تحدد شكل ومقدار تغيرات الشكل الجيومتري.

(3) قوانين سلوك مادة الجملة وهي كما ستُعرض لاحقاً، قوانين تجريبية تعرّف السلوك الميكانيكي لمادة الهيكل الحامل.

The kinematic relations represent the geometry of the deformation, whereas the behavior of the material is described by the constitutive law. The students will learn how to apply these equations and how to solve determinate as well as statically indeterminate problems. يعالج الطلبة مسائل مقررة سكونياً وأخرى غير مقررة سكونياً

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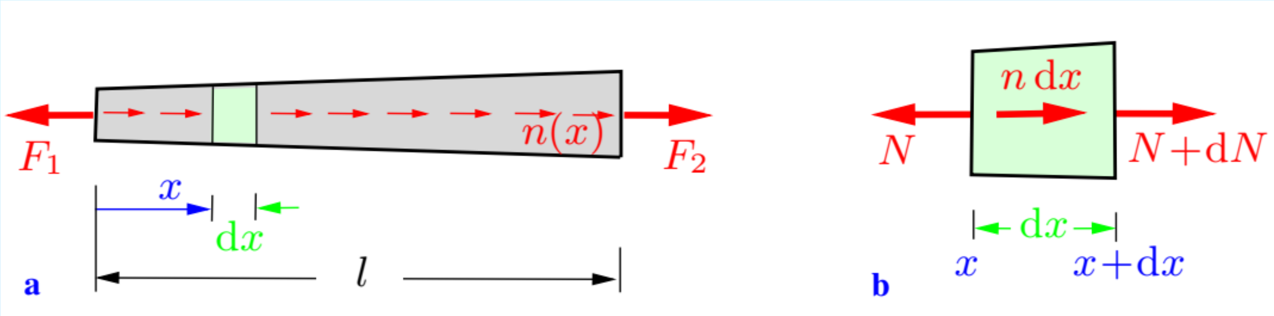
4 Single Bar under Tension or Compression

There are three different types of equations that allow us to determine the stresses & the strains in a bar: the equilibrium condition, the kinematic relation and Hooke's law.

Depending on the problem, the equilibrium condition may be formulated for the entire bar, a portion of the bar or for an element of the bar.

We will derive the equilibrium condition for an element. For this purpose we consider a bar which is subjected to two forces F_1 & F_2 at its ends and to a line load $n = n(x)$, see Fig.a.



The forces are assumed to be in equilibrium. We imagine a slice element of infinitesimal length dx separated from the bar as shown in Fig.b.

The F. B. D. shows the normal forces N and $N + dN$, respectively, at the ends of the element; the line load is replaced by its resultant ndx (note that n may be considered to be constant over the length dx).

Equilibrium of the forces in the direction of the axis of the bar

$$\Delta l = \int_0^l \frac{N(x)}{EA(x)} dx$$

In the special case of a bar (length l) with constant axial rigidity ($EA = const$) which is subjected only to forces at its end ($n \equiv 0, N = F$) the elongation is given by

$$\Delta l = \frac{l}{EA} F \quad \Leftrightarrow \quad F = \frac{EA}{l} \Delta l$$

Quantity $\frac{EA}{l}$ is the *axial rigidity (Stiffness) of the bar.* الصلابة المحورية

The Inverse $\frac{l}{EA}$ is the axial *flexibility* of the bar . ومقلوبها هو المطاوعة المحورية

5 Statically Determinate Systems of Bars

In the preceding section we calculated the stresses and deformations of single slender bars. We will now extend the investigation to trusses and to structures which consist of bars and rigid bodies.

In this section we will restrict ourselves to statically determinate systems where we can first calculate the forces in the bars with the aid of the equilibrium conditions.

- First, the stresses in the bars and the elongations are determined.
- Then, the displacements of arbitrary points of the structure can be found.

As it is assumed that the elongations are small compared with the lengths of the bars, equilibrium conditions are applied to the *undeformed* system.

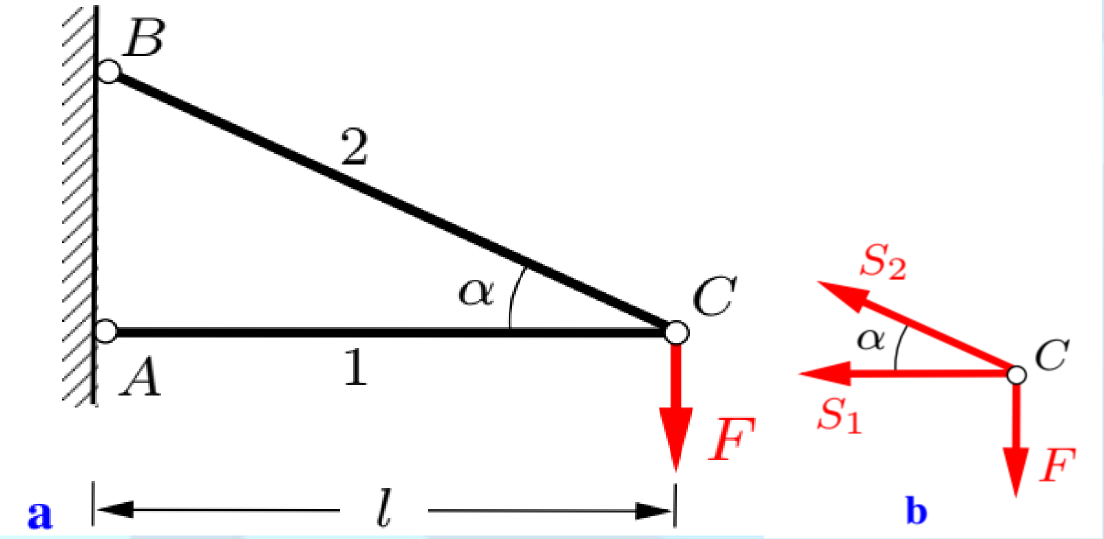
As an illustrative example let us consider the truss in Fig. a. Both bars have the axial rigidity EA.

Displacement of pin C due to the applied force F is to be determined.

First we calculate the forces S_1 and S_2 in the bars.

The equilibrium conditions, applied to the free-body diagram (Fig. b), yield

$\uparrow: S_2 \sin\alpha - F = 0$ and $\leftarrow: S_1 + S_2 \cos\alpha = 0$



↑: $S_2 \sin\alpha - F = 0$ and ←: $S_1 + S_2 \cos\alpha = 0 \quad \Rightarrow \quad S_1 = -F/\tan\alpha$ and $S_2 = F/\sin\alpha$

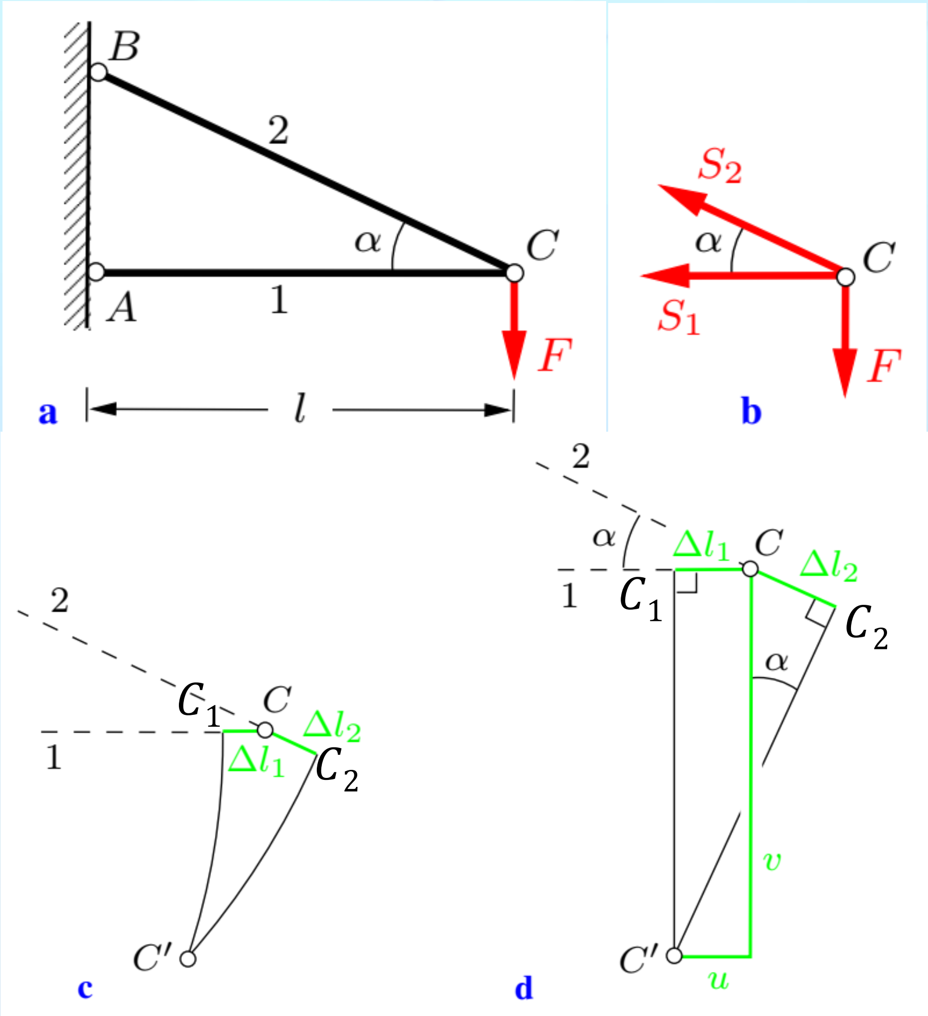
$$\Delta l_1 = \frac{l_1}{EA} S_1 = -\frac{Fl}{EA\tan\alpha} \quad \& \quad \Delta l_2 = \frac{l_2}{EA} S_2 = \frac{Fl}{EA\sin\alpha\cos\alpha}$$

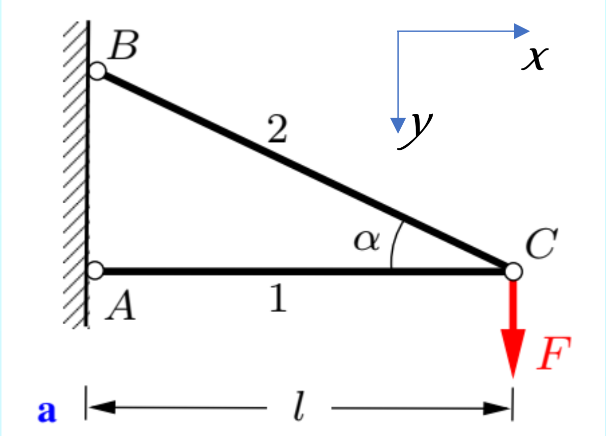
Bar1 becomes shorter (compression) and bar2, longer (tension).

The new position C' of pin C can be found as follows.
We consider the bars to be disconnected at C. Then the system becomes movable:
bar1 can rotate about point A; bar2 can rotate about point B.

The End of bar1 makes a circle of radius $l + \Delta l_1$ and the end of bar2 makes a circle of radius $(l/\cos\alpha) + \Delta l_2$ the two circles intersect at C', (Fig.c)

The two arcs C_1C' & C_2C' and are very small compared to l, so they can be approximated by the two tangents as in (Fig.d).

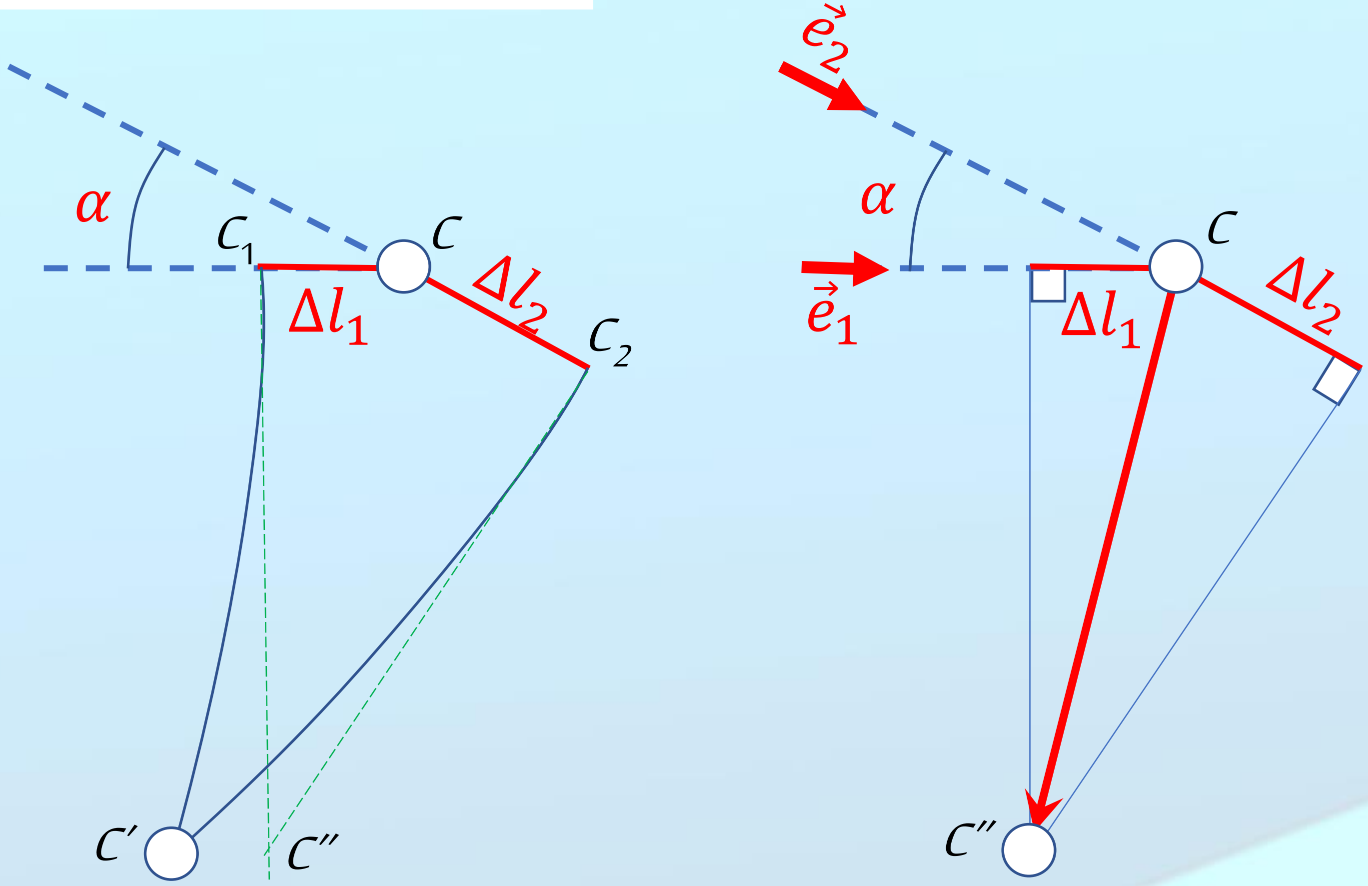




$\overrightarrow{CC''} = u\vec{i} + v\vec{j}, \quad \vec{e}_1 = \vec{i}\ , \vec{e}_2 = \cos\alpha\,\vec{i} + \sin\alpha\,\vec{j}.$

$\overrightarrow{CC''} \cdot \vec{e}_1 = u = \Delta l_1$ & $\overrightarrow{CC''} \cdot \vec{e}_2 = u\cos\alpha + v\sin\alpha = \Delta l_2$

$$u = -\frac{Fl\cos\alpha}{EA\sin\alpha}$$



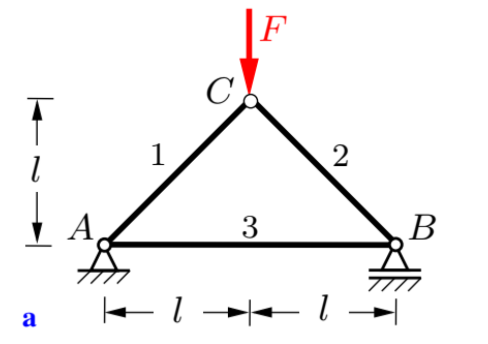
$$v\sin\alpha = \Delta l_2 - u\cos\alpha = \frac{Fl}{EA\sin\alpha\cos\alpha} + \frac{Fl\cos^2\alpha}{EA\sin\alpha} \Rightarrow$$

$$v = \frac{Fl(1+\cos^3\alpha)}{EA\sin\alpha\cos\alpha}$$

$E = 200\ GPA, A = 400\ mm^2$

$l = 2m, h = 1.5m, F = 20\ kN$

Example 1. (Design problem) The truss in Fig.a is under the action of the force $F = 20$ kN. If $E = 200$ Gpa (200000 MPa[≡N/mm^2]). Determine the required cross-sectional areas of the three members so that the stresses do not exceed the allowable stress $\sigma_{allow} = 150$ MPa and the displacement of B is smaller than 0.5% of the length of bar 3.



Solution

- First we calculate the (reactions if necessary) and forces in the members. The equilibrium conditions for the free-body diagrams of pin C and support B (Fig.b) yield:

At C; $\rightarrow: S_1 = S_2$; $\uparrow: S_1 = S_2 = -F/\sqrt{2}$ At B; $\rightarrow: -S_3 - S_2 \cos 45^0 = 0 \Rightarrow S_3 = F/2$

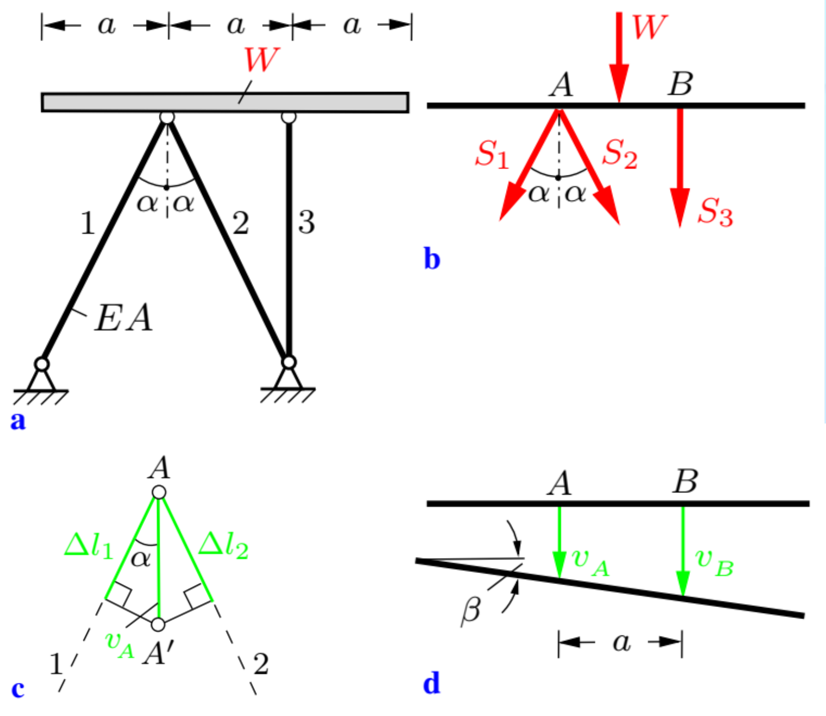
- Then we establish the design requirements (Conditions) متطلبات أو شروط التصميم

➢ Stress requirements: $\sigma_i = \frac{S_i}{A_i} \Rightarrow (A_i)_{\min} \geq \frac{S_i}{\sigma_{allow}}, i = 1,2,3.$

➢ Displacemen requirement: $u_B = (\Delta l)_3 = \frac{l_3 S_3}{E A_3} \leq 0.5 \times 10^{-2}(2l) \Rightarrow (A_3)_{\min} \geq \frac{(2l) S_3}{E (\Delta l)_3} = \frac{S_3}{0.5 \times 10^{-2} E}$

Member (length)	Normal Force: S[kN]	Min. Area [mm^2] $A_{\min} \geq S/\sigma_{all}$	Min. Area [mm^2] Displacement of B	Req. Min. Areas [mm^2]
1 ($l\sqrt{2}$)	$-F/\sqrt{2} = -14.1$	95	–	95
2 ($l\sqrt{2}$)	$-F/\sqrt{2} = -14.1$	95	–	95
3 ($2l$)	$F/2 = 10$	67	10	67

Example 2. A rigid beam (weight W) is mounted on three elastic bars (axial rigidity EA) as shown in Fig.a. Determine the angle of slope of the beam that is caused by its weight after the structure has been assembled.



Solution First we calculate the forces in the bars with the aid of the equilibrium conditions (Fig.b): $S_1 = S_2 = -W/4\cos\alpha\,, S_3 = -W/2$

With $l_1 = l_2 = a/\sin\alpha$ & $l_3 = a/\tan\alpha$, the elongations are: $\Delta l_1 = \Delta l_2 = \frac{l_1 S_1}{EA} = \frac{l_2 S_2}{EA} = -\frac{Wa}{4EA\sin\alpha\cos\alpha}, \Delta l_3 = \frac{l_3 S_{13}}{EA} = -\frac{Wa}{2EA\tan\alpha}$

Point B of the beam is displaced downward by $v_B = |\Delta l_3|$. To determine the vertical displacement v_A of point A we sketch the diagram (Fig.c). First we plot the changes Δl_1 & Δl_2 of the lengths in the direction of each bar. The lines perpendicular to these directions intersect at the displaced position A' of point A. So, its vertical displacement is $v_A = |\Delta l_1|/\cos\alpha$

Since v_A and v_B do not coincide, the beam does not stay horizontal.

The angle of slope β is obtained with the approximation $\tan\beta \approx \beta$ (small deformations) (Fig.d)

$$\beta = \frac{v_B - v_A}{a} = \frac{2\cos^3\alpha - 1}{4\cos^3\alpha}\,\frac{W\cot\alpha}{EA}.$$

$\cos^3\alpha = \frac{1}{2}$, the beam stays horizontal. $\Rightarrow \alpha = 37.5^0$

$\cos^3\alpha > \frac{1}{2}$ inclined to right

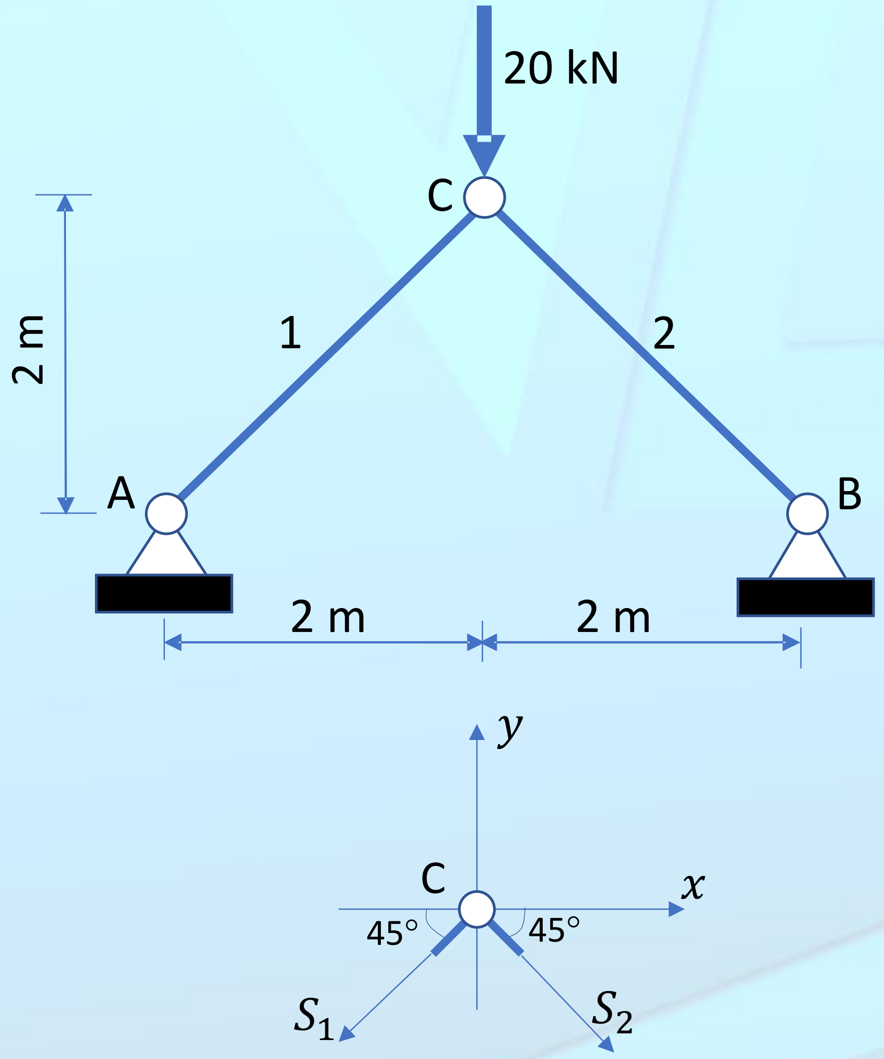
$\cos^3\alpha < \frac{1}{2}$ inclined to left

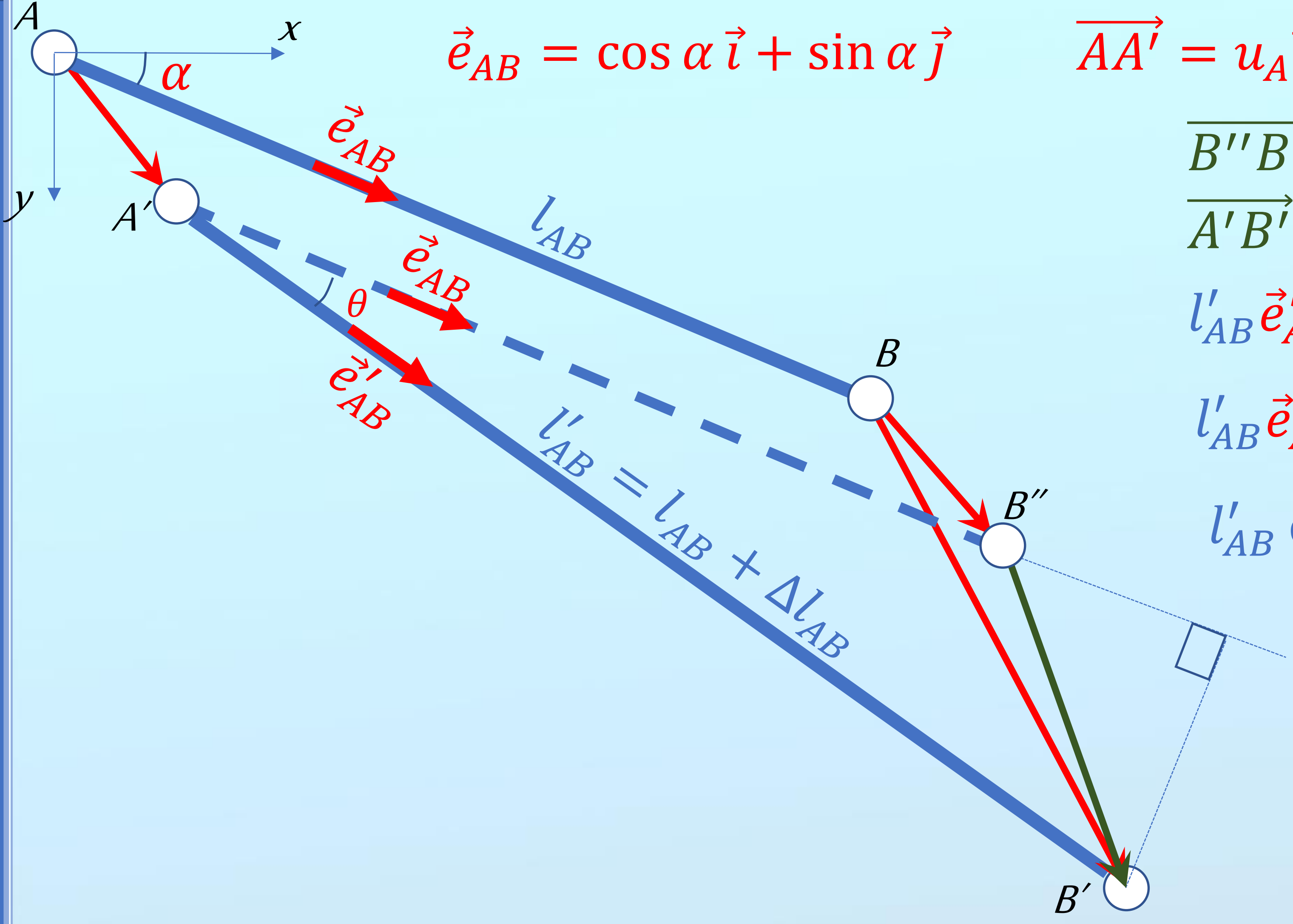
Problem 1. For the shown simple truss, if the two members have the same cross-section area $A = 100\ \text{mm}^2$, and the same modulus of elasticity $E = 200\ \text{GPa}$, find

1) The axial forces in members 1, 2.

2) The deflections in members 1, 2.

3) The displacements of joint C.





$$\vec{e}_{AB} = \cos\alpha\,\vec{\imath} + \sin\alpha\,\vec{\jmath}$$

$$\overrightarrow{AA'} = u_A\vec{\imath} + v_A\vec{\jmath} \qquad \overrightarrow{BB'} = u_B\vec{\imath} + v_B\vec{\jmath}$$

$$\overrightarrow{B''B'} = (u_B - u_A)\vec{\imath} + (v_B - v_A)\vec{\jmath}$$

$$\overrightarrow{A'B'} = \overrightarrow{A'B''} + \overrightarrow{B''B'}$$

$$l'_{AB}\vec{e}'_{AB} = l_{AB}\vec{e}_{AB} + \overrightarrow{B''B'}$$

$$l'_{AB}\vec{e}'_{AB}\cdot\vec{e}_{AB} = l_{AB}\vec{e}_{AB}\cdot\vec{e}_{AB} + \overrightarrow{B''B'}\cdot\vec{e}_{AB}$$

$$l'_{AB}\cos\theta = l_{AB} + \overrightarrow{B''B'}\cdot\vec{e}_{AB}$$

For $\theta \ll 1\,(rad)$:

$\cos\theta = 1$ & $\sin\theta \approx \theta$

$$l'_{AB} \approx l_{AB} + \overrightarrow{B''B'}\cdot\vec{e}_{AB}$$

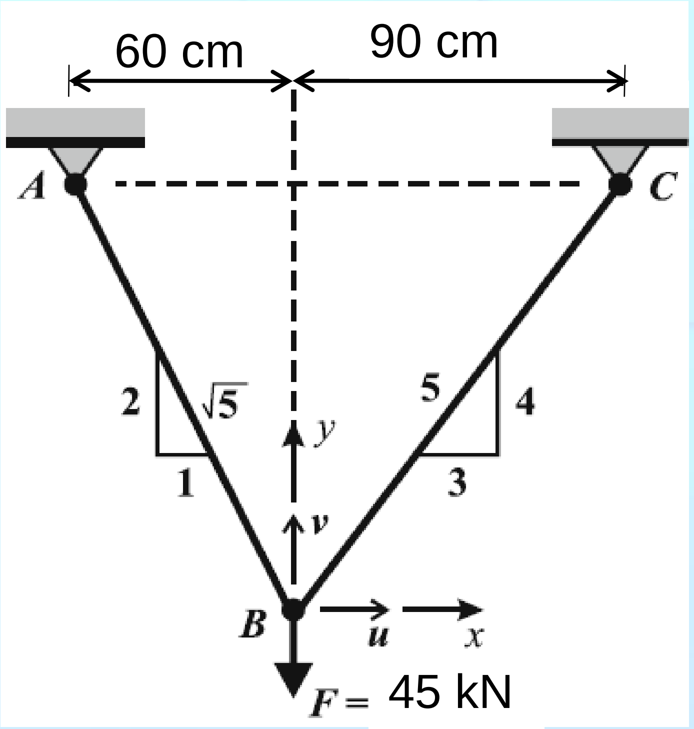
$$l'_{AB} - l_{AB} \approx \overrightarrow{B''B'}\cdot\vec{e}_{AB}$$

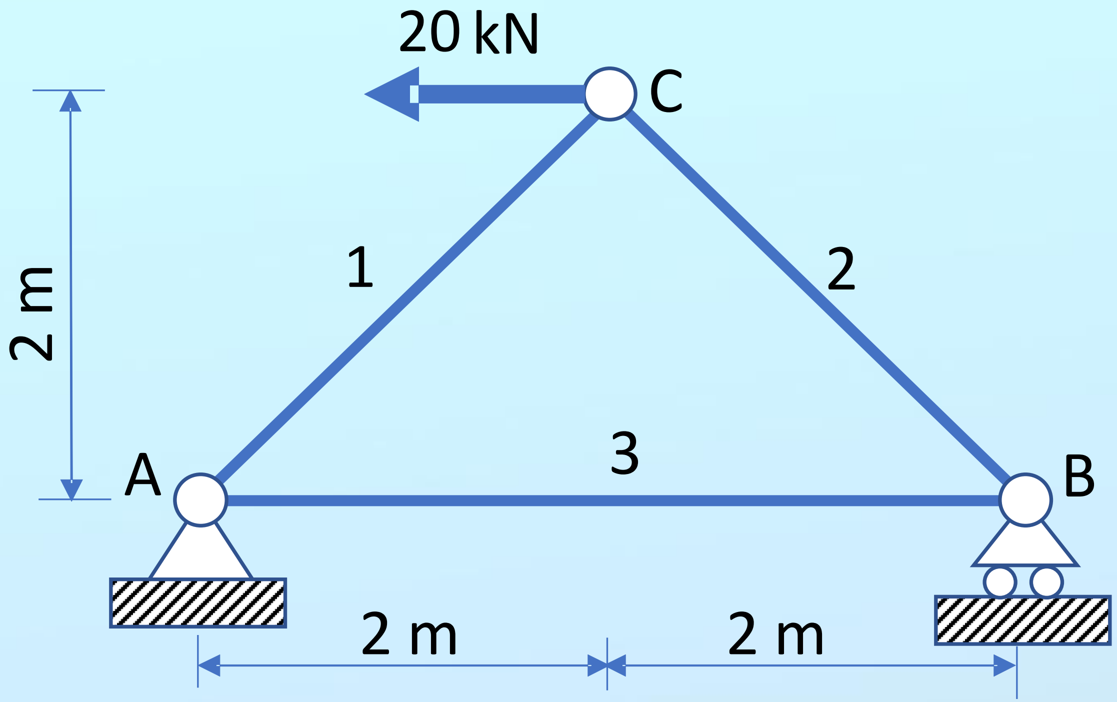
$$\Delta l_{AB} = (\vec{e}_{AB})\cdot[(u_B - u_A)\vec{\imath} + (v_B - v_A)\vec{\jmath}]$$

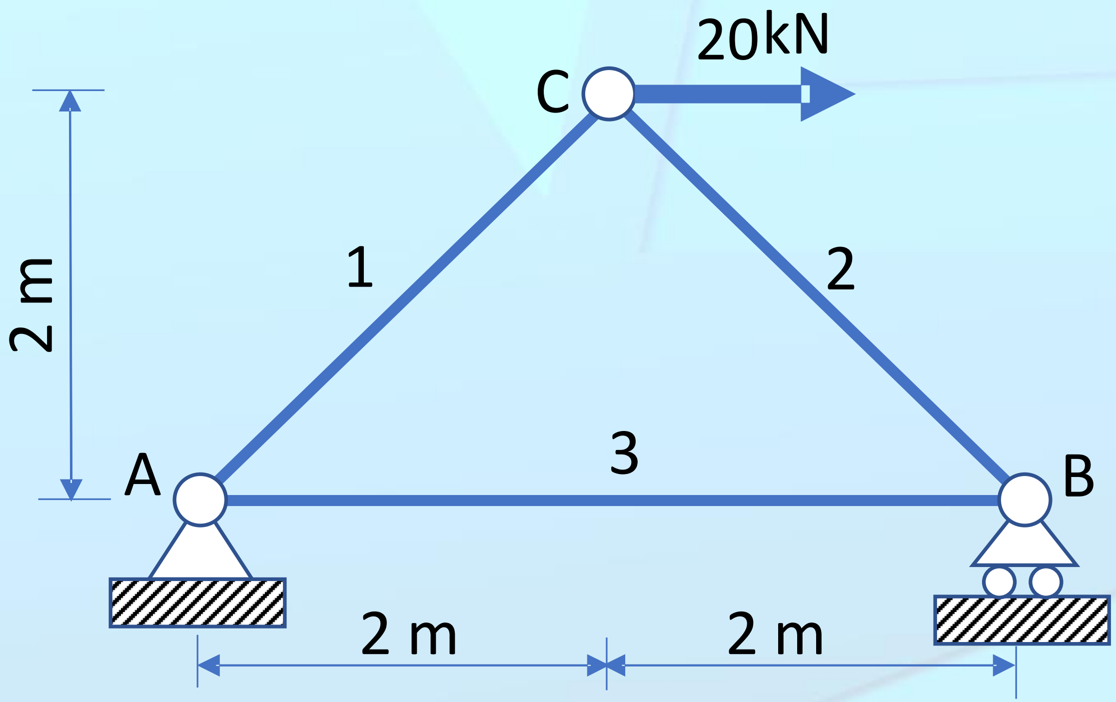
$$\Delta l_{AB} = (u_B - u_A)\cos\alpha + (v_B - v_A)\sin\alpha$$

Problem 2. An aluminum truss *ABC* is loaded at joint *B* by a point load of $F = 45$ kN. The cross-sectional areas of the bars are: $A_{AB} = 325$ mm^2 and $A_{BC} = 390$ mm^2. The modulus of aluminum is $E = 70$ GPa.

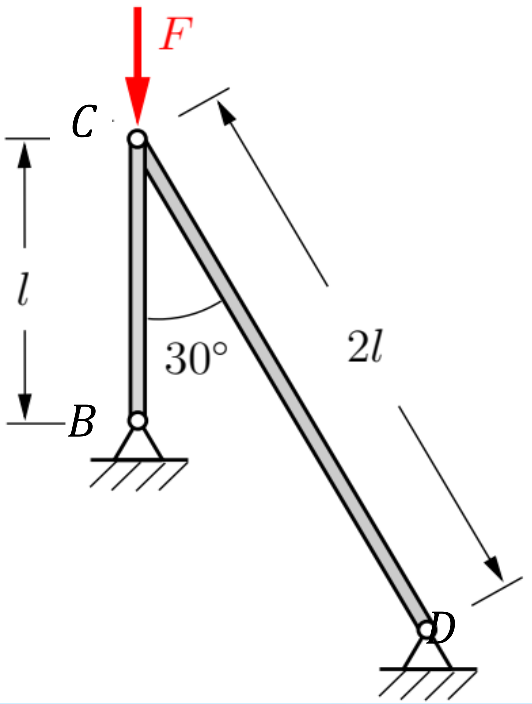
Determine the horizontal and vertical displacements of joint *B, u, and v.*



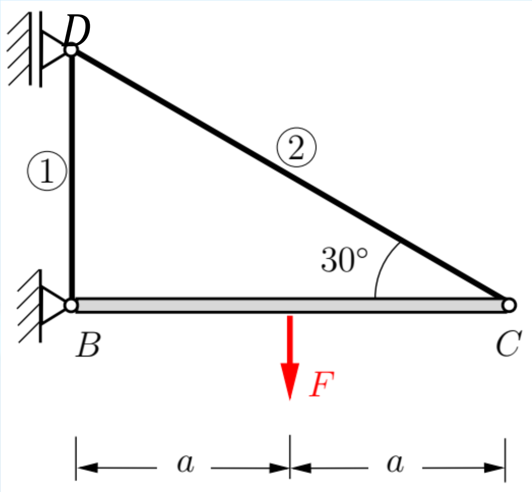

20 kN
C
2 m
1
2
A
3
B
2 m
2 m


20kN
C
2 m
1
2
A
3
B
2 m
2 m

Problem 5. The two rods (axial rigidity EA) shown in Fig. are pin-connected at C. The system is subjected to a vertical force F. Calculate in terms of F, l and EA, the displacement of pin C.



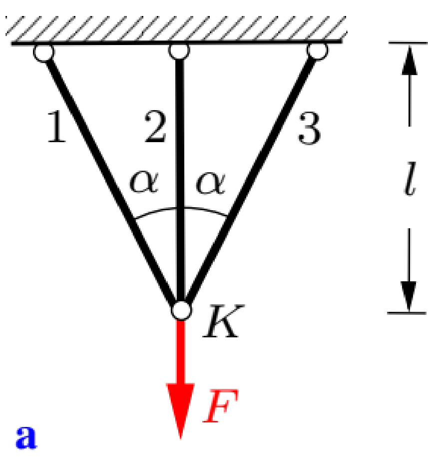
Problem 6. The structure shown in Fig. consists of a rigid beam BC and two elastic bars (axial rigidity EA). It is subjected to a force F. Calculate in terms of F, a and EA the displacement of pin C.



6 Statically Indeterminate Systems of Bars

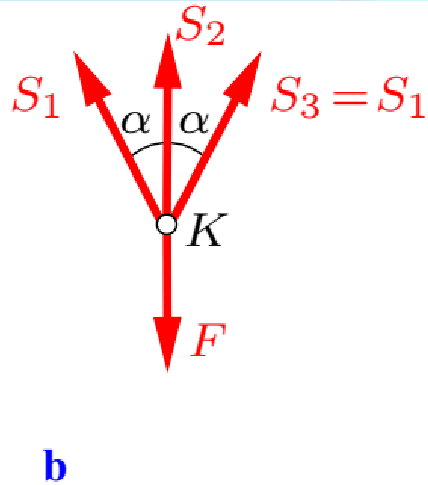
We will now investigate statically indeterminate systems for which the forces in the bars cannot be determined with the aid of the equilibrium conditions alone since the number of the unknown quantities exceeds the number of the equilibrium conditions.

In such systems the basic equations (1) Equilibrium conditions. (2) Kinematic equations (compatibility) & (3) Material behavior(Hooke's law), are coupled.



Let us consider the symmetrical truss shown in (Fig.a) It is stress-free before the load is applied. The axial rigidities $EA_1, EA_2, EA_3 = EA_1$ are given; the forces in the members are unknown.

The system is statically indeterminate to the first degree: The two equilibrium conditions applied to the free-body diagram of pin K (Fig.b) yield



$$\rightarrow: -S_1 \sin\alpha + S_3 \sin\alpha = 0 \;\Rightarrow\; S_1 = S_3,$$

$$\uparrow: S_1 \cos\alpha + S_2 + S_3 \cos\alpha - F = 0 \Rightarrow S_1 = S_3 = \frac{F - S_2}{2\cos\alpha}$$

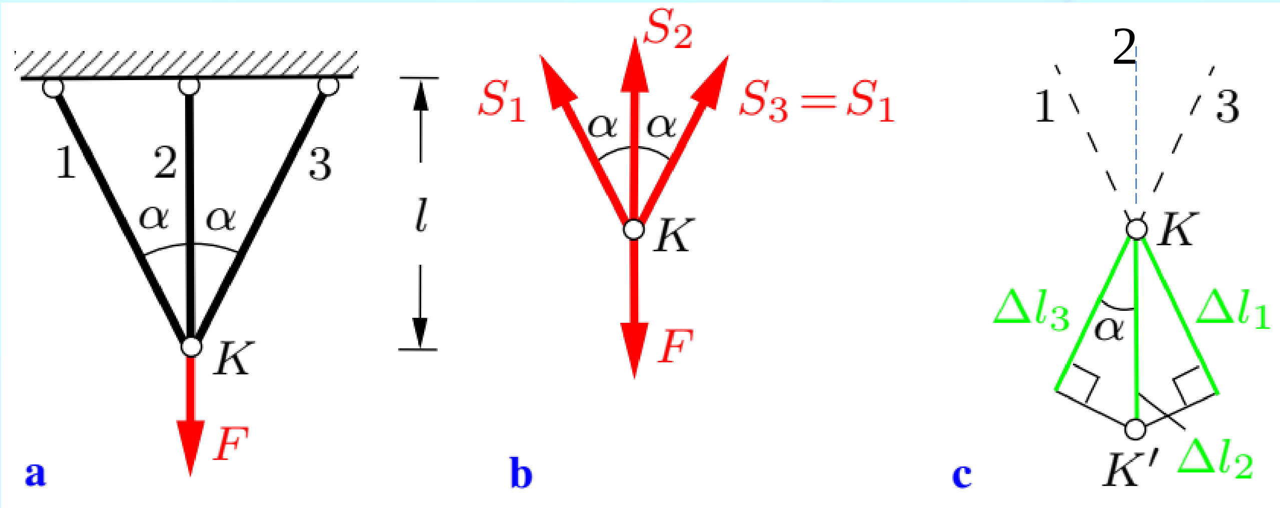
Number of static (force) unknowns is 3. Number of equilibrium equations is 2. So the number of indeterminacy is: 3-2=1.

The system is indeterminate to the first degree.

Hook's law gives the elongations of the bars by:

$$\Delta l_1 = \Delta l_3 = \frac{l_1}{EA_1} S_1$$

$$\Delta l_2 = \frac{l_2}{EA_2} S_2$$



The Kinematic (compatibility) condition is found by the displacement diagram (Fig.c):

$$\Delta l_1 \equiv \Delta l_3 = \Delta l_2 \cos\alpha$$

Substituting the material equations (Hook's law) in this compatibility equation, we write it in terms of the unknowns forces

$$\frac{l_1}{EA_1} S_1 = \frac{l_2}{EA_2} S_2 \cos\alpha$$

With the combination of the two equilibrium equations $S_1 = S_3 = \dfrac{F - S_2}{2\cos\alpha}$

And the geometric evidence: $l_1 = l/\cos\alpha$ and $l_2 = l$

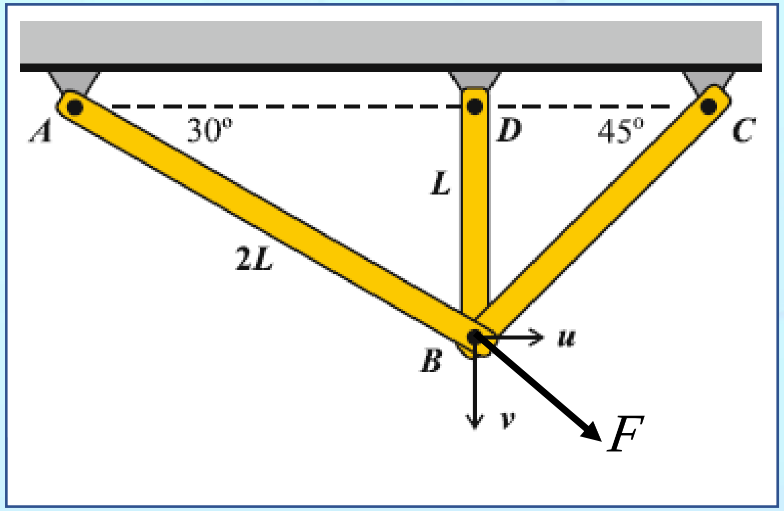
We obtain the three unknowns forces, then the elongations and displacements of *K*.

Problem 1. An aluminum truss is shown in the figure. Joint B is forced to move downward by v =2.0 mm; and to the right by u =1 mm. Length L = 1.00 m and the cross-sectional area of each bar is A =0.0008 m^2. The modulus is E = 70 GPa. Determine

(a) The elongation δ in each member.

(b) The force P in each member.

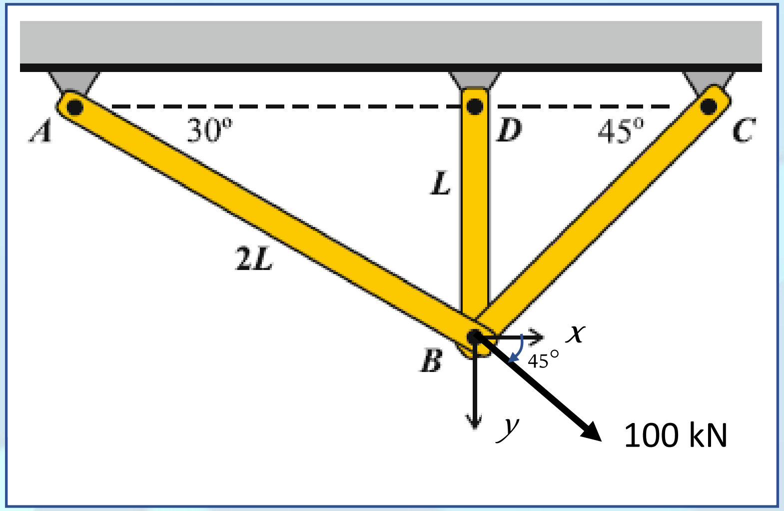
(c) The components, F_x and F_y, of the applied force that causes the displacement, and its magnitude F.



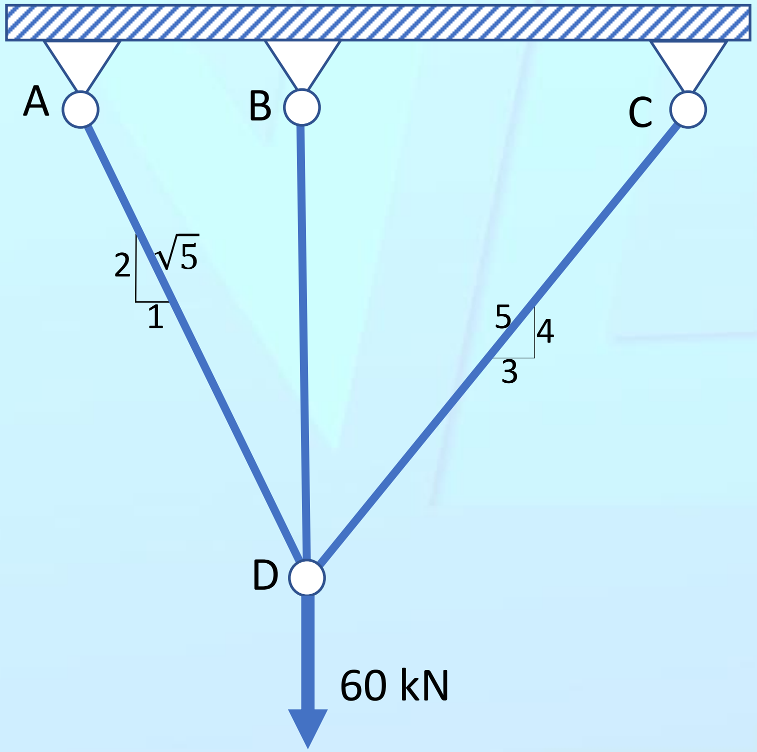
Problem 2. An aluminum truss is shown in the figure. Joint B is loaded by the shown force F. Length L = 1.00 m and the cross-sectional area of each bar is A =0.0008 m^2. The modulus is E = 70 GPa. Determine

(a) The displacements of joint.

(b) The axial stress in each bar.



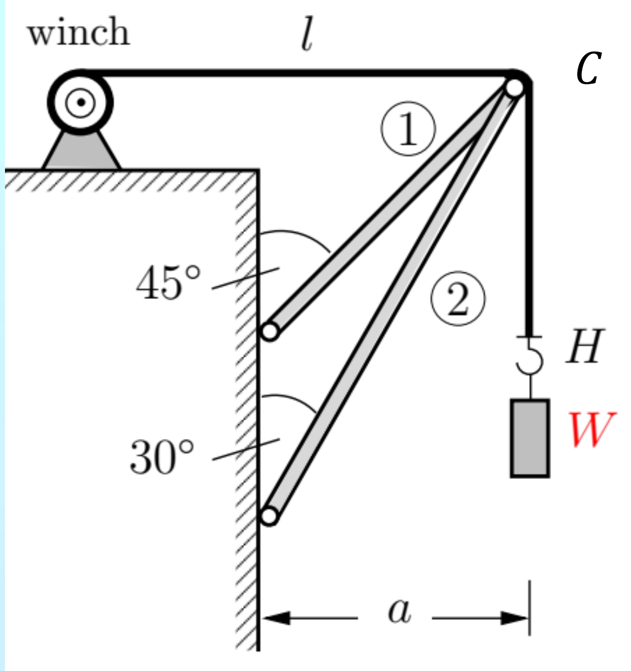
Problem 3. An aluminum truss *ABCD* is loaded at joint *D* by a point load F = 60 kN. The cross-sectional areas of the bars are: A_{AD} =325 mm^2 , A_{BD} = 390 mm^2 , and A_{CD} = 420 mm^2 . The modulus of aluminum is E =70 GPa.



Determine

1) the horizontal and vertical displacements of joint *D, u, and v.*
2) The axial stress in members *AD, BD,* and *CD.*

Problem 4. The Fig. shows a freight elevator. The cable (of total length is l and axial rigidity K) passes over a smooth pin C. A crate (of weight W) is suspended at the end of the cable. The axial rigidity EA of the two rods BC and DC is given. Determine the displacements of pin C and of the end of the cable (point H) due to the weight of the crate.



Problem 5. To assemble the truss (axial rigidity EA of the three bars) in Fig. the end point P of bar 2 has to be connected with pin K. Assume $\delta \ll h$. Determine the forces in the bars after the truss has been assembled.