جامعة المنارة

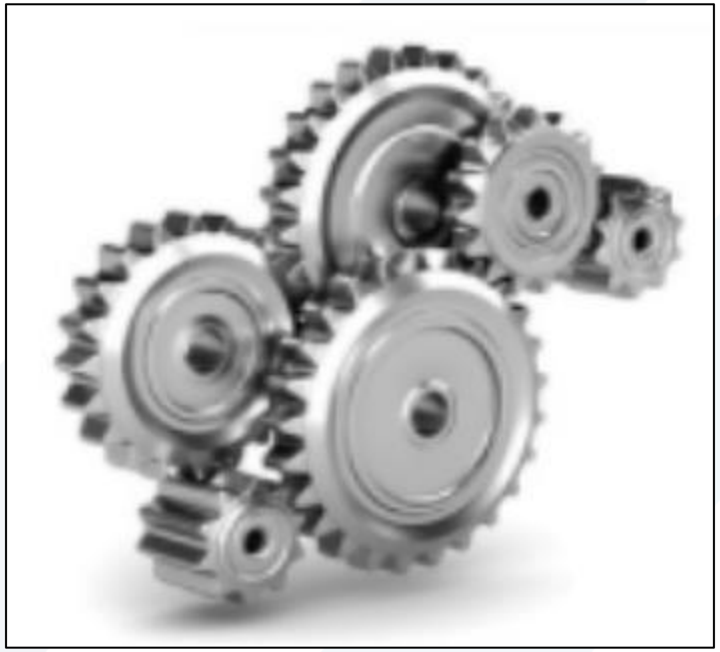
كلية الهندسة

قسم الروبوت و الأنظمة الذكية

مقرر ديناميك الآلات و الاهتزازات الميكانيكية

Planar Kinematics of a Rigid Body

Types of Motion



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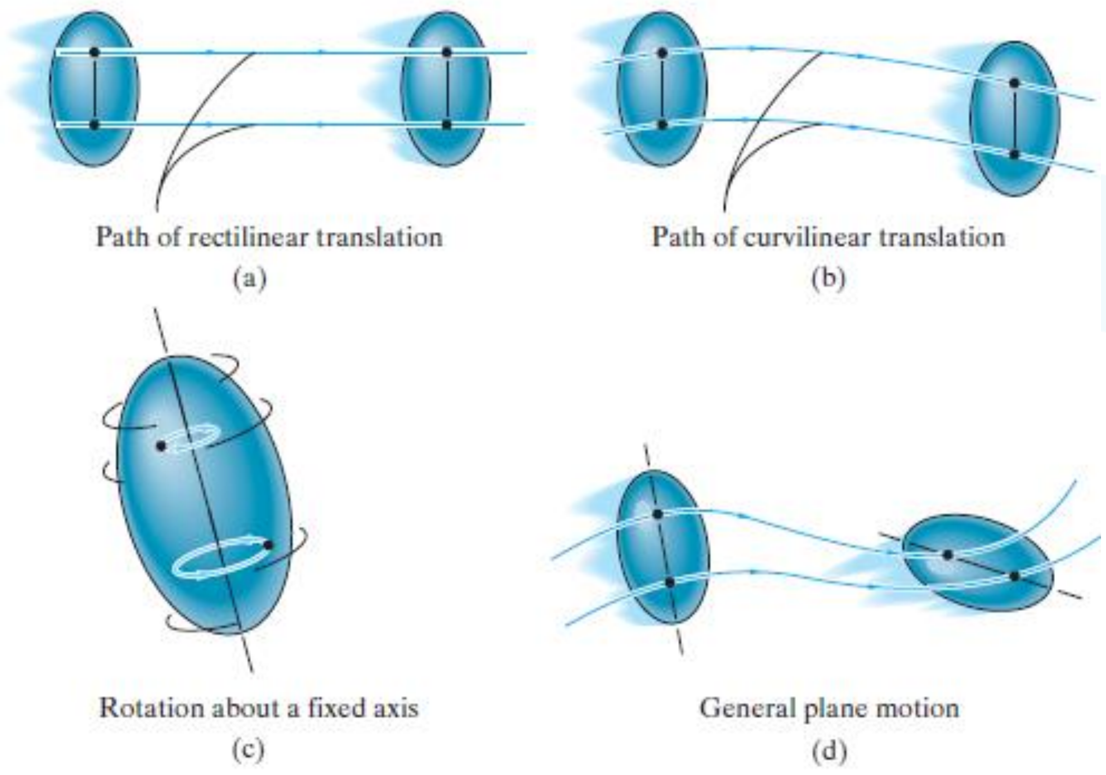
Planar Rigid-Body Motion

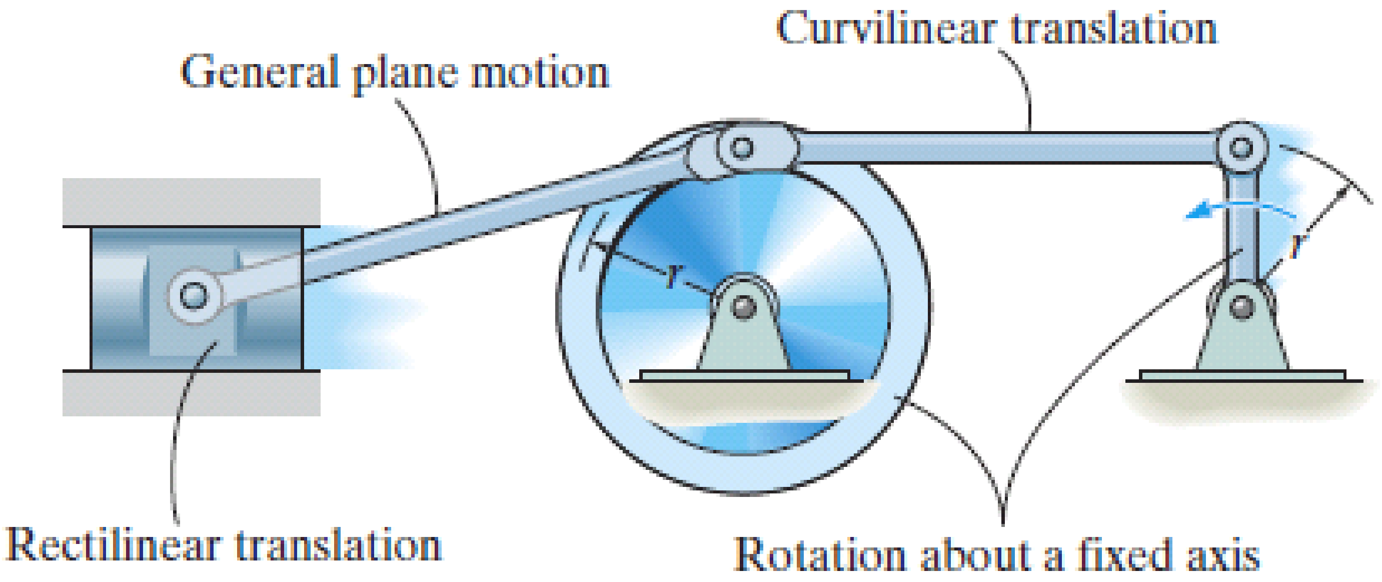
The *planar motion* of a body occurs when all the particles of a rigid body move along paths which are equidistant from a fixed plane. There are three types of rigid-body planar motion.



Translation. *This type of motion occurs when a line in the body* remains parallel to its original orientation throughout the motion. When the paths of motion for any two points on the body are parallel lines, the motion is called *rectilinear translation.* If the paths of motion are along curved lines, the motion is called *curvilinear translation.*

Rotation about a fixed axis. *When a rigid body rotates about a* fixed axis, all the particles of the body, except those which lie on the axis of rotation, move along circular paths.

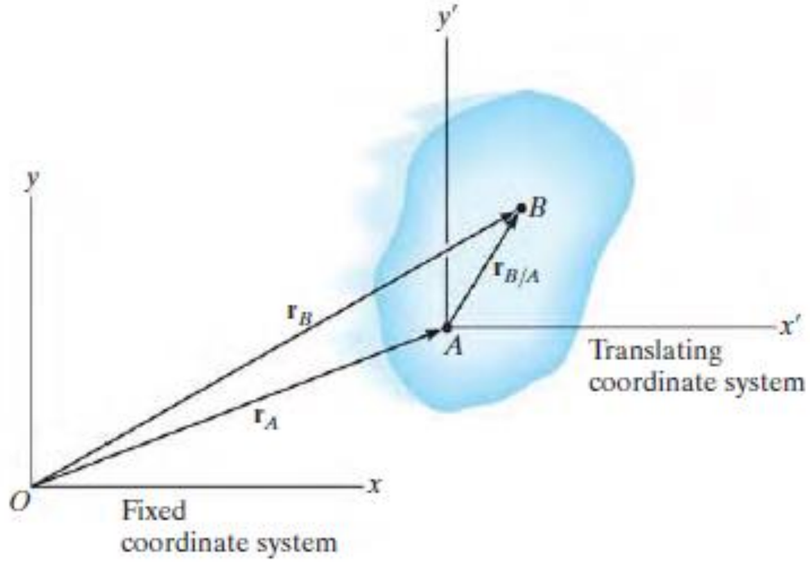
General plane motion. *When a body is subjected to general plane* motion, it undergoes a combination of translation *and rotation. The translation occurs within a reference plane, and the* rotation occurs about an axis perpendicular to the reference plane.


General plane motion
Curvilinear translation
r
r
Rectilinear translation
Rotation about a fixed axis

Translation

Position. The locations of points A and B on the body are defined with respect to fixed x, y reference frame using *position vectors* $\mathbf{r}_A$ and $\mathbf{r}_B$. The translating x', y' coordinate system is *fixed in the body* and has its origin at A, hereafter referred to as the *base point*. The position of B with respect to A is denoted by the *relative-position vector* $\mathbf{r}_{B/A}$ ("$\mathbf{r}$ of B with respect to A"). By vector addition,

$$\mathbf{r}_B = \mathbf{r}_A + \mathbf{r}_{B/A}$$



Velocity. A relation between the instantaneous velocities of A and B is obtained by taking the time derivative of this equation, which yields $\mathbf{v}_B = \mathbf{v}_A + d\mathbf{r}_{B/A}/dt$. Here $\mathbf{v}_A$ and $\mathbf{v}_B$ denote *absolute velocities* since these vectors are measured with respect to the x, y axes. The term $d\mathbf{r}_{B/A}/dt = \mathbf{0}$, since the *magnitude* of $\mathbf{r}_{B/A}$ is *constant* by definition of a rigid body, and because the body is translating the *direction* of $\mathbf{r}_{B/A}$ is also *constant*. Therefore,

$$\mathbf{v}_B = \mathbf{v}_A$$

Acceleration. Taking the time derivative of the velocity equation yields a similar relationship between the instantaneous accelerations of A and B:

$$\mathbf{a}_B = \mathbf{a}_A$$

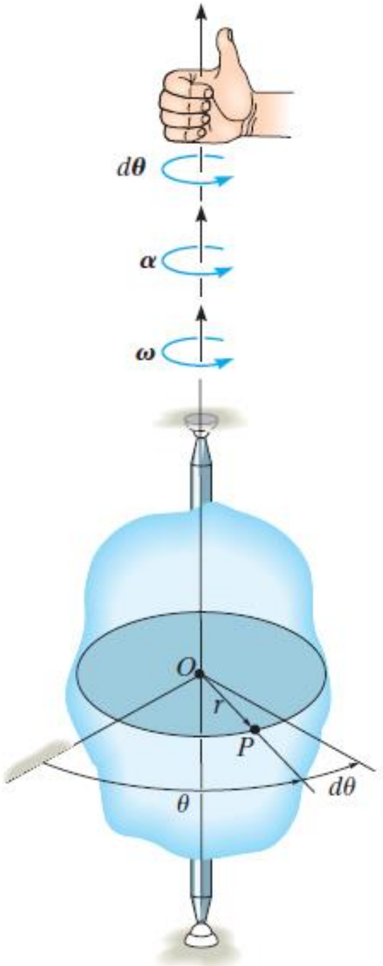
The above two equations indicate that *all points in a rigid body subjected to either rectilinear or curvilinear translation move with the same velocity and acceleration*.

Rotation about a Fixed Axis

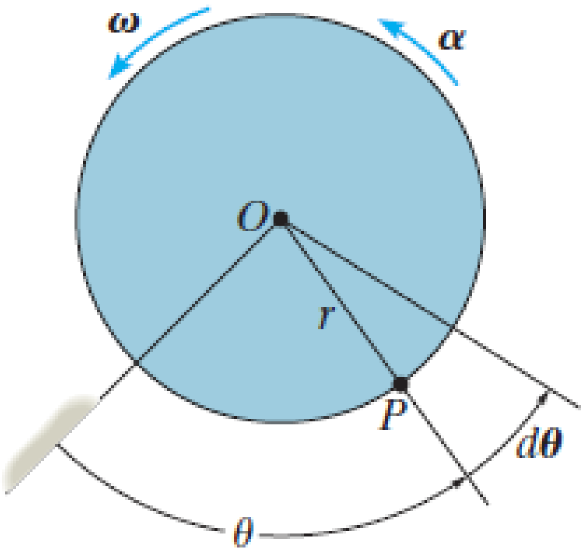
When a body rotates about a fixed axis, any point P located in the body travels along a *circular path*. To study this motion it is first necessary to discuss the angular motion of the body about the axis.

Angular Motion. Since a point is without dimension, it cannot have angular motion. *Only lines or bodies undergo angular motion*. For example, consider the body shown and the angular motion of a radial line r located within the shaded plane.

Angular Position. At the instant shown, the *angular position* of r is defined by the angle θ, measured from a *fixed* reference line to r.



Angular Displacement. The change in the angular position, which can be measured as a differential $d\boldsymbol{\theta}$, is called the *angular displacement*. This vector has a *magnitude* of $d\theta$, measured in degrees, radians, or revolutions, where 1 rev $= 2\pi$ rad. Since motion is about a *fixed axis*, the direction of $d\boldsymbol{\theta}$ is *always* along this axis. Specifically, the *direction* is determined by the right-hand rule; that is, the fingers of the right hand are curled with the sense of rotation, so that in this case the thumb, or $d\boldsymbol{\theta}$, points upward. In two dimensions, as shown by the top view of the shaded plane, both θ and $d\theta$ are counterclockwise, and so the thumb points outward from the page.



Angular Velocity. The time rate of change in the angular position is called the *angular velocity* $\boldsymbol{\omega}$ (omega). Since $d\boldsymbol{\theta}$ occurs during an instant of time dt, then,

$$(\circlearrowleft +) \qquad \omega = \frac{d\theta}{dt}$$

This vector has a *magnitude* which is often measured in rad/s. It is expressed here in scalar form since its *direction* is also along the axis of rotation . When indicating the angular motion in the shaded plane, we can refer to the sense of rotation as clockwise or counterclockwise. Here we have *arbitrarily* chosen counterclockwise rotations as *positive* and indicated this by the curl shown in parentheses next to Eq. Realize, however, that the directional sense of $\boldsymbol{\omega}$ is actually outward from the page.

Angular Acceleration. The *angular acceleration* **α** (alpha) measures the time rate of change of the angular velocity. The *magnitude* of this vector is

$$(\circlearrowleft +) \qquad \alpha = \frac{d\omega}{dt}$$

$$(\circlearrowleft +) \qquad \alpha = \frac{d^2\theta}{dt^2}$$

The line of action of **α** is the same as that for **ω**, however, its sense of *direction* depends on whether **ω** is increasing or decreasing. If **ω** is decreasing, then **α** is called an *angular deceleration* and therefore has a sense of direction which is opposite to **ω**.

By eliminating *dt* we obtain a differential relation between the angular acceleration, angular velocity, and angular displacement, namely,

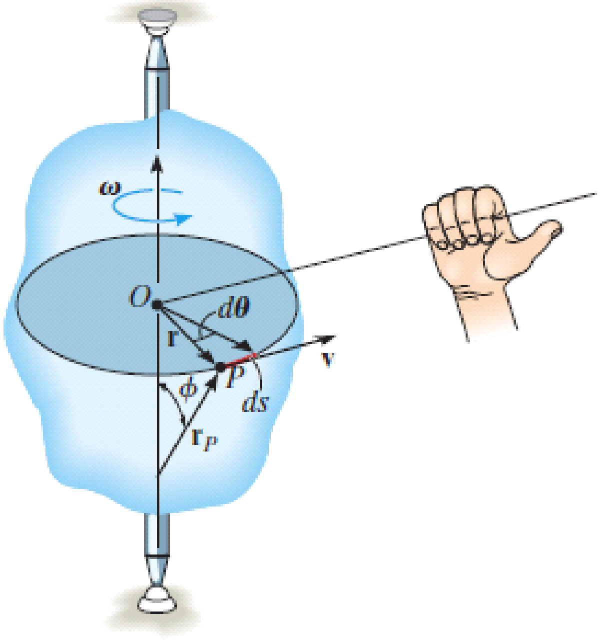
$$(\circlearrowleft +) \qquad \alpha\, d\theta = \omega\, d\omega$$

Constant Angular Acceleration.

$(\circlearrowleft +)$	$\omega = \omega_0 + \alpha_c t$
$(\circlearrowleft +)$	$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha_c t^2$
$(\circlearrowleft +)$	$\omega^2 = \omega_0^2 + 2\alpha_c(\theta - \theta_0)$
	Constant Angular Acceleration

Here θ_0 and ω_0 are the initial values of the body's angular position and angular velocity, respectively.

Motion of Point P. As the rigid body rotates, point P travels along a *circular path* of radius r with center at point O. This path is contained within the shaded plane shown in top view.

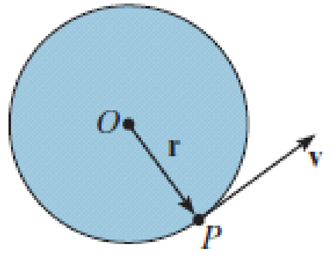


Position and Displacement. The position of P is defined by the position vector **r**, which extends from O to P. If the body rotates $d\theta$ then P will displace $ds = r d\theta$.

Velocity. The velocity of P has a magnitude which can be found by dividing $ds = r d\theta$ by dt so that

$$v = \omega r$$

As shown, the *direction* of **v** is *tangent* to the circular path.



Both the magnitude and direction of **v** can also be accounted for by using the cross product of $\boldsymbol{\omega}$ and $\mathbf{r}_P$. Here, $\mathbf{r}_P$ is directed from *any point* on the axis of rotation to point P. We have

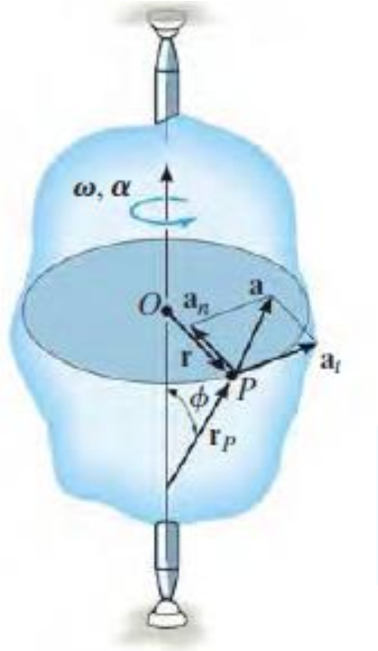
$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}_P$$

The order of the vectors in this formulation is important, since the cross product is not commutative, i.e., $\boldsymbol{\omega} \times \mathbf{r}_P \neq \mathbf{r}_P \times \boldsymbol{\omega}$.
The fingers of the right hand are curled from $\boldsymbol{\omega}$ toward $\mathbf{r}_P$ ($\boldsymbol{\omega}$ "cross" $\mathbf{r}_P$).
The thumb indicates the correct direction of **v**, which is tangent to the path in the direction of motion.

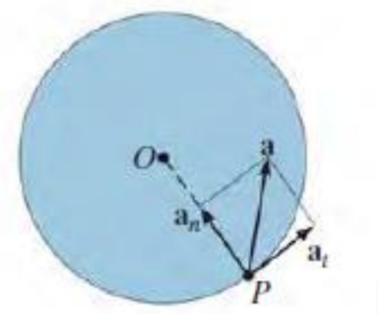
Acceleration. The acceleration of P can be expressed in terms of its normal and tangential components. Applying $a_t = dv/dt$ and $a_n = v^2/\rho$, where $\rho = r$, $v = \omega r$, and $\alpha = d\omega/dt$, we get

$$a_t = \alpha r$$

$$a_n = \omega^2 r$$



The *tangential component of acceleration* represents the time rate of change in the velocity's magnitude. If the speed of P is increasing, then $\mathbf{a}_t$ acts in the same direction as $\mathbf{v}$; if the speed is decreasing, $\mathbf{a}_t$ acts in the opposite direction of $\mathbf{v}$; and finally, if the speed is constant, $\mathbf{a}_t$ is zero.



The *normal component of acceleration* represents the time rate of change in the velocity's direction. The *direction* of $\mathbf{a}_n$ is always toward O, the center of the circular path

Like the velocity, the acceleration of point P can be expressed in terms of the vector cross product.
we have

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d\boldsymbol{\omega}}{dt} \times \mathbf{r}_P + \boldsymbol{\omega} \times \frac{d\mathbf{r}_P}{dt}$$

Recalling that $\boldsymbol{\alpha} = d\boldsymbol{\omega}/dt$, and $(d\mathbf{r}_P/dt = \mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}_P)$yields

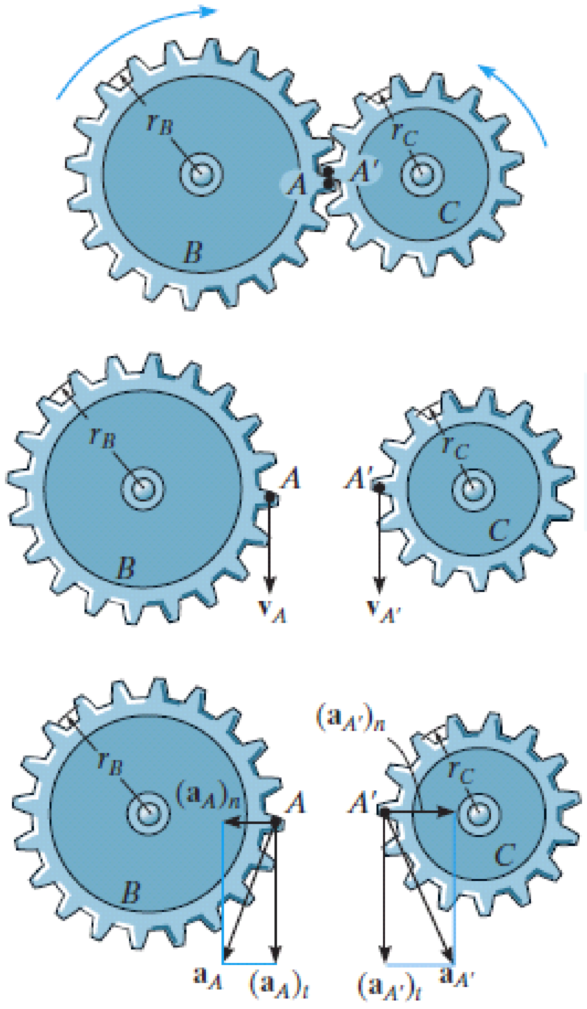
$$\mathbf{a} = \boldsymbol{\alpha} \times \mathbf{r}_P + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_P)$$

From the definition of the cross product, the first term on the right has a magnitude $a_t = \alpha r_P \sin\phi = \alpha r$, and by the right-hand rule, $\boldsymbol{\alpha} \times \mathbf{r}_P$ is in the direction of $\mathbf{a}_t$. Likewise, the second term has a magnitude $a_n = \omega^2 r_P \sin\phi = \omega^2 r$, and applying the right-hand rule twice, first to determine the result $\mathbf{v}_P = \boldsymbol{\omega} \times \mathbf{r}_P$ then $\boldsymbol{\omega} \times \mathbf{v}_P$, it can be seen that this result is in the same direction as $\mathbf{a}_n$, shown. Noting that this is also the *same* direction as $-\mathbf{r}$, which lies in the plane of motion, we can express $\mathbf{a}_n$ in a much simpler form as $\mathbf{a}_n = -\omega^2\mathbf{r}$.

$$\begin{aligned}\mathbf{a} &= \mathbf{a}_t + \mathbf{a}_n \\ &= \boldsymbol{\alpha} \times \mathbf{r} - \omega^2\mathbf{r}\end{aligned}$$

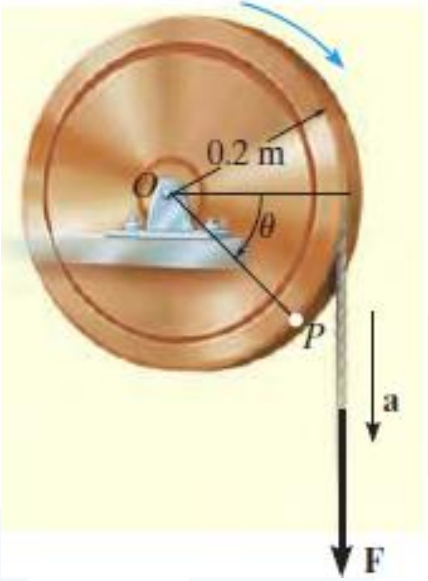
Since $\mathbf{a}_t$ and $\mathbf{a}_n$ are perpendicular to one another, if needed the magnitude of acceleration can be determined from the Pythagorean theorem; namely, $a = \sqrt{a_n^2 + a_t^2}$.

If two rotating bodies contact one another, then the *points in contact* move along *different circular paths*, and the velocity and *tangential components* of acceleration of the points will be the *same*: however, the *normal components* of acceleration will *not* be the same. For example, consider the two meshed gears . Point A is located on gear B and a coincident point A' is located on gear C. Due to the rotational motion, $\mathbf{v}_A = \mathbf{v}_{A'}$, and as a result, $\omega_B r_B = \omega_C r_C$ or $\omega_B = \omega_C(r_C/r_B)$.
Also, $(\mathbf{a}_A)_t = (\mathbf{a}_{A'})_t$, so that $\alpha_B = \alpha_C(r_C/r_B)$; however, since both points follow different circular paths, $(\mathbf{a}_A)_n \neq (\mathbf{a}_{A'})_n$ and therefore, as shown, $\mathbf{a}_A \neq \mathbf{a}_{A'}$.



EXAMPLE

A cord is wrapped around a wheel, which is initially at rest when $\theta = 0$. If a force is applied to the cord and gives it an acceleration $a = (4t)\ \mathrm{m/s^2}$, where t is in seconds, determine, as a function of time, (a) the angular velocity of the wheel, and (b) the angular position of line OP in radians.



SOLUTION

Part (a). The wheel is subjected to rotation about a fixed axis passing through point O. Thus, point P on the wheel has motion about a circular path, and the acceleration of this point has *both* tangential and normal components. The tangential component is $(a_P)_t = (4t)\ \mathrm{m/s^2}$, since the cord is wrapped around the wheel and moves *tangent* to it. Hence the angular acceleration of the wheel is

$$(\circlearrowright +) \qquad (a_P)_t = \alpha r$$

$$(4t)\ \text{m/s}^2 = \alpha(0.2\ \text{m})$$

$$\alpha = (20t)\ \text{rad/s}^2 \circlearrowright$$

Using this result, the wheel's angular velocity ω can now be determined from $\alpha = d\omega/dt$, since this equation relates α, t, and ω. Integrating, with the initial condition that $\omega = 0$ when $t = 0$, yields

$$(\circlearrowright +) \qquad \alpha = \frac{d\omega}{dt} = (20t)\ \text{rad/s}^2$$

$$\int_0^{\omega} d\omega = \int_0^{t} 20t\, dt$$

$$\omega = 10t^2\ \text{rad/s} \circlearrowright \qquad \textit{Ans.}$$

Part (b). Using this result, the angular position θ of OP can be found from $\omega = d\theta/dt$, since this equation relates θ, ω, and t. Integrating, with the initial condition $\theta = 0$ when $t = 0$, we have

$$(\circlearrowright +) \qquad \frac{d\theta}{dt} = \omega = (10t^2)\ \text{rad/s}$$

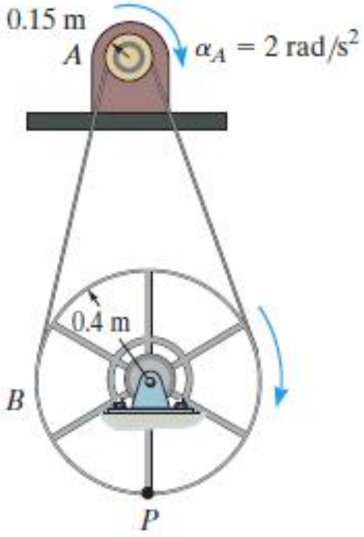
$$\int_0^\theta d\theta = \int_0^t 10t^2\, dt$$

$$\theta = 3.33t^3\ \text{rad}$$ *Ans.*

NOTE: We cannot use the equation of constant angular acceleration, since α is a function of time.

EXAMPLE

The motor shown in the photo is used to turn a wheel and attached blower contained within the housing.
If the pulley A connected to the motor begins to rotate from rest with a constant angular acceleration of $\alpha_A = 2 \text{ rad/s}^2$, determine the magnitudes of the velocity and acceleration of point P on the wheel, after the pulley has turned two revolutions. Assume the transmission belt does not slip on the pulley and wheel.



SOLUTION

Angular Motion. First we will convert the two revolutions to radians. Since there are 2π rad in one revolution, then

$$\theta_A = 2 \text{ rev}\left(\frac{2\pi \text{ rad}}{1 \text{ rev}}\right) = 12.57 \text{ rad}$$

Since α_A is constant, the angular velocity of pulley A is therefore

$$(\circlearrowright +) \qquad \omega^2 = \omega_0^2 + 2\alpha_c(\theta - \theta_0)$$

$$\omega_A^2 = 0 + 2(2 \text{ rad/s}^2)(12.57 \text{ rad} - 0)$$

$$\omega_A = 7.090 \text{ rad/s}$$

The belt has the same speed and tangential component of acceleration as it passes over the pulley and wheel. Thus,

$$v = \omega_A r_A = \omega_B r_B; \quad 7.090 \text{ rad/s } (0.15 \text{ m}) = \omega_B (0.4 \text{ m})$$

$$\omega_B = 2.659 \text{ rad/s}$$

$$a_t = \alpha_A r_A = \alpha_B r_B; \quad 2 \text{ rad/s}^2 \, (0.15 \text{ m}) = \alpha_B (0.4 \text{ m})$$

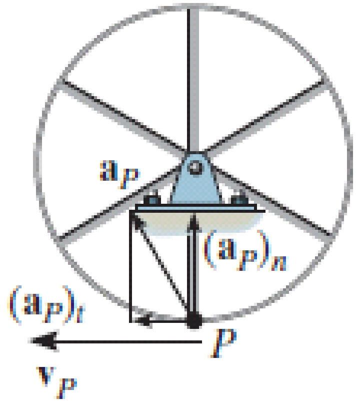
$$\alpha_B = 0.750 \text{ rad/s}^2$$

Motion of *P*. As shown on the kinematic diagram we have

$$v_P = \omega_B r_B = 2.659 \text{ rad/s } (0.4 \text{ m}) = 1.06 \text{ m/s} \qquad \textit{Ans.}$$

$$(a_P)_t = \alpha_B r_B = 0.750 \text{ rad/s}^2 (0.4 \text{ m}) = 0.3 \text{ m/s}^2$$

$$(a_P)_n = \omega_B^2 r_B = (2.659 \text{ rad/s})^2 (0.4 \text{ m}) = 2.827 \text{ m/s}^2$$

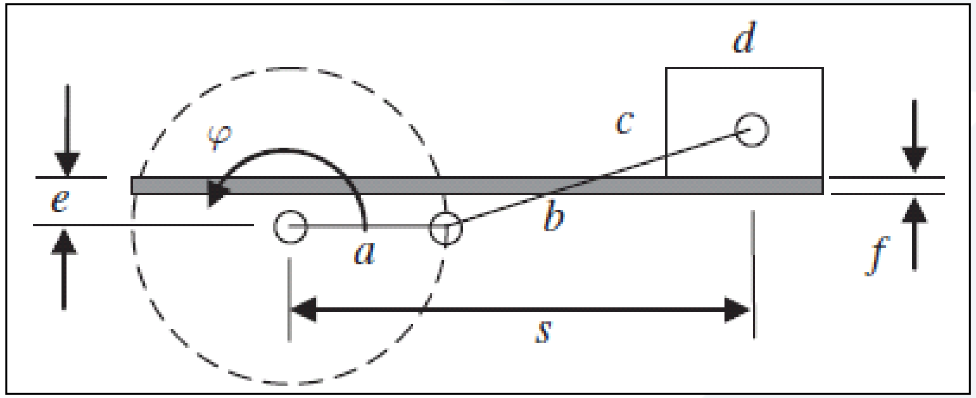


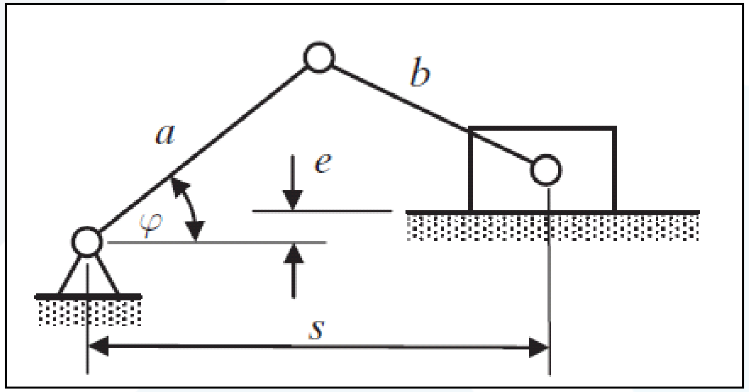
Thus

$$a_P = \sqrt{(0.3 \text{ m/s}^2)^2 + (2.827 \text{ m/s}^2)^2} = 2.84 \text{ m/s}^2 \qquad \textit{Ans.}$$

EXAMPLE

Animation of a slider–crank mechanism using Matlab

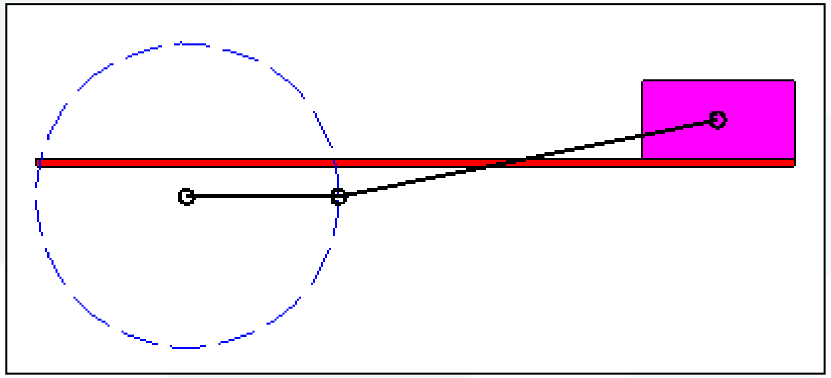




$$s = a\cos(\varphi) + \sqrt{b^2 - (a\sin(\varphi) - e)^2}$$

We shall assume the following numerical values: $0 \le \varphi \le 2\pi$ at $n = 40$ equally spaced positions, $a = 1$, $b = 2.5$, $e = 0.25$, $c = 0.5$, $d = 1$, and $f = 0.06$.

```
clc
clear
n = 40;
phi = linspace(0, 2*pi, n);
a = 1;
b = 2.5;
e = 0.25;
nF = 5;
c = 0.5;
d = 1;
f = 0.06;
ax = a*cos(phi);
ay = a*sin(phi);
s = ax+sqrt(b^2-(ay-e).^2);
v = [1.1*min(ax), 1.1*(max(s)+d/2) 1.1*min(ay), 1.1*max(ay)];
xgnd = [min(ax),min(ax),max(s)+d/2,max(s)+d/2, min(ax)];
ygnd = [e-f,e, e, e-f, e-f];
slidery = [e, e+c, e+c, e,e]; % Vertical component of slider is constant
for k = 1:n
    fill(xgnd, ygnd, 'r') % Thin horizontal bar
    hold on
    plot(ax, ay, 'b--', 0, 0, 'ko'); % Dashed circle and center of circle
    sliderx = [s(k)-d/2, s(k)-d/2, s(k)+d/2, s(k)+d/2,s(k)-d/2];
    fill(sliderx, slidery, 'm'); % Slider position
    plot([0 ax(k)], [0 ay(k)], 'ko-', 'LineWidth', 2);
    plot([ax(k), s(k)], [ay(k), e+c/2], 'ko-', 'LineWidth', 2);
    axis(v)
    axis off equal
    SliderCrankFrame(k) = getframe;
    hold off
end
movie(SliderCrankFrame,nF)
```



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