

## Modeling of Rigid-Body Mechanical Systems Part 1





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## MECHANICAL ENERGY

Conservation of mechanical energy is a direct consequence of Newton's second law. Consider the scalar case, where the force  $f$  can be a function of displacement  $x$ .

$$m\dot{v} = f(x)$$

Multiply both sides by  $v dt$  and use the fact that  $v = dx/dt$ .

$$mv dv = v f(x) dt = \frac{dx}{dt} f(x) dt = f(x) dx$$

Integrate both sides.

$$\int mv dv = \frac{mv^2}{2} = \int f(x) dx + C$$

where  $C$  is a constant of integration.

Work is force times displacement, so the integral on the right represents the total work done on the mass by the force  $f(x)$ . The term on the left-hand side of the equal sign is called the *kinetic energy* (KE).

If the work done by  $f(x)$  is independent of the path and depends only on the end points, then the force  $f(x)$  is derivable from a function  $V(x)$  as follows:

$$f(x) = -\frac{dV}{dx}$$

Then, in this case,  $f(x)$  is called a *conservative force*. If we integrate both sides of the last equation, we obtain

$$V(x) = \int dV = - \int f(x) dx$$

or

$$\frac{mv^2}{2} + V(x) = C$$

This equation shows that  $V(x)$  has the same units as kinetic energy.  $V(x)$  is called the *potential energy (PE) function*.

Equation states that the sum of the kinetic and potential energies must be constant, if no force other than the conservative force is applied. If  $v$  and  $x$  have the values  $v_0$  and  $x_0$  at the time  $t_0$ , then

$$\frac{mv_0^2}{2} + V(x_0) = C$$

Comparing gives

$$\frac{mv^2}{2} - \frac{mv_0^2}{2} + V(x) - V(x_0) = 0$$

which can be expressed as

$$\Delta KE + \Delta PE = 0$$

where the change in kinetic energy is  $\Delta KE = m(v^2 - v_0^2)/2$  and the change in potential energy is  $\Delta PE = V(x) - V(x_0)$ . For some problems, the following form of the principle is more convenient to use:

$$\frac{mv_0^2}{2} + V(x_0) = \frac{mv^2}{2} + V(x)$$

In the form , we see that conservation of mechanical energy states that the change in kinetic energy plus the change in potential energy is zero. Note that the potential energy has a relative value only. The choice of reference point for measuring  $x$  determines only the value of  $C$ , which shows to be irrelevant.

Gravity is an example of a conservative force, for which  $f = -mg$ . The gravity force is conservative because the work done lifting an object depends only on the change in height and not on the path taken. Thus, if  $x$  represents vertical displacement,

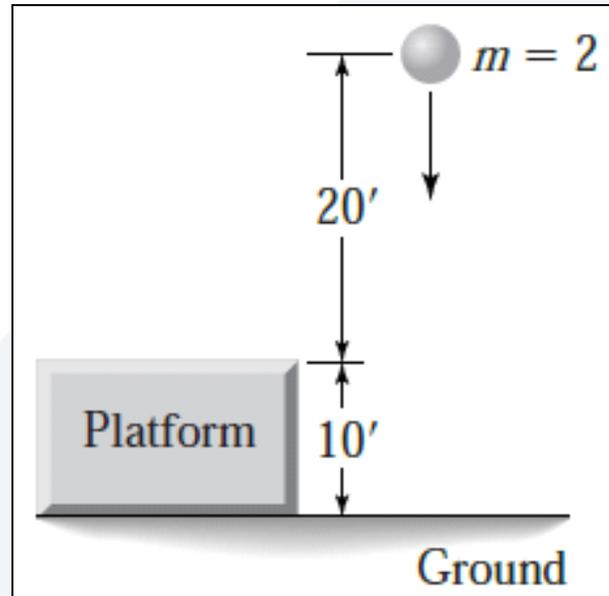
$$V(x) = mgx$$

and

$$\frac{mv^2}{2} + mgx = C$$
$$\frac{mv^2}{2} - \frac{mv_0^2}{2} + mg(x - x_0) = 0$$

**EXAMPLE**

An object with a mass of  $m = 2$  slugs drops from a height of 30 ft above the ground. Determine its speed after it drops 20 ft to a platform that is 10 ft above the ground.



### Solution

Measuring  $x$  from the ground gives  $x_0 = 30$  ft and  $x = 10$  ft at the platform. If the object is dropped from rest, then  $v_0 = 0$ .

$$\frac{m}{2}(v^2 - 0) + mg(10 - 30) = 0$$

or  $v^2 = 40g$ . Using  $g = 32.2$  ft/sec<sup>2</sup>, we obtain the speed of the object when it reaches the platform.

Note that if we had chosen to measure  $x$  from the platform instead of the ground, then  $v_0 = 0$ ,  $x_0 = 20$ , and  $x = 0$  at the platform.

$$\frac{m}{2}(v^2 - 0) + mg(0 - 20) = 0$$

or  $v^2 = 40g$ , which gives the same answer as before.

## DRY FRICTION FORCE

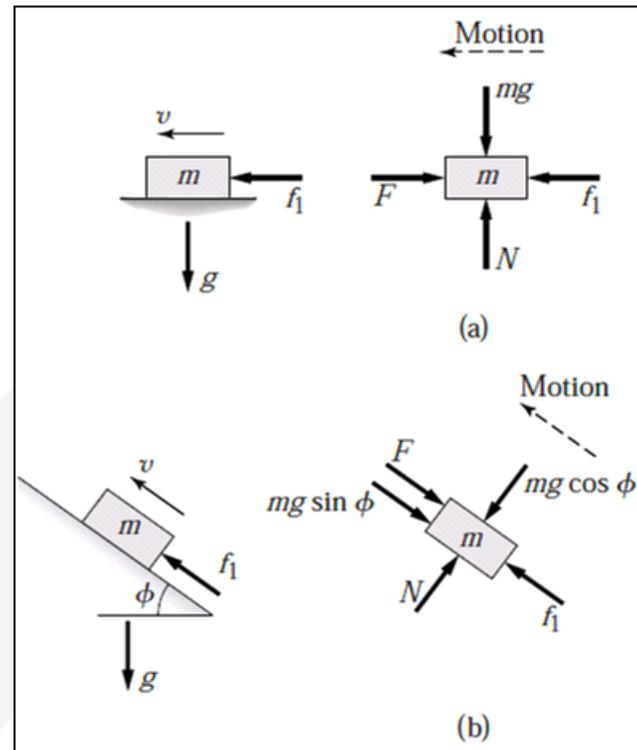
Not every constant force is conservative. A common example of a *non-conservative* force is the *dry* friction force. This force is non-conservative because the work done by the force depends on the path taken. The dry friction force  $F$  is directly proportional to the force  $N$  normal to the frictional surface. Thus  $F = \mu N$ . The proportionality constant is  $\mu$ , the *coefficient of friction*.

The dry friction force that exists before motion begins is called *static friction* (sometimes shortened to *stiction*). The static friction coefficient has the value  $\mu_s$  to distinguish it from the *dynamic* friction coefficient  $\mu_d$ , which describes the friction after motion begins. Dynamic friction is also called *sliding* friction, *kinetic* friction, or *Coulomb* friction. In general,  $\mu_s > \mu_d$ , which explains why it is more difficult to start an object sliding than to keep it moving. We will use the symbol  $\mu$  rather than  $\mu_d$  because most of our applications involve motion.

Because Coulomb friction cannot be derived from a potential energy function, the conservation of mechanical energy principle does not apply. This makes sense physically because the friction force dissipates the energy as heat, and thus mechanical energy, which consists of kinetic plus potential energy, is not conserved. Total energy, of course, is conserved.

**EXAMPLE**

Derive the equation of motion (a) for the block of mass  $m$  shown in Figure a and (b) for the mass  $m$  on an incline, shown in Figure b. In both cases, a force  $f_1$ , which is not the friction force, is applied to move the mass.



### Solution

- a. The free body diagrams are shown in Figure a  
The normal force  $N$  is the weight  $mg$ . Thus the friction force  $F$  is  $\mu N$ , or  $F = \mu mg$ . If  $v > 0$ , the equation of motion is

$$m\dot{v} = f_1 - \mu mg$$

- b. The friction force depends on the force normal to the surface. For the mass  $m$  shown in Figure b, the normal force  $N$  must equal  $mg \cos \phi$  as long as the mass is in contact with the surface. The free body diagrams are shown in the figure

$$m\dot{v} = f_1 - mg \sin \phi - \mu mg \cos \phi$$

### EXAMPLE

For the mass shown in Figure b,  $m = 2 \text{ kg}$ ,  $\phi = 30^\circ$ ,  $v(0) = 3 \text{ m/s}$ , and  $\mu = 0.5$ . Determine whether the mass comes to rest if (a)  $f_1 = 50 \text{ N}$  and (b)  $f_1 = 5 \text{ N}$ .

### Solution

Because the velocity is initially positive [ $v(0) = 3$ ],

$$2\dot{v} = f_1 - (\sin 30^\circ + 0.5 \cos 30^\circ)(2)(9.81) = f_1 - 18.3$$

For part (a),  $f_1 = 50$  and thus  $\dot{v} = (50 - 18.3)/2 = 15.85$  and the acceleration is positive. Thus, because  $v(0) > 0$ , the speed is always positive for  $t \geq 0$  and the mass never comes to rest.

For part (b),  $f_1 = 5$ ,  $\dot{v} = (5 - 18.3)/2 = -6.65$ , and thus the mass is decelerating. Because  $v(t) = -6.65t + 3$ , the speed becomes zero at  $t = 3/6.65 = 0.45 \text{ s}$ .

## ROTATION ABOUT A FIXED AXIS

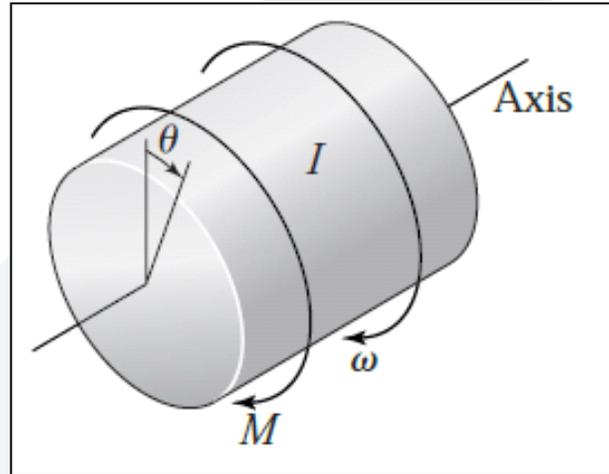
In this section, we consider the dynamics of rigid bodies whose motion is constrained to allow only rotation about an axis through a nonaccelerating point.

For planar motion, which means that the object can translate in two dimensions and can rotate only about an axis that is perpendicular to the plane, Newton's second law can be used to show that

$$I_O \dot{\omega} = M_O$$

where  $\omega$  is the *angular velocity* of the mass about an axis through a point  $O$ ,  $I_O$  is the *mass moment of inertia* of the body about the point  $O$ , and  $M_O$  is the sum of the moments applied to the body about the point  $O$ .

This situation is illustrated in the Figure . The angular displacement is  $\theta$ , and  $\dot{\theta} = \omega$ . The term *torque* and the symbol  $T$  are often used instead of *moment* and  $M$ . Also, when the context is unambiguous, we use the term “inertia” as an abbreviation for “mass moment of inertia.”



## CALCULATING INERTIA

The mass moment of inertia  $I$  about a specified reference axis is defined as

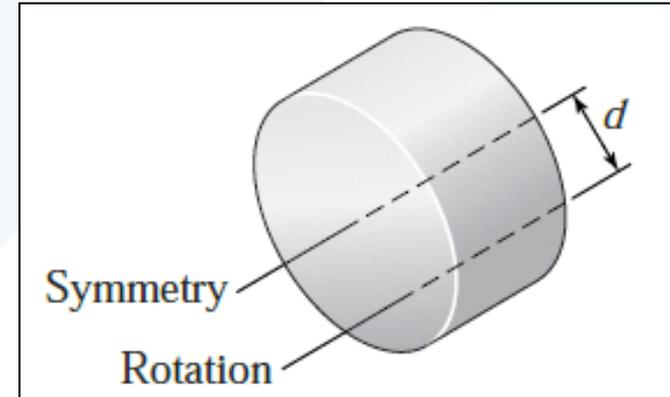
$$I = \int r^2 dm$$

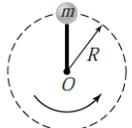
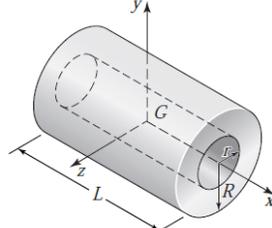
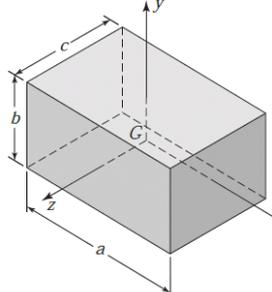
where  $r$  is the distance from the reference axis to the mass element  $dm$ . The expressions for  $I$  for some common shapes are given in Table

If the rotation axis of a homogeneous rigid body does not coincide with the body's axis of symmetry, but is parallel to it at a distance  $d$ , then the mass moment of inertia about the rotation axis is given by the *parallel-axis theorem*,

$$I = I_s + md^2$$

where  $I_s$  is the inertia about the symmetry axis



Sphere	$I_G = \frac{2}{5}mR^2$
	
Mass rotating about point $O$	$I_O = mR^2$
	
Hollow cylinder	$I_x = \frac{1}{2}m(R^2 + r^2)$
	$I_y = I_z = \frac{1}{12}m(3R^2 + 3r^2 + L^2)$
	
Rectangular prism	$I_x = \frac{1}{12}m(b^2 + c^2)$
	

## ENERGY AND ROTATIONAL MOTION

The work done by a moment  $M$  causing a rotation through an angle  $\theta$  is

$$W = \int_0^{\theta} M d\theta$$

Multiply both sides of  $I_O \dot{\omega} = M_O$  by  $\omega dt$ , and note that  $\omega = d\theta/dt$ .

$$I\omega d\omega = M d\theta$$

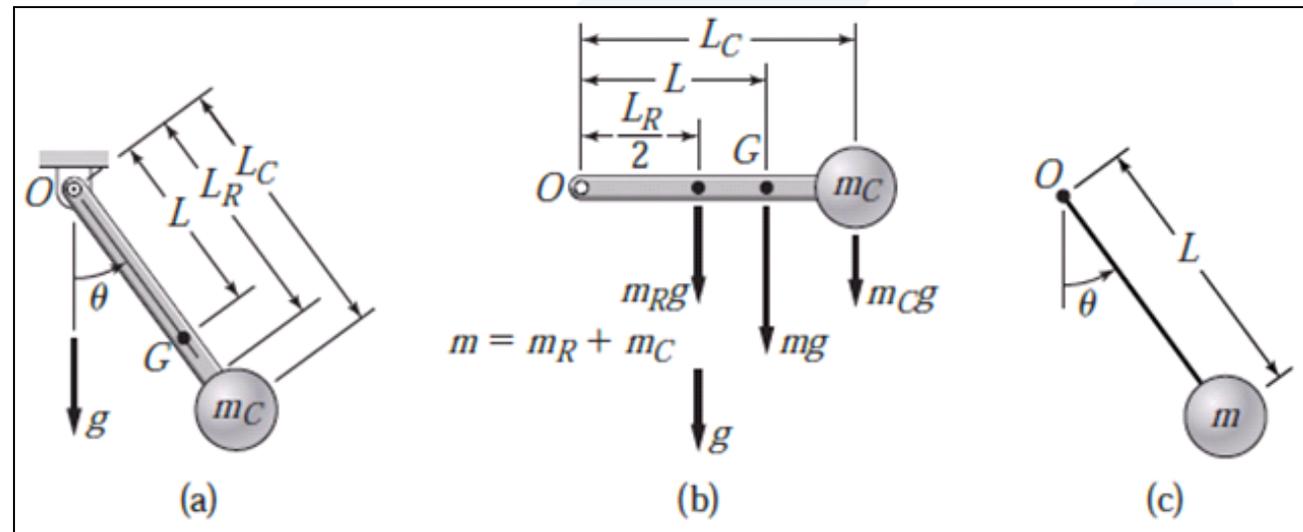
Integrating both sides gives

$$\int_0^{\omega} I\omega d\omega = \frac{1}{2} I\omega^2 = \int_0^{\theta} M d\theta$$

We thus see that the work done by the moment  $M$  produces the kinetic energy of rotation:  
 $KE = I\omega^2/2$ .

### EXAMPLE

The pendulum shown in Figure a consists of a concentrated mass  $m_C$  (the bob) a distance  $L_C$  from point  $O$ , attached to a rod of length  $L_R$  and inertia  $I_{RG}$  about its mass center. (a) Obtain its equation of motion. (b) Discuss the case where the rod's mass  $m_R$  is small compared to the concentrated mass. (c) Determine the equation of motion for small angles  $\theta$ .



### Solution

- a. For the pendulum shown in Figure a, the inertia of the concentrated mass  $m_C$  about point  $O$  is  $m_C L_C^2$  (see Table ). From the parallel-axis theorem, the rod's inertia about point  $O$  is

$$I_{RO} = I_{RG} + m_R \left( \frac{L_R}{2} \right)^2$$

and thus the entire pendulum's inertia about point  $O$  is

$$I_O = I_{RO} + m_C L_C^2 = I_{RG} + m_R \left( \frac{L_R}{2} \right)^2 + m_C L_C^2$$

With the moment equation about point  $O$ , the moment  $M_O$  is caused by the perpendicular component of the weight  $mg$  acting through the mass center at  $G$ . Thus the desired equation of motion is

$$I_O \ddot{\theta} = -mgL \sin \theta \quad (1)$$

The distance  $L$  between point  $O$  and the mass center  $G$  of the entire pendulum is not given, but can be calculated as follows. If the entire pendulum mass were concentrated at  $G$ , the weight force would produce the same moment about point  $O$  as the real pendulum. Thus, taking moments about point  $O$ , we have

$$mgL = m_C g L_C + m_R g \frac{L_R}{2}$$

where  $m = m_C + m_R$ . Solve for  $L$  to obtain

$$L = \frac{m_C L_C + m_R (L_R/2)}{m_C + m_R} \quad (2)$$

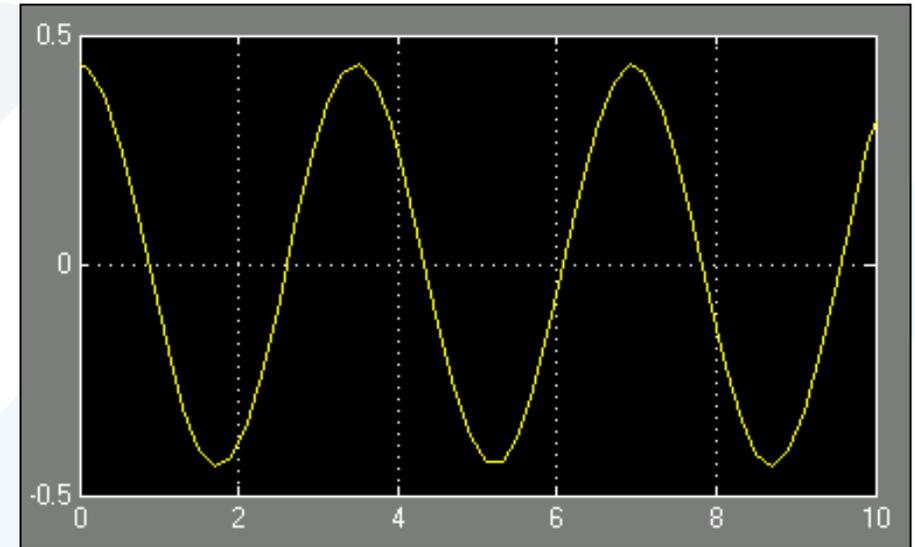
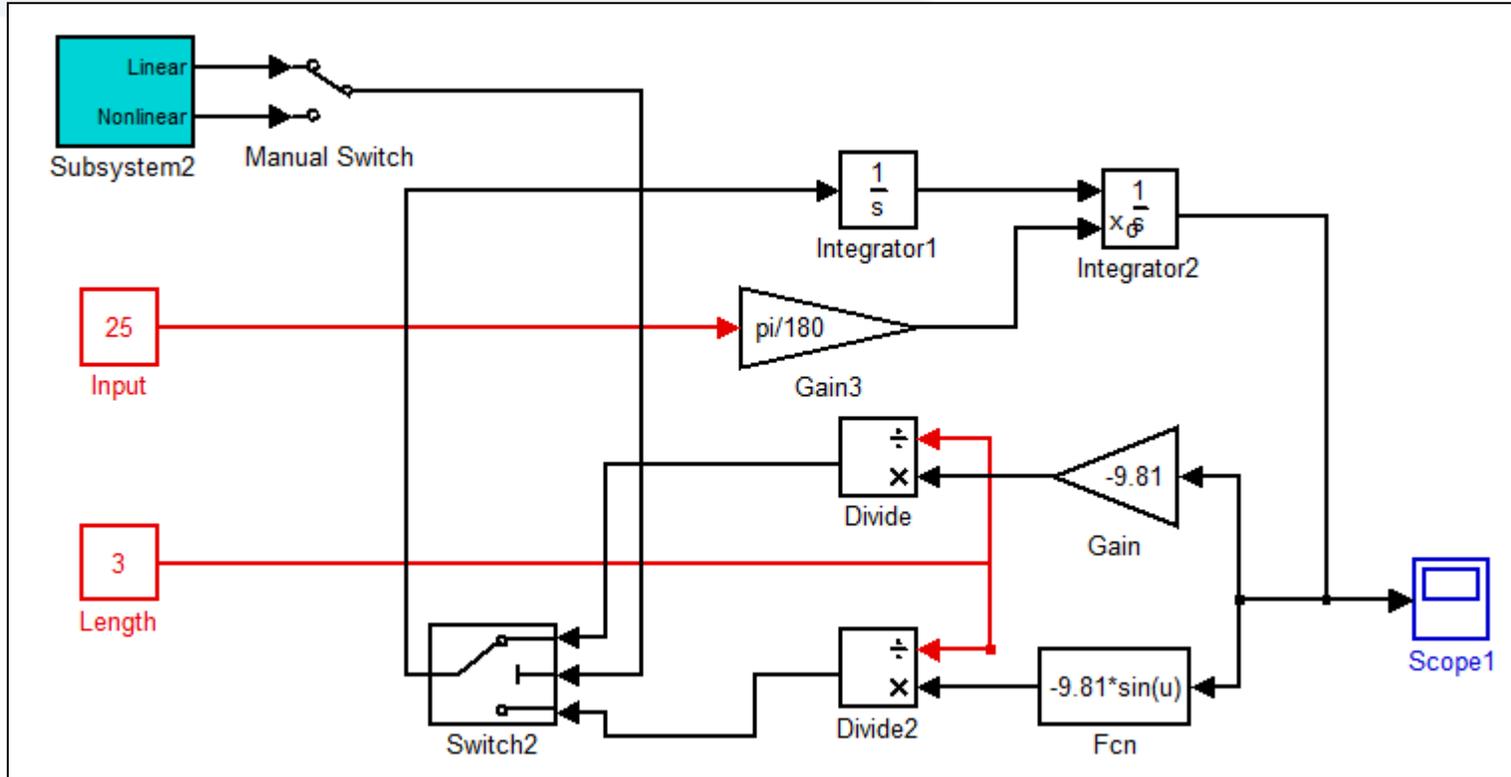
- b. If we neglect the rod's mass  $m_R$  compared to the concentrated mass  $m_C$ , then we can take  $m_R = I_{RG} = 0$ ,  $m = m_C$ ,  $L = L_C$ , and  $I_O = mL^2$ . In this case, the equation of motion reduces to

$$L\ddot{\theta} + g \sin \theta = 0 \quad (3)$$

This is a model for a pendulum whose mass is concentrated at a distance  $L$  from the pivot point, like that shown in Figure. Note that this equation of motion is independent of the value of  $m$ .

- c. For small angles,  $\sin \theta \approx \theta$  if  $\theta$  is in radians. Substituting this approximation into equation (3) gives

$$L\ddot{\theta} + g\theta = 0 \quad (4)$$



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