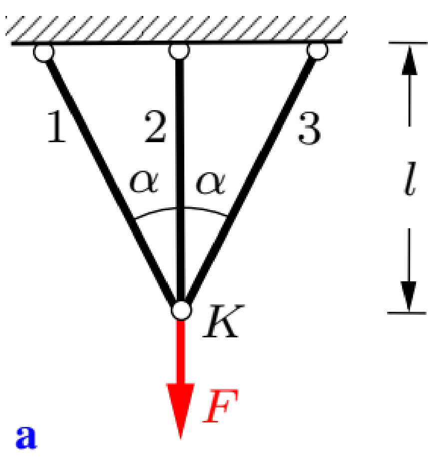
6 Statically Indeterminate Systems of Bars

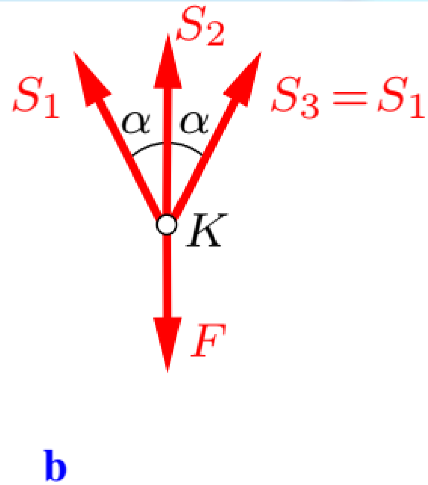
We will now investigate statically indeterminate systems for which the forces in the bars cannot be determined with the aid of the equilibrium conditions alone since the number of the unknown quantities exceeds the number of the equilibrium conditions.

In such systems the basic equations (1) Equilibrium conditions. (2) Kinematic equations (compatibility) & (3) Material behavior(Hooke's law), are coupled.



Let us consider the symmetrical truss shown in (Fig.a) It is stress-free before the load is applied. The axial rigidities $EA_1, EA_2, EA_3 = EA_1$ are given; the forces in the members are unknown.

The system is statically indeterminate to the first degree: The two equilibrium conditions applied to the free-body diagram of pin K (Fig.b) yield



$$\rightarrow: -S_1 \sin\alpha + S_3 \sin\alpha = 0 \;\Rightarrow\; S_1 = S_3,$$

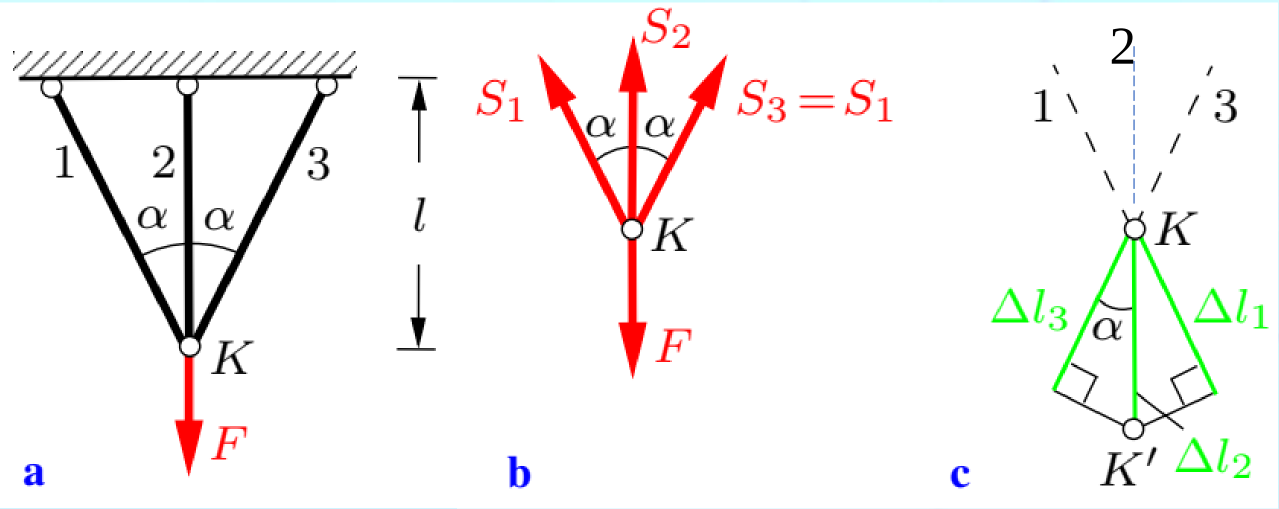
$$\uparrow: \; S_1 \cos\alpha + S_2 + S_3 \cos\alpha - F = 0 \Rightarrow S_2 + 2S_1 \cos\alpha = F$$

Number of static (force) unknowns is 3. Number of equilibrium equations is 2. So the number of indeterminacy is: 3-2=1.

The system is indeterminate to the first degree.

Hook's law gives the elongations of the bars by:

$$\Delta l_1 \equiv \Delta l_3 = \frac{l_1}{EA_1} S_1 \quad \& \quad \Delta l_2 = \frac{l_2}{EA_2} S_2$$



The Kinematic (compatibility) condition is found by the displacement diagram (Fig. c):

$$\Delta l_1 \equiv \Delta l_3 = \Delta l_2 \cos\alpha$$

Substituting the material equations (Hook's law) in this compatibility equation, we write it in terms of the unknowns forces

$$\frac{l_1}{EA_1} S_1 = \frac{l_2}{EA_2} S_2 \cos\alpha$$

And the geometric evidence: $l_1 = l/\cos\alpha$ and $l_2 = l$

$$\Rightarrow S_1 = \frac{A_1}{A_2} \cos^2\alpha \, S_2$$

With the combination of the two equilibrium equations

$$S_2 + 2S_1 \cos\alpha = F$$

We obtain the three unknowns forces, then the elongations and displacements of K.

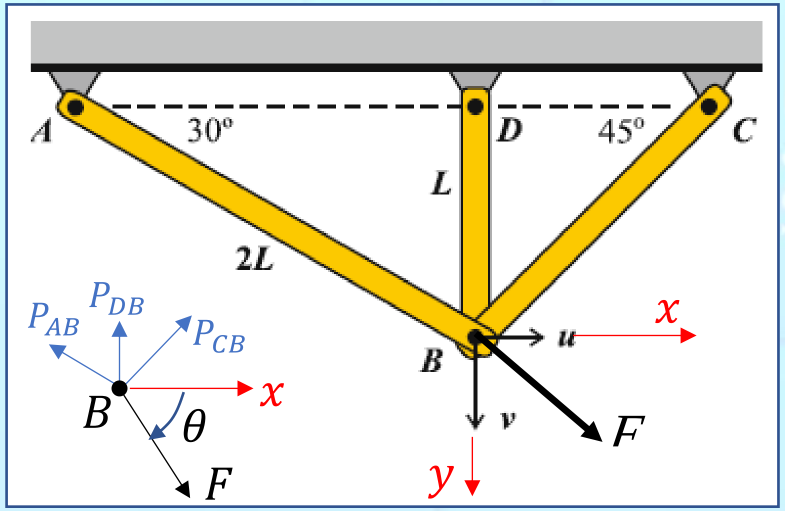
$$S_2 = \frac{A_2}{A_2 + 2A_1\cos^3\alpha} F \;\&\; S_1 = S_3 = \frac{A_1\cos^2\alpha}{A_2 + 2A_1\cos^3\alpha} F \;\&\; v_K = \Delta l_2 = \frac{l_2}{EA_2} S_2 = \frac{l}{E(A_2 + 2A_1\cos^3\alpha)} F$$

Problem 1. An aluminum truss is shown in the figure. Joint B is forced to move downward by $v = 2.0$ mm; and to the right by $u = 1$ mm. Length $L = 1$ m and the cross-sectional area of each bar is $A = 8 \times 10^{-4}\ m^2$. The modulus is $E = 70$ GPa. Determine

(a) The elongation Δ in each member.

(b) The force P in each member.

(c) The components, F_x & F_y, of the applied force that causes the displacement, and its magnitude F.



Solution:

(a) The elongations in the members

$$\Delta_{AB} = (u_B - u_A)\cos\theta_{AB} + (v_B - v_A)\sin\theta_{AB} = (u-0)\cos 30^\circ + (v-0)\sin 30^\circ = (1)\left(\sqrt{3}/2\right) + (2)(1/2) = 1.866 \text{ mm}$$

$$\Delta_{DB} = (u_B - u_D)\cos\theta_{DB} + (v_B - v_D)\sin\theta_{DB} = (u-0)\cos 90^\circ + (v-0)\sin 90^\circ = (1)(0) + (2)(1) = 2 \text{ mm}$$

$$\Delta_{CB} = (u_B - u_C)\cos\theta_{CB} + (v_B - v_C)\sin\theta_{CB} = (u-0)\cos 135^\circ + (v-0)\sin 135^\circ = -(1)\left(\sqrt{2}/2\right) + (2)\left(\sqrt{2}/2\right) = 0.707 \text{ mm}$$

(b) The Forces in the members

$$P_{AB} = \frac{EA}{L_{AB}}\Delta_{AB} = \frac{70\times10^6\times8\times10^{-4}}{2}\times1.866\times10^{-3} = 52.2 \text{ kN}$$

$$P_{DB} = \frac{EA}{L_{DB}}\Delta_{DB} = \frac{70\times10^6\times8\times10^{-4}}{1}\times2\times10^{-3} = 112 \text{ kN}$$

$$P_{CB} = \frac{EA}{L_{CB}}\Delta_{CB} = \frac{70\times10^6\times8\times10^{-4}}{\sqrt{2}}\times0.707\times10^{-3} = 28 \text{ kN}$$

(c) The force F, causing the displacements of B.

Eq. Eqs. of joint B:

$$\rightarrow: -P_{AB}\cos 30^\circ + 0 + P_{CB}\cos 45^\circ + F_x = 0 \quad \Rightarrow \quad F_x = 25.4 \text{ kN}$$

$$\downarrow: -P_{AB}\sin 30^\circ - P_{DB} - P_{CB}\sin 45^\circ + F_y = 0 \Rightarrow \quad F_y = 158 \text{ kN}$$

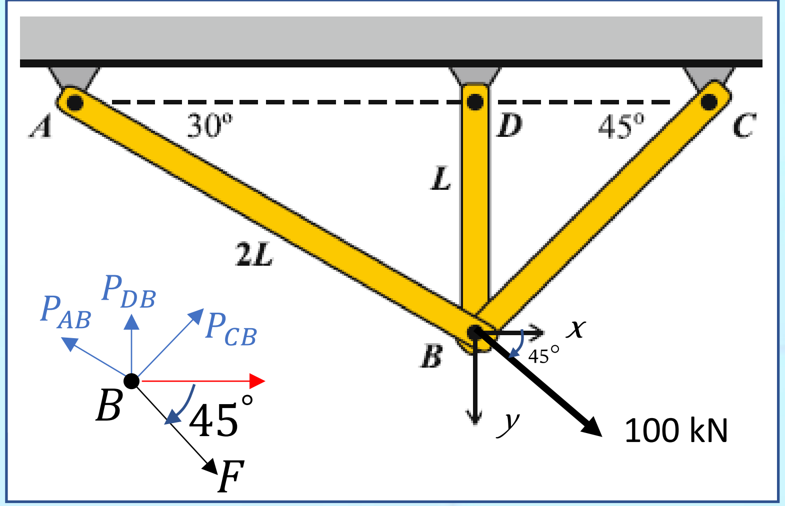
$$F = \sqrt{F_x^2 + F_y^2} = \sqrt{25.4^2 + 158^2} = 160\ kN$$

$$\theta = \tan^{-1}(F_y/F_x) = \tan^{-1}(6.22) = 80.9^\circ$$

Problem 2. An aluminum truss is shown in the figure. Joint B is loaded by the shown force F. Length $L = 1.00$ m and the cross-sectional area of each bar is $A = 0.0008$ m^2. The modulus is $E = 70$ GPa. Determine

(a) The displacements of joint B.

(b) The axial stress in each bar.



Solution: *First for displacements, then forces and stresses*

Eq. Eqs. of joint B:

$$\rightarrow: -P_{AB}\cos 30^\circ + 0 + P_{CB}\cos 45^\circ + F\cos 45^\circ = 0 \Rightarrow \sqrt{3}P_{AB} - \sqrt{2}P_{CB} = \sqrt{2}F$$

$$\downarrow: -P_{AB}\sin 30^\circ - P_{DB} - P_{CB}\sin 45^\circ + F\sin 45^\circ = 0 \Rightarrow P_{AB} + 2P_{DB} + \sqrt{2}P_{CB} = \sqrt{2}F$$

2 equations with 3 unknowns

From the behavior law and the kinematics equations:

$$P_{AB} = \frac{EA}{L_{AB}}\Delta_{AB} = \frac{EA}{2L}[(u_B - u_A)\cos\theta_{AB} + (v_B - v_A)\sin\theta_{AB}] = \frac{EA}{2L}\left(u\cos 30^\circ + v\sin 30^\circ\right) = \frac{EA}{4L}(u\sqrt{3} + v)$$

$$P_{DB} = \frac{EA}{L_{DB}}\Delta_{DB} = \frac{EA}{L}[(u_B - u_D)\cos\theta_{DB} + (v_B - v_D)\sin\theta_{DB}] = \frac{EA}{L}\left(u\cos 90^\circ + v\sin 90^\circ\right) = \frac{EA}{L}(v)$$

$$P_{CB} = \frac{EA}{L_{CB}}\Delta_{CB} = \frac{EA}{L\sqrt{2}}[(u_B - u_C)\cos\theta_{CB} + (v_B - v_C)\sin\theta_{CB}] = \frac{EA}{L\sqrt{2}}\left(u\cos 135^\circ + v\sin 135^\circ\right) = \frac{EA}{2L}(-u + v)$$

Sub. In Eq. Eqs.

$$\Rightarrow \sqrt{3}\left[\frac{EA}{4L}(u\sqrt{3} + v)\right] - \sqrt{2}\left[\frac{EA}{2L}(-u + v)\right] = \sqrt{2}F \Rightarrow \sqrt{3}(u\sqrt{3} + v) - 2\sqrt{2}(-u + v) = \frac{4\sqrt{2}L}{EA}F \Rightarrow (3 + 2\sqrt{2})u + (\sqrt{3} - 2\sqrt{2})v = \frac{4\sqrt{2}L}{EA}F$$

$$\Rightarrow \frac{EA}{4L}(u\sqrt{3} + v) + 2\left[\frac{EA}{L}(v)\right] + \sqrt{2}\left[\frac{EA}{2L}(-u + v)\right] = \sqrt{2}F \Rightarrow (u\sqrt{3} + v) + 8v + 2\sqrt{2}(-u + v) = \frac{4\sqrt{2}L}{EA}F \quad \Rightarrow (\sqrt{3} - 2\sqrt{2})u + (9 + 2\sqrt{2})v = \frac{4\sqrt{2}L}{EA}F$$

Solving to get: $u = 0.1908\dfrac{4\sqrt{2}L}{EA}F = 0.1908\dfrac{4\sqrt{2}\times 10^3 \times 100 \times 10^3}{70\times 10^3 \times 8 \times 10^2} = 1.927\ mm$ & $v = 0.1022\dfrac{4\sqrt{2}L}{EA}F = 1.032\ mm$

$$u = 0.1908\frac{4\sqrt{2}L}{EA}F = 0.1908\frac{4\sqrt{2}\times 10^3\times 100\times 10^3}{70\times 10^3\times 8\times 10^2} = 1.927\ mm\ \&\ v = 0.1022\frac{4\sqrt{2}L}{EA}F = 1.032\ mm$$

$$P_{AB} = \frac{EA}{4L}(u\sqrt{3}+v) = \sqrt{2}F(0.1908\sqrt{3}+0.1022) = 61.2\text{ kN} \Rightarrow \sigma_{AB} = \frac{P_{AB}}{A} = \frac{61.2\times 10^3}{8\times 10^2} = 76.5\text{ MPa}$$

$$P_{DB} = \frac{EA}{L}(v) = 4\sqrt{2}F(0.1022) = 57.8\text{ kN} \Rightarrow \sigma_{DB} = \frac{P_{DB}}{A} = \frac{57.8\times 10^3}{8\times 10^2} = 72.3\text{ MPa}$$

$$P_{CB} = \frac{EA}{2L}(-u+v) = 2\sqrt{2}F(-0.1908+0.1022) = -25.1\text{ kN} \Rightarrow \sigma_{CB} = \frac{P_{CB}}{A} = -\frac{25.1\times 10^3}{8\times 10^2} = -31.4\text{MPa}$$

Solution: *First for Forces and stresses, then displacements*

Eq. Eqs. of joint B: $\sqrt{3}P_{AB} - \sqrt{2}P_{CB} = \sqrt{2}F$ & $P_{AB} + 2P_{DB} + \sqrt{2}P_{CB} = \sqrt{2}F$

Kin. Eqs. Elongations in terms of displacements, then eliminating displacements to get compatibility Eq.:

$$\left.\begin{aligned}\Delta_{AB} &= \tfrac{\sqrt{3}}{2}u + \tfrac{1}{2}v\\ \Delta_{DB} &= v\\ \Delta_{CB} &= -\tfrac{\sqrt{2}}{2}u + \tfrac{\sqrt{2}}{2}v\end{aligned}\right\} \Rightarrow \left\{\begin{aligned}\Delta_{AB} &= \tfrac{\sqrt{3}}{2}u + \tfrac{1}{2}\Delta_{DB}\\ \Delta_{CB} &= -\tfrac{\sqrt{2}}{2}u + \tfrac{\sqrt{2}}{2}\Delta_{DB}\end{aligned}\right\} \Rightarrow$$

$$\left\{\begin{aligned}\tfrac{2}{\sqrt{3}}\Delta_{AB} &= u + \tfrac{1}{\sqrt{3}}\Delta_{DB}\\ \sqrt{2}\Delta_{CB} &= -u + \Delta_{DB}\end{aligned}\right\} \Rightarrow \frac{2}{\sqrt{3}}\Delta_{AB} + \sqrt{2}\Delta_{CB} = \frac{1}{\sqrt{3}}\Delta_{DB} + \Delta_{DB}$$

⇒ compatibility Eq.: $2\Delta_{AB} - (1+\sqrt{3})\Delta_{DB} + \sqrt{6}\Delta_{CB} = 0$

Then replacing the elongations by the member forces

$$\frac{4L}{EA}P_{AB} - (1+\sqrt{3})\frac{L}{EA}P_{DB} + \sqrt{6}\frac{L\sqrt{2}}{EA}P_{CB} = 0 \Rightarrow$$

Comp. Eq. in terms of forces: $4P_{AB} - (1+\sqrt{3})P_{DB} + 2\sqrt{3}P_{CB} = 0$

With the two Eq. Eqs. of joint B: $\sqrt{3}P_{AB} + 0 - \sqrt{2}P_{CB} = \sqrt{2}F$

$P_{AB} + 2P_{DB} + \sqrt{2}P_{CB} = \sqrt{2}F$

Solving to get forces: $P_{AB} = 61.2\text{ kN}, P_{DB} = 57.8\text{ kN}, P_{CB} = -25.1\text{ kN}.$

Then to get forces: $\sigma_{AB} = 76.5, \sigma_{DB} = 72.3, \sigma_{CB} = -31.4\text{MPa}.$

Then elongations

$$\left\{\begin{aligned}\Delta_{AB} &= \frac{2L}{EA}P_{AB} = \frac{2\times 10^3\times 76.5\times 10^3}{70\times 10^3\times 8\times 10^2} = 2.732\ mm\\ \Delta_{DB} &= \frac{L}{EA}P_{DB} = \frac{10^3\times 57.8\times 10^3}{70\times 10^3\times 8\times 10^2} = 1.032\ mm\\ \Delta_{CB} &= \frac{L\sqrt{2}}{EA}P_{DB} = \frac{\sqrt{2}\times 10^3\times(-25.1\times 10^3)}{70\times 10^3\times 8\times 10^2} = -0.634\ mm\end{aligned}\right.$$

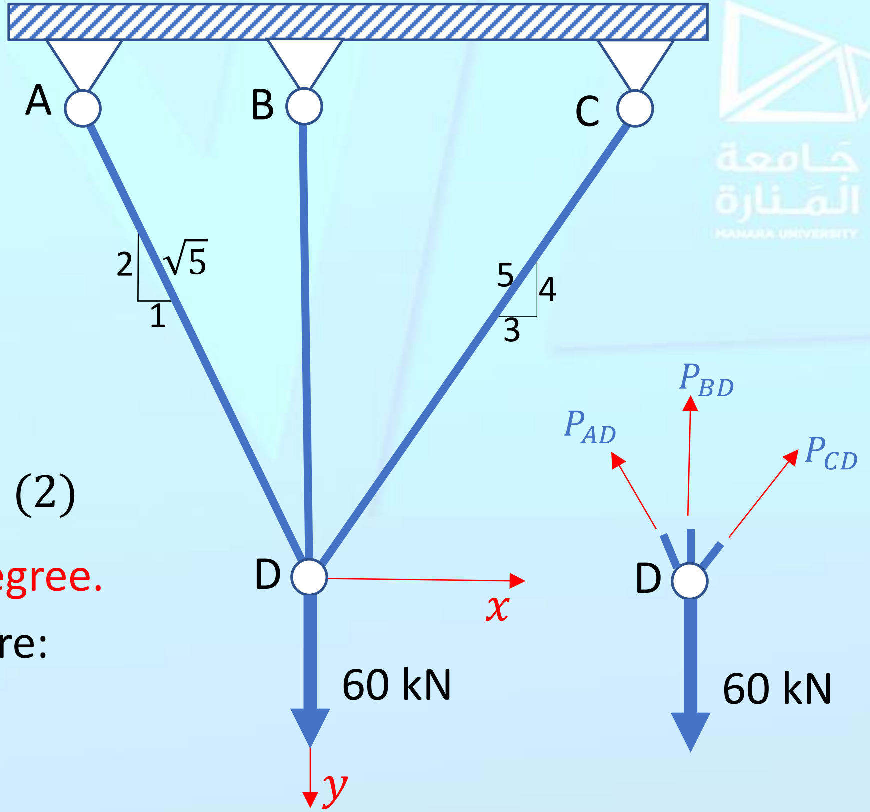
Finally back to the Kin. Eqs. to get:

$v = \Delta_{DB} = 1.032\ mm$ & $u = -\sqrt{2}\Delta_{CB} + \Delta_{DB} = 1.927\ mm$

Problem 3. An aluminum truss is loaded at joint D by $F = 60$ kN. the length of BD is 1 m. The cross-sectional areas are: $A_{AD} = 360\ mm^2$, $A_{BD} = 400\ mm^2$, and $A_{CD} = 450\ mm^2$. The modulus of aluminum is $E = 70$ GPa. Determine

1) The axial stress in members *AD, BD,* and *CD.*

2) The horizontal and vertical displacements of joint *D, u, and v.*



Solution (1) Eq. Eqs. of joint D:

$\sum F_x = 0: -\frac{1}{\sqrt{5}}P_{AD} + 0 + \frac{3}{5}P_{CD} = 0 \ \ (1)$ $\qquad \sum F_y = 0: \frac{2}{\sqrt{5}}P_{AD} + P_{BD} + \frac{4}{5}P_{CD} = 60 \ \ (2)$

Two Eq. Eqs. with three unknowns, so the system is indeterminate to the first degree.

Kin. Eqs. giving members elongations in terms of joint D displacements u & v , are:

$\Delta_{AD} = \frac{1}{\sqrt{5}}u + \frac{2}{\sqrt{5}}v \ \ (4)$ $\qquad \Delta_{BD} = v \ \ (5)$ $\qquad \Delta_{CB} = -\frac{3}{5}u + \frac{4}{5}v$ (6)

$u + 2v = \sqrt{5}\,\Delta_{AD}\ (4')$ $\qquad v = \Delta_{BD}\ (5')$ $\qquad -3u + 4v = 5\Delta_{CB}$ (6')

$u + 2\Delta_{BD} = \sqrt{5}\,\Delta_{AD}\ (4'')$ $\qquad -3u + 4\Delta_{BD} = 5\Delta_{CB}$ (6'')

(4'')X3+(6'') ➡ $10\Delta_{BD} = 3\sqrt{5}\,\Delta_{AD} + 5\Delta_{CB}$ ➡ $3\sqrt{5}\,\Delta_{AD} - 10\Delta_{BD} + 5\Delta_{CB} = 0$ ➡

$3\sqrt{5}\,\frac{L_{AD}}{EA_{AD}}P_{AD} - 10\frac{L_{BD}}{EA_{BD}}P_{BD} + 5\frac{L_{CD}}{EA_{CD}}P_{CD} = 0$ ➡ $3\sqrt{5}\,\frac{\sqrt{5}/2}{360/400}P_{AD} - 10P_{BD} + 5\frac{5/4}{450/400}P_{CD} = 0$

$\frac{15}{1.8}P_{AD} - 10P_{BD} + \frac{25}{4.5}P_{CD} = 0 \ \ (3)$ Solving (1), (2) & (3) to get $P_{AD} = 21.9\ kN, P_{BD} = 27.3\ kN, P_{CD} = 16.3\ kN$

$$\sigma_{AD} = \frac{21900}{360} = 60.8\ MPa, \sigma_{BD} = \frac{27300}{400} = 68.3\ MPa, \sigma_{CD} = \frac{16300}{450} = 36.2\ MPa.$$

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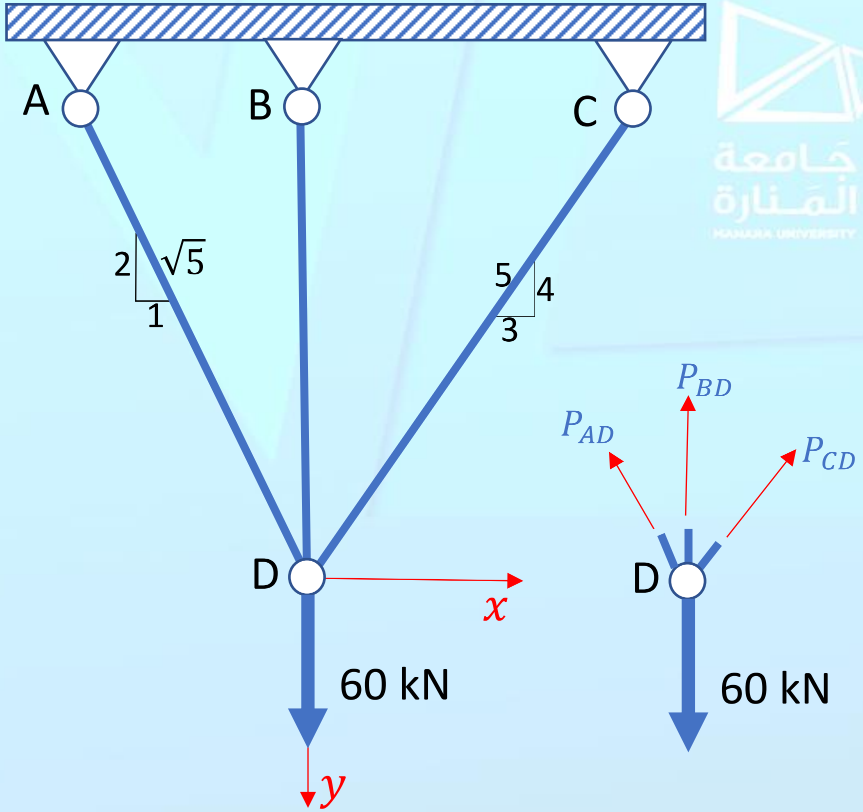


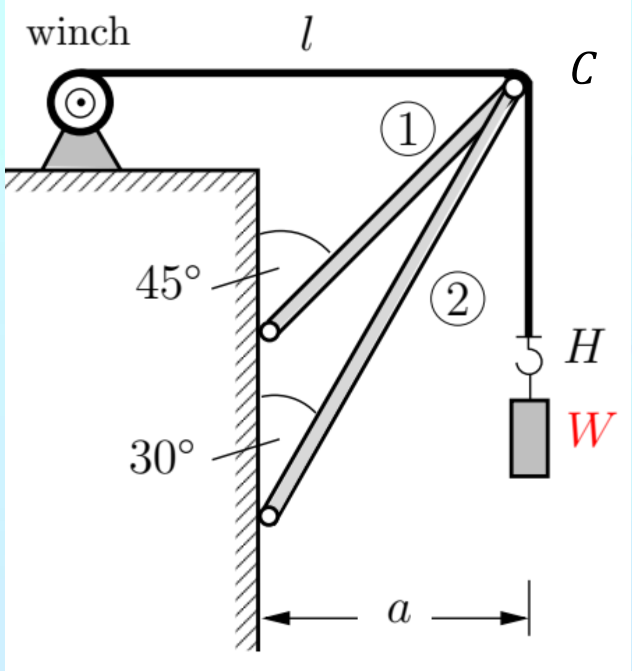
Solution (2) From Eq. (5'):

$$v = \Delta_{BD} = \frac{L_{BD}}{EA_{BD}} P_{BD} = \frac{1000 \times 27300}{70000 \times 400} = 0.975\ mm$$

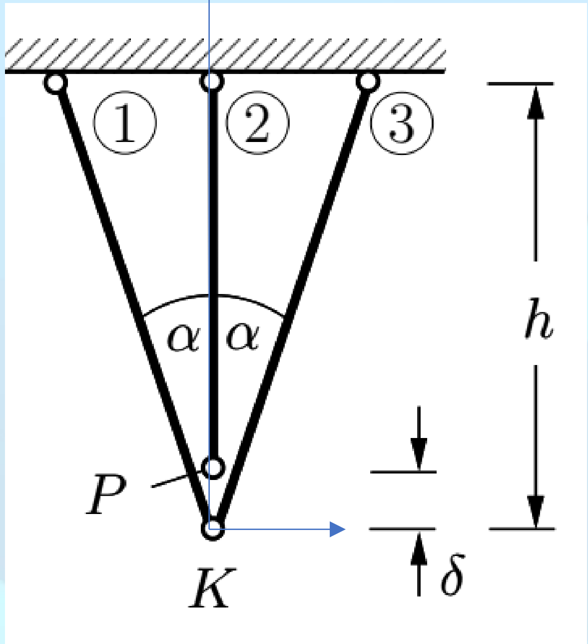
From Eq. (4'):

$$u = \sqrt{5}\,\Delta_{AD} - 2v = \sqrt{5}\frac{L_{AD}}{EA_{AD}} P_{AD} - 2v = \sqrt{5}\frac{(\sqrt{5}/2) \times 1000 \times 27300}{70000 \times 360} - 2 \times 0.975 = 0.758\ mm$$

Problem 4. The Fig. shows a freight elevator. The cable (of total length is l and axial rigidity K) passes over a smooth pin C. A crate (of weight W) is suspended at the end of the cable. The axial rigidity EA of the two rods BC and DC is given. Determine the displacements of pin C and of the end of the cable (point H) due to the weight of the crate.



Problem 5. To assemble the truss (axial rigidity EA of the three bars) in Fig. the end point P of bar 2 has to be connected with pin K. Assume $\delta \ll h$. Determine the forces in the bars after the truss has been assembled.



Eq. Eqs.: $P_1 = P_3, 2P_1 \cos\alpha + P_2 = 0$

Kin. Eqs.: $\Delta_1 = \Delta_3 = -v\cos\alpha\,, \Delta_2 = \delta - v.$

Comp. Eq.: $\Delta_2 = \delta - .\dfrac{\Delta_1}{\cos\alpha} \Rightarrow \Delta_1 - \Delta_2\cos\alpha = -\delta\cos\alpha$

Comp. Eq. in terms of member forces: $\dfrac{h}{EA\ \cos\alpha}P_1 - \dfrac{h\cos\alpha}{EA}P_2 = -\delta\cos\alpha$

Simplifying to get: $P_1 - P_2\cos^2\alpha = -\dfrac{EA\cos^2\alpha}{h}\delta$

Solving with the second Eq. Eq. to get $P_1 = -\dfrac{EA\cos^2\alpha}{h(1+\cos^3\alpha)}\delta,\quad P_2 = \dfrac{2EA\cos^3\alpha}{h(1+\cos^3\alpha)}\delta$