

Planar Kinematics of a Rigid Body

Motion Analysis :Acceleration





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An equation that relates the accelerations of two points on a bar (rigid body) subjected to general plane motion may be determined by differentiating $\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A}$ with respect to time. This yields

$$\frac{d\mathbf{v}_B}{dt} = \frac{d\mathbf{v}_A}{dt} + \frac{d\mathbf{v}_{B/A}}{dt}$$

The terms $d\mathbf{v}_B/dt = \mathbf{a}_B$ and $d\mathbf{v}_A/dt = \mathbf{a}_A$ represent the *absolute accelerations* of points B and A .

The last term represents the acceleration of B with respect to A as measured by an observer fixed to translating x' , y' axes which have their origin at the base point A . It was shown that to this observer point B appears to move along a *circular arc* that has a radius of curvature

$r_{B/A}$. Consequently, $\mathbf{a}_{B/A}$ can be expressed in terms of its tangential and normal components; i.e., $\mathbf{a}_{B/A} = (\mathbf{a}_{B/A})_t + (\mathbf{a}_{B/A})_n$, where $(\mathbf{a}_{B/A})_t = \alpha r_{B/A}$ and $(\mathbf{a}_{B/A})_n = \omega^2 r_{B/A}$. Hence, the relative-acceleration equation can be written in the form

$$\mathbf{a}_B = \mathbf{a}_A + (\mathbf{a}_{B/A})_t + (\mathbf{a}_{B/A})_n$$

where

\mathbf{a}_B = acceleration of point B

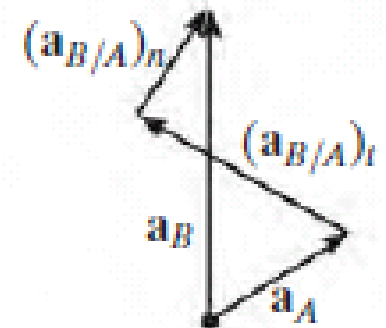
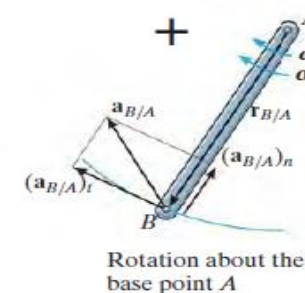
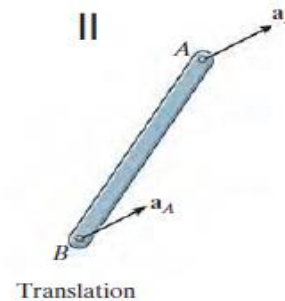
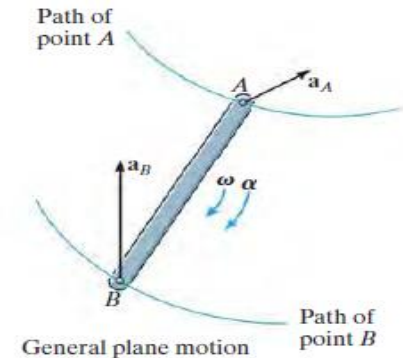
\mathbf{a}_A = acceleration of point A

$(\mathbf{a}_{B/A})_t$ = tangential acceleration component of B with respect to A . The *magnitude* is $(a_{B/A})_t = \alpha r_{B/A}$, and the *direction* is perpendicular to $\mathbf{r}_{B/A}$.

$(\mathbf{a}_{B/A})_n$ = normal acceleration component of B with respect to A . The *magnitude* is $(a_{B/A})_n = \omega^2 r_{B/A}$, and the *direction* is always from B toward A .

It is seen that at a given instant the acceleration of B is determined by considering the bar to translate with an acceleration \mathbf{a}_A , and simultaneously rotate about the base point A with an instantaneous angular velocity ω and angular acceleration α .

Vector addition of these two effects, applied to B , yields \mathbf{a}_B . It should be noted that since points A and B move along curved paths, the accelerations of these points will have both tangential and normal components.



Since the relative-acceleration components represent the effect of *circular motion* observed from translating axes having their origin at the base point A , these terms can be expressed as $(\mathbf{a}_{B/A})_t = \boldsymbol{\alpha} \times \mathbf{r}_{B/A}$ and $(\mathbf{a}_{B/A})_n = -\omega^2 \mathbf{r}_{B/A}$

$$\mathbf{a}_B = \mathbf{a}_A + \boldsymbol{\alpha} \times \mathbf{r}_{B/A} - \omega^2 \mathbf{r}_{B/A}$$

where

\mathbf{a}_B = acceleration of point B

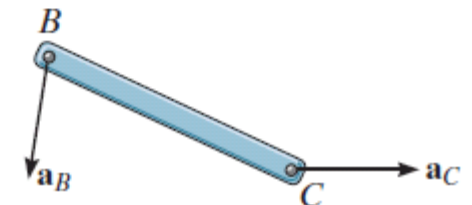
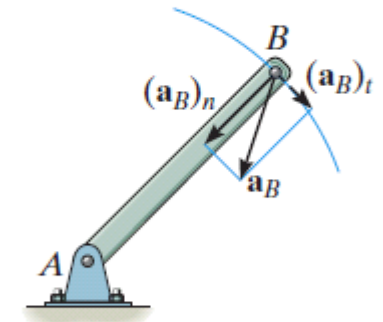
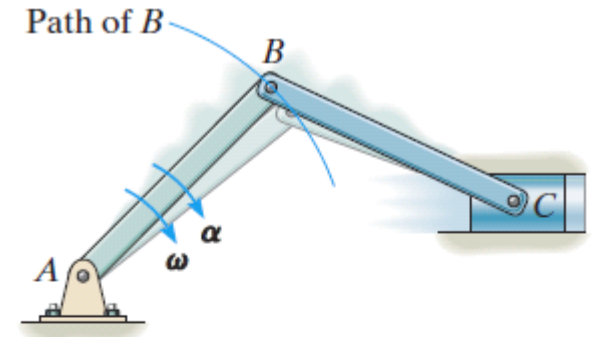
\mathbf{a}_A = acceleration of the base point A

$\boldsymbol{\alpha}$ = angular acceleration of the body

ω = angular velocity of the body

$\mathbf{r}_{B/A}$ = position vector directed from A to B

If equations are applied in a practical manner to study the accelerated motion of a rigid body which is *pin connected to two other bodies*, it should be realized that points which are *coincident at the pin* move with the *same acceleration, since the path of motion over which they travel is the same*. For example, point *B* lying on either rod *BA* or *BC* of the crank mechanism shown has the same acceleration, since the rods are pin connected at *B*. Here the motion of *B* is along a *circular path*, so that ***a_B can be expressed in terms of its*** tangential and normal components. At the other end of rod *BC* point *C* moves along a *straight-lined path*, which is defined by the piston. Hence, ***a_c is horizontal***



Finally, consider a disk that rolls without slipping as shown. As a result, $v_A = 0$ and so from the kinematic diagram, the velocity of the mass center G is

$$\mathbf{v}_G = \mathbf{v}_A + \boldsymbol{\omega} \times \mathbf{r}_{G/A} = \mathbf{0} + (-\omega \mathbf{k}) \times (r \mathbf{j})$$

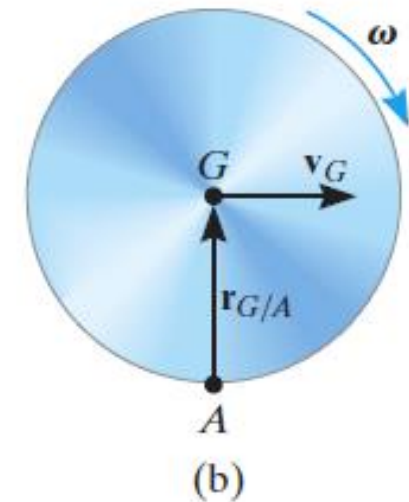
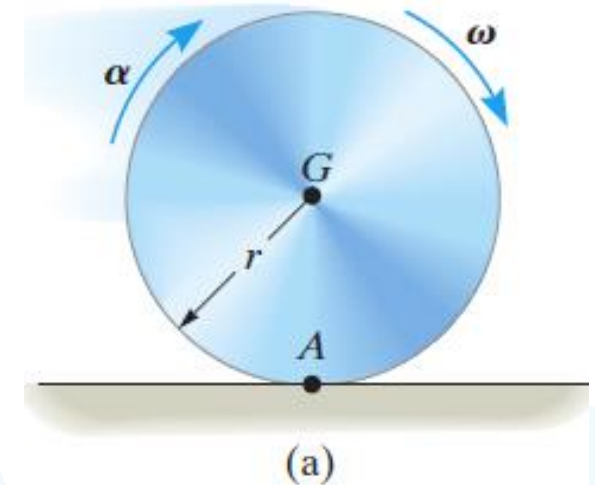
So that

$$v_G = \omega r$$

This same result can also be determined using the IC method where point A is the *IC*.

Since G moves along a *straight line*, its acceleration in this case can be determined from the time derivative of its velocity.

$$\begin{aligned} \frac{dv_G}{dt} &= \frac{d\omega}{dt} r \\ a_G &= \alpha r \end{aligned}$$



Procedure for Analysis

The relative acceleration equation can be applied between any two points A and B on a body either by using a Cartesian vector analysis, or by writing the x and y scalar component equations directly.

Velocity Analysis.

- Determine the angular velocity ω of the body by using a velocity analysis . Also, determine velocities \mathbf{v}_A and \mathbf{v}_B

Vector Analysis

Kinematic Diagram.

- Establish the directions of the fixed x, y coordinates and draw the kinematic diagram of the body. Indicate on it \mathbf{a}_A , \mathbf{a}_B , $\boldsymbol{\omega}$, $\boldsymbol{\alpha}$, and $\mathbf{r}_{B/A}$.
- If points A and B move along *curved paths*, then their accelerations should be indicated in terms of their tangential and normal components, i.e., $\mathbf{a}_A = (\mathbf{a}_A)_t + (\mathbf{a}_A)_n$ and $\mathbf{a}_B = (\mathbf{a}_B)_t + (\mathbf{a}_B)_n$.

Acceleration Equation.

- To apply $\mathbf{a}_B = \mathbf{a}_A + \boldsymbol{\alpha} \times \mathbf{r}_{B/A} - \omega^2 \mathbf{r}_{B/A}$, express the vectors in Cartesian vector form and substitute them into the equation. Evaluate the cross product and then equate the respective \mathbf{i} and \mathbf{j} components to obtain two scalar equations.
- If the solution yields a *negative* answer for an *unknown* magnitude, it indicates that the sense of direction of the vector is opposite to that shown on the kinematic diagram.

Scalar Analysis Kinematic Diagram.

- If the acceleration equation is applied in scalar form, then the magnitudes and directions of the relative-acceleration components $(\mathbf{a}_{B/A})_t$ and $(\mathbf{a}_{B/A})_n$ must be established. To do this draw a kinematic diagram such as shown. Since the body is considered to be momentarily “pinned” at the base point A , the *magnitudes* of these components are $(a_{B/A})_t = \alpha r_{B/A}$ and $(a_{B/A})_n = \omega^2 r_{B/A}$. Their *sense of direction* is established from the diagram such that $(\mathbf{a}_{B/A})_t$ acts perpendicular to $\mathbf{r}_{B/A}$, in accordance with the rotational motion α of the body, and $(\mathbf{a}_{B/A})_n$ is directed from B toward A .

Acceleration Equation.

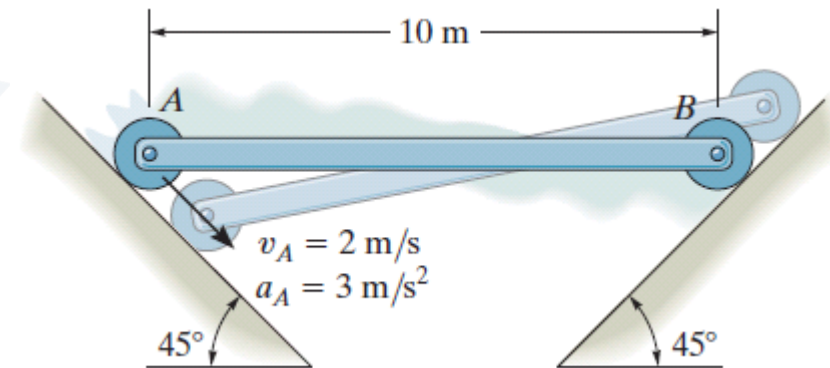
- Represent the vectors in $\mathbf{a}_B = \mathbf{a}_A + (\mathbf{a}_{B/A})_t + (\mathbf{a}_{B/A})_n$ graphically by showing their magnitudes and directions underneath each term. The scalar equations are determined from the x and y components of these vectors.

EXAMPLE

The rod AB shown is confined to move along the inclined planes at A and B . If point A has an acceleration of 3 m/s^2 and a velocity of 2 m/s , both directed down the plane at the instant the rod is horizontal, determine the angular acceleration of the rod at this instant.

SOLUTION I (VECTOR ANALYSIS)

We will apply the acceleration equation to points A and B on the rod. To do so it is first necessary to determine the angular velocity of the rod. Show that it is $\omega = 0.283 \text{ rad/s}$ using either the velocity equation or the method of instantaneous centers.



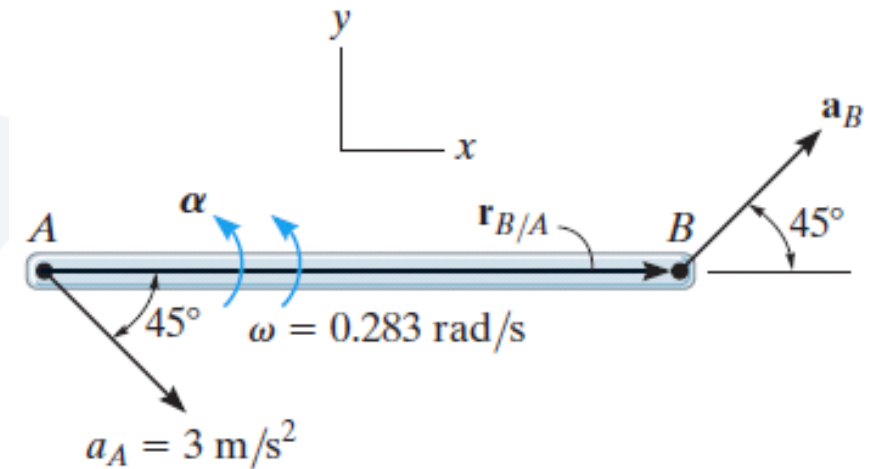
Kinematic Diagram. Since points A and B both move along straight-line paths, they have *no* components of acceleration normal to the paths. There are two unknowns, namely, a_B and α .

Acceleration Equation.

$$\mathbf{a}_B = \mathbf{a}_A + \boldsymbol{\alpha} \times \mathbf{r}_{B/A} - \omega^2 \mathbf{r}_{B/A}$$

$$a_B \cos 45^\circ \mathbf{i} + a_B \sin 45^\circ \mathbf{j} = 3 \cos 45^\circ \mathbf{i} - 3 \sin 45^\circ \mathbf{j} + (\alpha \mathbf{k}) \times (10 \mathbf{i}) - (0.283)^2 (10 \mathbf{i})$$

Carrying out the cross product and equating the \mathbf{i} and \mathbf{j} components yields



$$a_B \cos 45^\circ = 3 \cos 45^\circ - (0.283)^2(10) \quad (1)$$

$$a_B \sin 45^\circ = -3 \sin 45^\circ + \alpha(10) \quad (2)$$

Solving, we have

$$a_B = 1.87 \text{ m/s}^2 \nearrow 45^\circ$$

$$\alpha = 0.344 \text{ rad/s}^2 \curvearrowright$$

Ans.

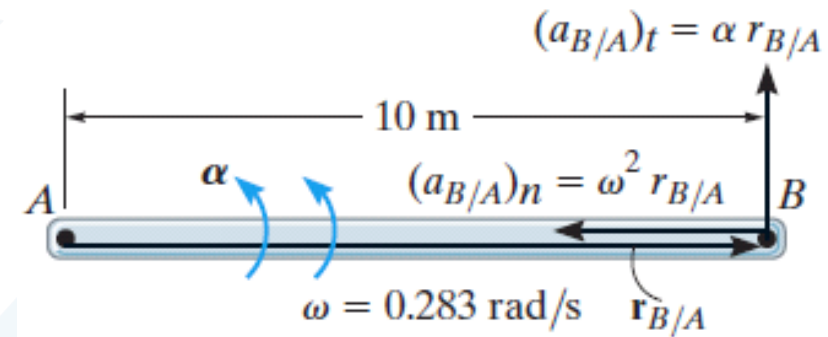
SOLUTION II (SCALAR ANALYSIS)

From the kinematic diagram, showing the relative-acceleration components $(\mathbf{a}_{B/A})_t$ and $(\mathbf{a}_{B/A})_n$, we have

$$\mathbf{a}_B = \mathbf{a}_A + (\mathbf{a}_{B/A})_t + (\mathbf{a}_{B/A})_n$$

$$\begin{bmatrix} a_B \\ \nearrow 45^\circ \end{bmatrix} = \begin{bmatrix} 3 \text{ m/s}^2 \\ \searrow 45^\circ \end{bmatrix} + \begin{bmatrix} \alpha(10 \text{ m}) \\ \uparrow \end{bmatrix} + \begin{bmatrix} (0.283 \text{ rad/s})^2(10 \text{ m}) \\ \leftarrow \end{bmatrix}$$

Equating the x and y components yields Eqs. 1 and 2, and the solution proceeds as before.



EXAMPLE

The disk rolls without slipping and has the angular motion shown. Determine the acceleration of point A at this instant.

SOLUTION I (VECTOR ANALYSIS)

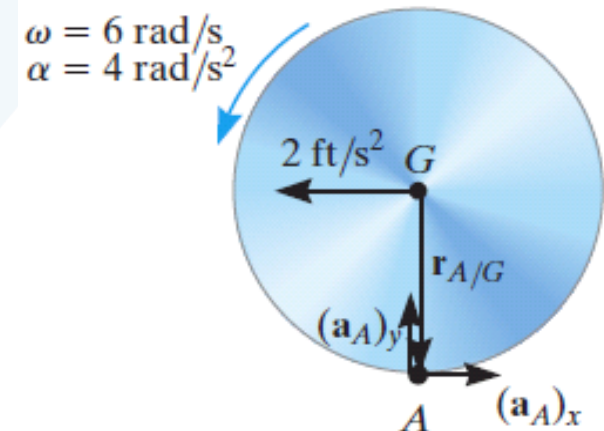
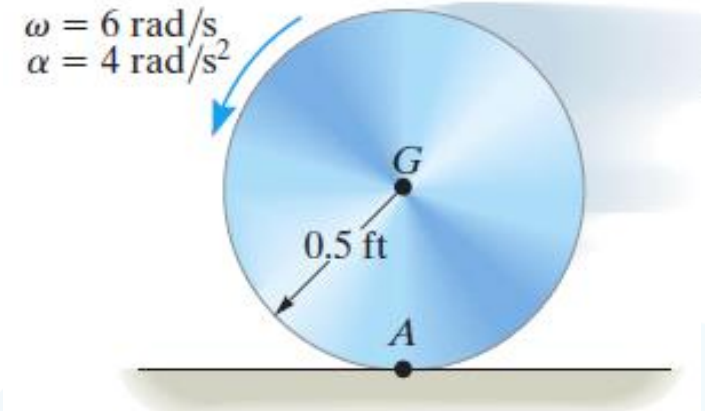
Kinematic Diagram. Since no slipping occurs,

$$a_G = \alpha r = (4 \text{ rad/s}^2)(0.5 \text{ ft}) = 2 \text{ ft/s}^2$$

Acceleration Equation.

We will apply the acceleration equation to points G and A

$$\begin{aligned} \mathbf{a}_A &= \mathbf{a}_G + \boldsymbol{\alpha} \times \mathbf{r}_{A/G} - \omega^2 \mathbf{r}_{A/G} \\ \mathbf{a}_A &= -2\mathbf{i} + (4\mathbf{k}) \times (-0.5\mathbf{j}) - (6)^2(-0.5\mathbf{j}) \\ &= \{18\mathbf{j}\} \text{ ft/s}^2 \end{aligned}$$



SOLUTION II (SCALAR ANALYSIS)

Using the result for $a_G = 2 \text{ ft/s}^2$ determined above, and from the kinematic diagram, showing the relative motion $\mathbf{a}_{A/G}$, we have

$$\mathbf{a}_A = \mathbf{a}_G + (\mathbf{a}_{A/G})_x + (\mathbf{a}_{A/G})_y$$

$$\begin{bmatrix} (a_A)_x \\ \rightarrow \end{bmatrix} + \begin{bmatrix} (a_A)_y \\ \uparrow \end{bmatrix} = \begin{bmatrix} 2 \text{ ft/s}^2 \\ \leftarrow \end{bmatrix} + \begin{bmatrix} (4 \text{ rad/s}^2)(0.5 \text{ ft}) \\ \rightarrow \end{bmatrix} + \begin{bmatrix} (6 \text{ rad/s})^2(0.5 \text{ ft}) \\ \uparrow \end{bmatrix}$$

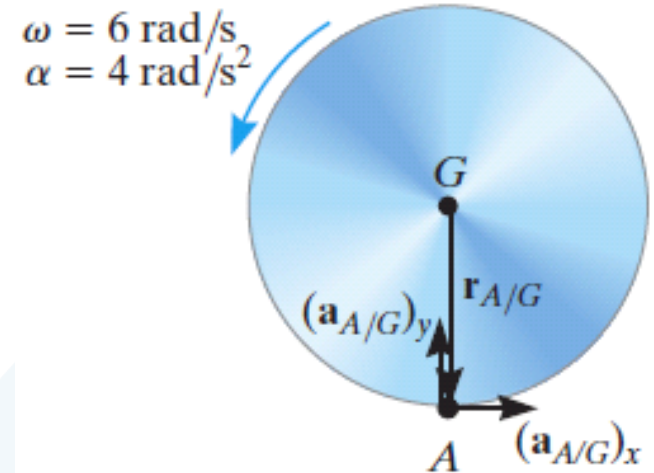
$$\xrightarrow{+} \quad (a_A)_x = -2 + 2 = 0$$

$$+\uparrow \quad (a_A)_y = 18 \text{ ft/s}^2$$

Therefore,

$$a_A = \sqrt{(0)^2 + (18 \text{ ft/s}^2)^2} = 18 \text{ ft/s}^2$$

Ans.



EXAMPLE

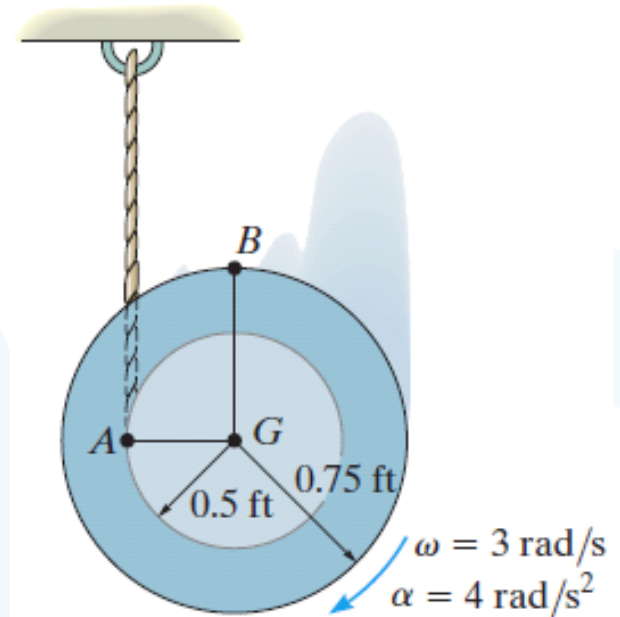
The spool shown unravels from the cord, such that at the instant shown it has an angular velocity of 3 rad/s and an angular acceleration of 4 rad/s^2 . Determine the acceleration of point B .

SOLUTION I (VECTOR ANALYSIS)

The spool “appears” to be rolling downward without slipping at point A . Therefore, we can determine the acceleration of point G

$$a_G = \alpha r = (4 \text{ rad/s}^2)(0.5 \text{ ft}) = 2 \text{ ft/s}^2$$

We will apply the acceleration equation to points G and B .



Kinematic Diagram. Point B moves along a *curved path* having an *unknown* radius of curvature. Its acceleration will be represented by its unknown x and y components as shown

Acceleration Equation.

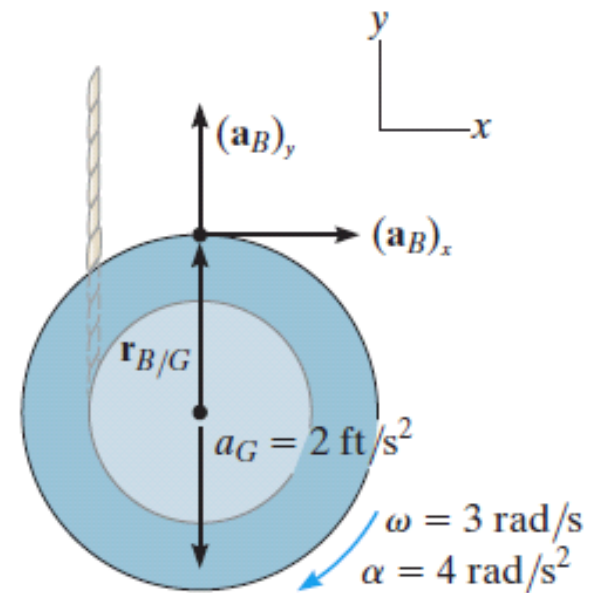
$$\mathbf{a}_B = \mathbf{a}_G + \boldsymbol{\alpha} \times \mathbf{r}_{B/G} - \omega^2 \mathbf{r}_{B/G}$$

$$(a_B)_x \mathbf{i} + (a_B)_y \mathbf{j} = -2\mathbf{j} + (-4\mathbf{k}) \times (0.75\mathbf{j}) - (3)^2(0.75\mathbf{j})$$

Equating the \mathbf{i} and \mathbf{j} terms, the component equations are

$$(a_B)_x = 4(0.75) = 3 \text{ ft/s}^2 \rightarrow \quad (1)$$

$$(a_B)_y = -2 - 6.75 = -8.75 \text{ ft/s}^2 = 8.75 \text{ ft/s}^2 \downarrow \quad (2)$$



The magnitude and direction of \mathbf{a}_B are therefore

$$a_B = \sqrt{(3)^2 + (8.75)^2} = 9.25 \text{ ft/s}^2$$

Ans.

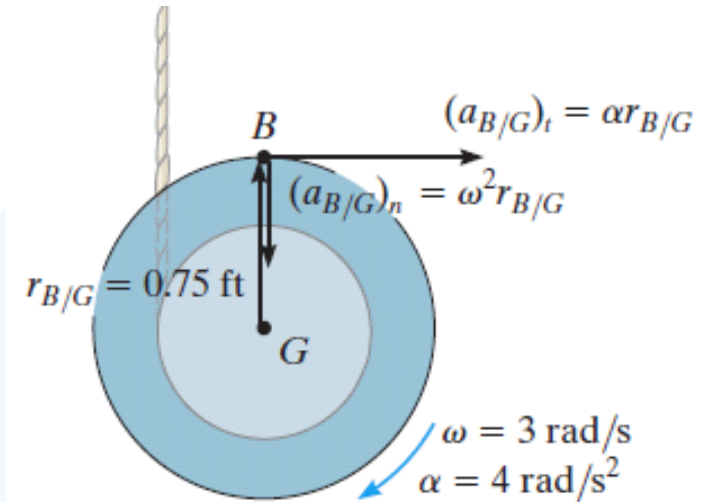
$$\theta = \tan^{-1} \frac{8.75}{3} = 71.1^\circ \searrow$$

Ans.

SOLUTION II (SCALAR ANALYSIS)

This problem may be solved by writing the scalar component equations directly. The kinematic diagram shows the relative-acceleration components $(\mathbf{a}_{B/G})_t$ and $(\mathbf{a}_{B/G})_n$. Thus,

$$\mathbf{a}_B = \mathbf{a}_G + (\mathbf{a}_{B/G})_t + (\mathbf{a}_{B/G})_n$$



$$\begin{aligned} \left[\begin{array}{c} (a_B)_x \\ \rightarrow \end{array} \right] + \left[\begin{array}{c} (a_B)_y \\ \uparrow \end{array} \right] \\ = \left[\begin{array}{c} 2 \text{ ft/s}^2 \\ \downarrow \end{array} \right] + \left[\begin{array}{c} 4 \text{ rad/s}^2 (0.75 \text{ ft}) \\ \rightarrow \end{array} \right] + \left[\begin{array}{c} (3 \text{ rad/s})^2 (0.75 \text{ ft}) \\ \downarrow \end{array} \right] \end{aligned}$$

The x and y components yield Eqs. 1 and 2 above.

EXAMPLE

The collar C moves downward with an acceleration of 1 m/s^2 . At the instant shown, it has a speed of 2 m/s which gives links CB and AB an angular velocity $\omega_{AB} = \omega_{CB} = 10 \text{ rad/s}$. Determine the angular accelerations of CB and AB at this instant.

SOLUTION (VECTOR ANALYSIS)

Kinematic Diagram. The kinematic diagrams of *both* links AB and CB are shown. To solve, we will apply the appropriate kinematic equation to each link.

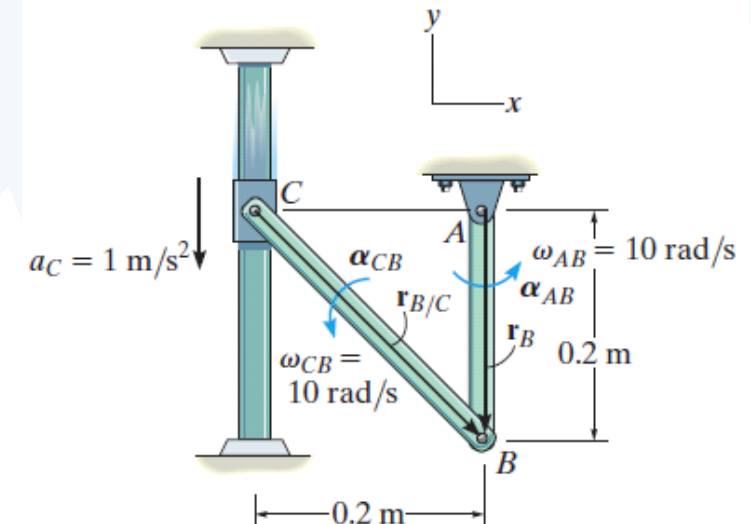
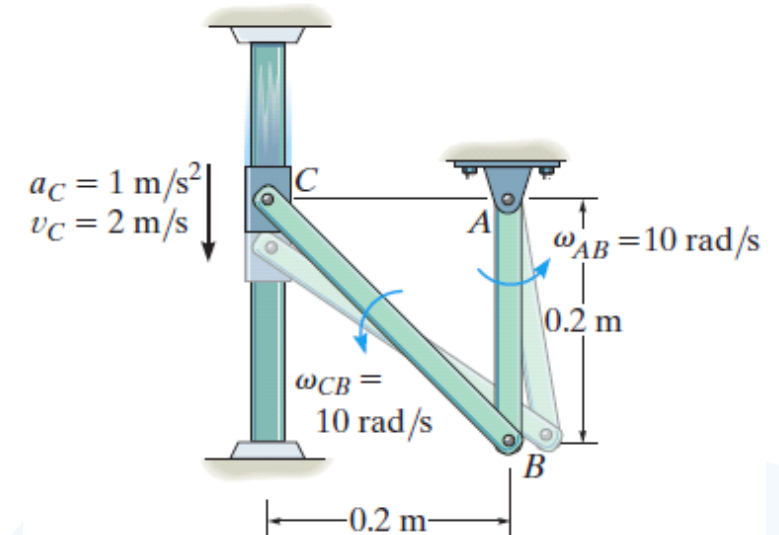
Acceleration Equation.

Link AB (rotation about a fixed axis):

$$\mathbf{a}_B = \boldsymbol{\alpha}_{AB} \times \mathbf{r}_B - \omega_{AB}^2 \mathbf{r}_B$$

$$\mathbf{a}_B = (\alpha_{AB} \mathbf{k}) \times (-0.2 \mathbf{j}) - (10)^2 (-0.2 \mathbf{j})$$

$$\mathbf{a}_B = 0.2 \alpha_{AB} \mathbf{i} + 20 \mathbf{j}$$



Note that \mathbf{a}_B has n and t components since it moves along a *circular path*.

Link BC (general plane motion):

$$\mathbf{a}_B = \mathbf{a}_C + \boldsymbol{\alpha}_{CB} \times \mathbf{r}_{B/C} - \omega_{CB}^2 \mathbf{r}_{B/C}$$

$$0.2\alpha_{AB}\mathbf{i} + 20\mathbf{j} = -1\mathbf{j} + (\alpha_{CB}\mathbf{k}) \times (0.2\mathbf{i} - 0.2\mathbf{j}) - (10)^2(0.2\mathbf{i} - 0.2\mathbf{j})$$

$$0.2\alpha_{AB}\mathbf{i} + 20\mathbf{j} = -1\mathbf{j} + 0.2\alpha_{CB}\mathbf{j} + 0.2\alpha_{CB}\mathbf{i} - 20\mathbf{i} + 20\mathbf{j}$$

Thus,

$$0.2\alpha_{AB} = 0.2\alpha_{CB} - 20$$

$$20 = -1 + 0.2\alpha_{CB} + 20$$

Solving,

$$\alpha_{CB} = 5 \text{ rad/s}^2 \curvearrowright$$

Ans.

$$\alpha_{AB} = -95 \text{ rad/s}^2 = 95 \text{ rad/s}^2 \curvearrowleft$$

Ans.

EXAMPLE

The crankshaft AB turns with a clockwise angular acceleration of 20 rad/s^2 . Determine the acceleration of the piston at the instant AB is in the position shown. At this instant $\omega_{AB} = 10 \text{ rad/s}$ and $\omega_{BC} = 2.43 \text{ rad/s}$.

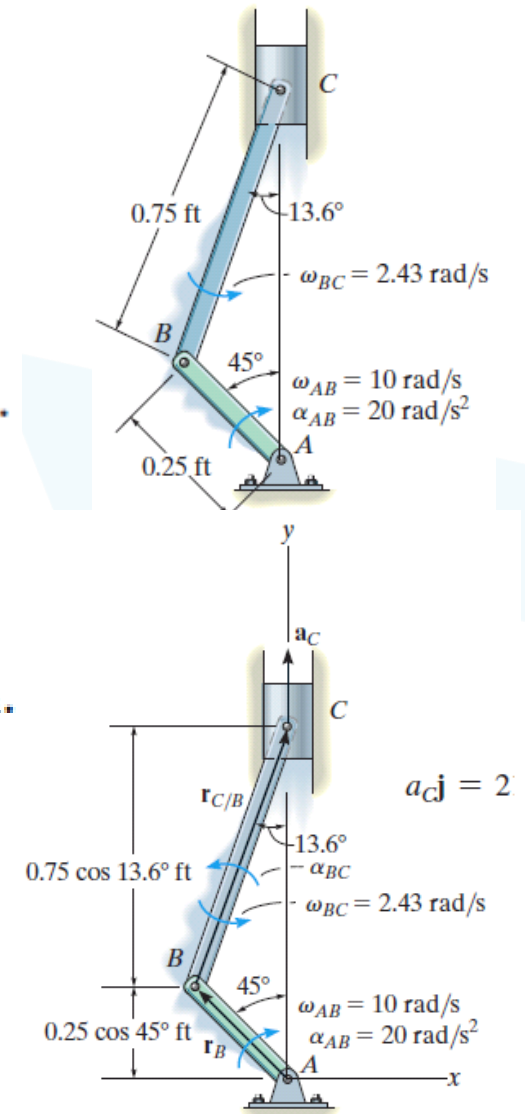
SOLUTION (VECTOR ANALYSIS)

Kinematic Diagram. The kinematic diagrams for both AB and BC are shown. Here \mathbf{a}_C is vertical since C moves along a straight-line path.

Acceleration Equation. Expressing each of the position vectors in Cartesian vector form

$$\mathbf{r}_B = \{-0.25 \sin 45^\circ \mathbf{i} + 0.25 \cos 45^\circ \mathbf{j}\} \text{ ft} = \{-0.177 \mathbf{i} + 0.177 \mathbf{j}\} \text{ ft}$$

$$\mathbf{r}_{C/B} = \{0.75 \sin 13.6^\circ \mathbf{i} + 0.75 \cos 13.6^\circ \mathbf{j}\} \text{ ft} = \{0.177 \mathbf{i} + 0.729 \mathbf{j}\} \text{ ft}$$



Crankshaft **AB** (rotation about a fixed axis):

$$\begin{aligned}\mathbf{a}_B &= \boldsymbol{\alpha}_{AB} \times \mathbf{r}_B - \omega_{AB}^2 \mathbf{r}_B \\ &= (-20\mathbf{k}) \times (-0.177\mathbf{i} + 0.177\mathbf{j}) - (10)^2(-0.177\mathbf{i} + 0.177\mathbf{j}) \\ &= \{21.21\mathbf{i} - 14.14\mathbf{j}\} \text{ ft/s}^2\end{aligned}$$

Connecting Rod **BC** (general plane motion): Using the result for \mathbf{a}_B and noting that \mathbf{a}_C is in the vertical direction, we have

$$\begin{aligned}\mathbf{a}_C &= \mathbf{a}_B + \boldsymbol{\alpha}_{BC} \times \mathbf{r}_{C/B} - \omega_{BC}^2 \mathbf{r}_{C/B} \\ a_C \mathbf{j} &= 21.21\mathbf{i} - 14.14\mathbf{j} + (\alpha_{BC}\mathbf{k}) \times (0.177\mathbf{i} + 0.729\mathbf{j}) - (2.43)^2(0.177\mathbf{i} + 0.729\mathbf{j}) \\ a_C \mathbf{j} &= 21.21\mathbf{i} - 14.14\mathbf{j} + 0.177\alpha_{BC}\mathbf{j} - 0.729\alpha_{BC}\mathbf{i} - 1.04\mathbf{i} - 4.30\mathbf{j} \\ 0 &= 20.17 - 0.729\alpha_{BC} \\ a_C &= 0.177\alpha_{BC} - 18.45\end{aligned}$$

Solving yields

$$\alpha_{BC} = 27.7 \text{ rad/s}^2 \curvearrowright$$

$$a_C = -13.5 \text{ ft/s}^2$$

Ans.

NOTE: Since the piston is moving upward, the negative sign for a_C indicates that the piston is decelerating, i.e., $\mathbf{a}_C = \{-13.5\mathbf{j}\} \text{ ft/s}^2$. This causes the speed of the piston to decrease until AB becomes vertical, at which time the piston is momentarily at rest.

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