جامعة المنارة

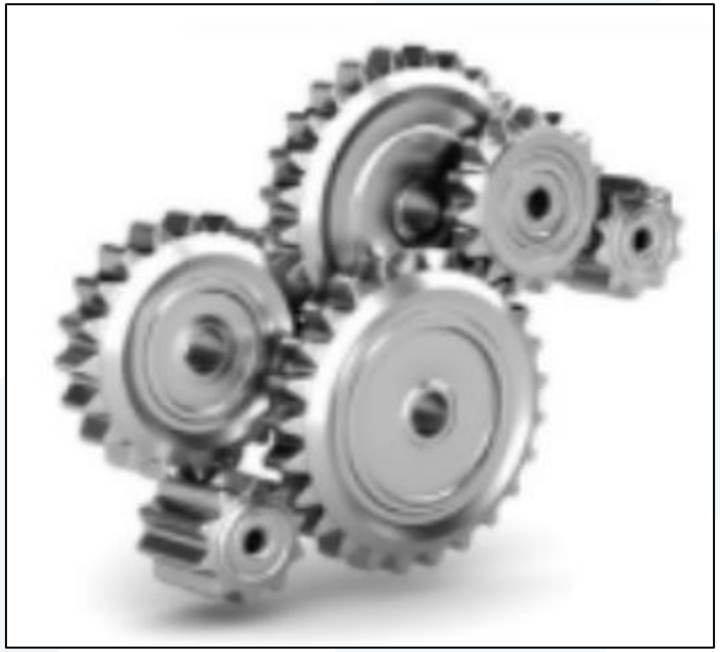
كلية الهندسة

قسم الروبوتيك و الأنظمة الذكية

مقرر نمذجة الميكانيزمات

Modeling of Rigid-Body Mechanical Systems

Part 2



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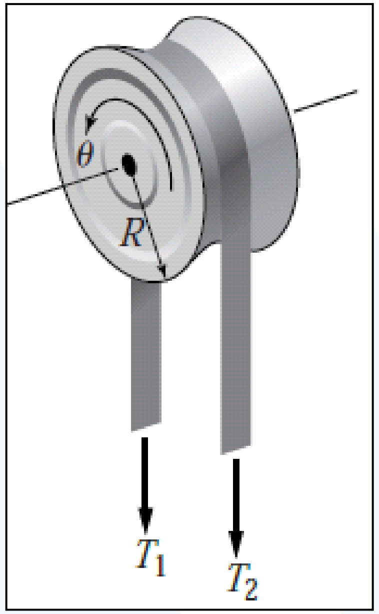
PULLEY DYNAMICS

Pulleys can be used to change the direction of an applied force or to amplify forces. In our examples, we will assume that the cords, ropes, chains, and cables drive the pulleys without slipping and are inextensible; if not, then they must be modeled as springs. Figure shows a pulley of inertia I whose center is fixed to a support. Assume that the tension forces in the cable are different on each side of the pulley.



Then

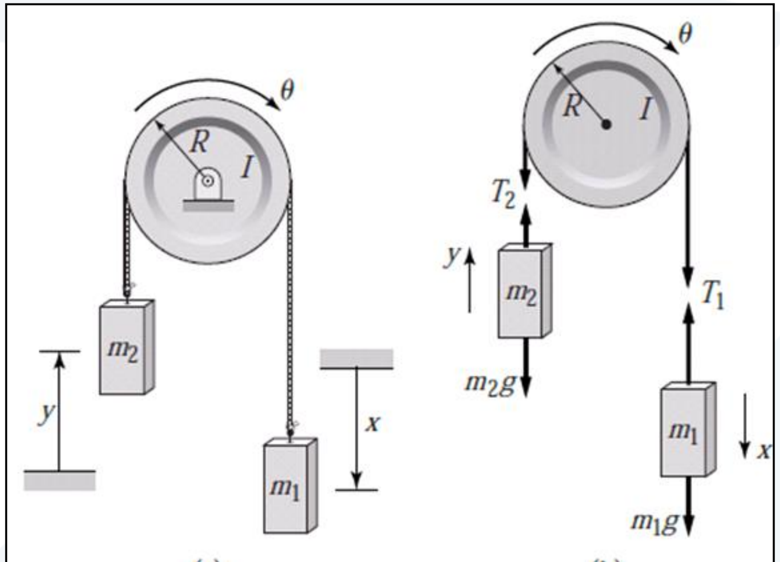
$$I\ddot{\theta} = RT_1 - RT_2 = R(T_1 - T_2)$$

An immediate result of practical significance is that the tension forces are approximately equal if $I\ddot{\theta}$ is negligible. This condition is satisfied if either the pulley rotates at a constant speed or if the pulley inertia is negligible compared to the other inertias in the system. The pulley inertia will be negligible if either its mass or its radius is small. Thus, when we neglect the mass, radius, or inertia of a pulley, the tension forces in the cable may be taken to be the same on both sides of the pulley.

The force on the support at the pulley center is $T_1 + T_2$. If the mass, radius, or inertia of the pulley are negligible, then the support force is $2T_1$.

EXAMPLE

Consider the pulley system shown in Figure a. Obtain the equation of motion in terms of x and obtain an expression for the tension forces in the cable.



(a) (b)

Solution

The free body diagrams of the three bodies are shown in part (b) of the figure. Newton's law for mass m_1 gives

$$m_1\ddot{x} = m_1 g - T_1 \quad (1)$$

Newton's law for mass m_2 gives

$$m_2\ddot{y} = T_2 - m_2 g \tag{2}$$

$$I\ddot{\theta} = RT_1 - RT_2 = R(T_1 - T_2) \tag{3}$$

Because the cable is assumed inextensible, $x = y$ and thus $\ddot{x} = \ddot{y}$. We can then solve (1) and (2) for the tension forces.

$$T_1 = m_1 g - m_1\ddot{x} = m_1(g - \ddot{x}) \tag{4}$$

$$T_2 = m_2\ddot{y} + m_2 g = m_2(\ddot{y} + g) = m_2(\ddot{x} + g) \tag{5}$$

Substitute these expressions into (3).

$$I\ddot{\theta} = (m_1 - m_2)gR - (m_1 + m_2)R\ddot{x} \tag{6}$$

Because $x = R\theta$, $\ddot{x} = R\ddot{\theta}$, and (6) becomes

$$I\frac{\ddot{x}}{R} = (m_1 - m_2)gR - (m_1 + m_2)R\ddot{x}$$

which can be rearranged as

$$\left(m_1 + m_2 + \frac{I}{R^2}\right)\ddot{x} = (m_1 - m_2)g \quad (7)$$

This is the desired equation of motion. We can solve it for $\ddot{x}$ and substitute the result into equations (4) and (5) to obtain T_1 and T_2 as functions of the parameters m_1, m_2, I, R, and g.

Equation (7) can be solved for $\dot{x}(t)$ and $x(t)$ by direct integration.

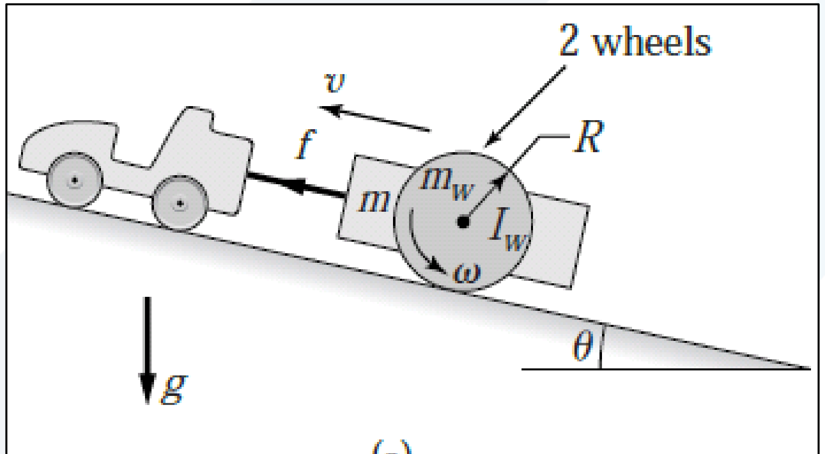
EQUIVALENT MASS AND INERTIA

Some systems composed of translating and rotating parts whose motions are directly coupled can be modeled as a purely translational system or as a purely rotational system, by using the concepts of equivalent mass and equivalent inertia. These models can be derived using kinetic energy equivalence.

Equivalent mass and equivalent inertia are complementary concepts. A system should be viewed as an equivalent mass if an external force is applied, and as an equivalent inertia if an external torque is applied.

EXAMPLE

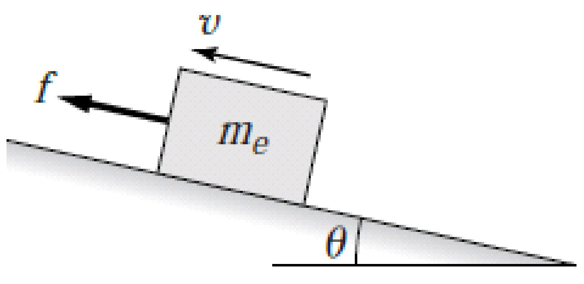
A tractor pulls a cart up a slope, starting from rest and accelerating to 20 m/s in 15 s (Figure a). The force in the cable is f, and the body of the cart has a mass m. The cart has two identical wheels, each with radius R, mass m_w, and inertia I_w about the wheel center. The two wheels are coupled with an axle whose mass is negligible. Assume that the wheels do not slip or bounce. Derive an expression for the force f using kinetic energy equivalence.



(a)

Solution

The assumption of no slip and no bounce means that the wheel rotation is directly coupled to the cart translation. This means that if we know the cart translation x, we also know the wheel rotation θ, because $x = R\theta$ if the wheels do not slip or bounce. Because the input is the force f, we will derive an equivalent mass, and thus we will visualize the system as a block of mass m_e being pulled up the incline by the force f, as shown in Figure b. The wheels will roll without



(b)

slipping (pure rolling). Therefore, in our equivalent model, Figure b, there is no friction between the block and the surface. The kinetic energy of the system is

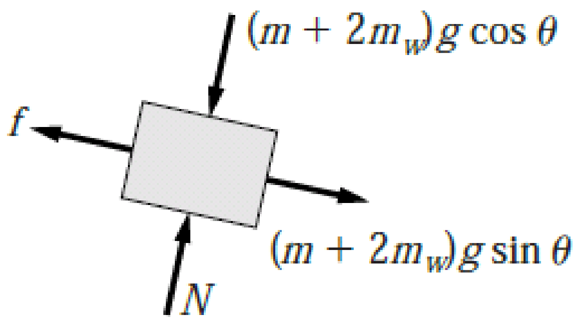
$$KE = \frac{1}{2}mv^2 + \frac{1}{2}(2m_w v^2) + \frac{1}{2}(2I_w\omega^2)$$

Because $v = R\omega$, we obtain

$$KE = \frac{1}{2}\left(m + 2m_w + 2\frac{I_w}{R^2}\right)v^2 \quad (1)$$

For the block in Figure b, $KE = 0.5m_e v^2$. Comparing this with equation (1) we see that the equivalent mass is given by

$$m_e = m + 2m_w + 2\frac{I_w}{R^2} \quad (2)$$



(c)

From the free body diagram we obtain the following equation of motion.

$$m_e\dot{v} = f - (m + 2m_w)g\sin\theta \tag{3}$$

where m_e is given by equation (2).

The acceleration is $\dot{v} = 20/15 = 4/3\ \mathrm{m/s^2}$. Substitute this into equation (3) and solve for f:

$$f = \frac{4}{3}m_e + (m + 2m_w)g\sin\theta \tag{4}$$

This is the force required to provide the specified acceleration.

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MECHANICAL DRIVES

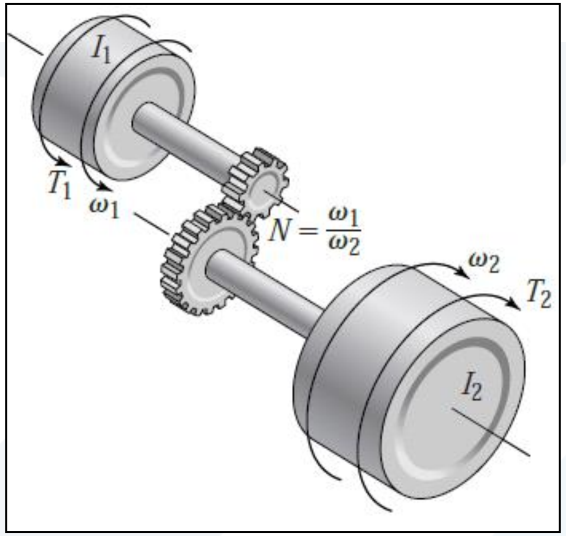
Gears, belts, levers, and pulleys transform an input motion, force, or torque into another motion, force, or torque at the output. For example, a gear pair can be used to reduce speed and increase torque, and a lever can increase force.

Several types of gears are used in mechanical drives. These include helical, spur, rack-and-pinion, worm, bevel, and planetary gears. Other mechanical drives use belts or chains. We now use a spur gear pair, a rack-and-pinion gear pair, and a belt drive to demonstrate the use of kinetic energy equivalence to obtain a model. This approach can be used to analyze other gear and drive types.

A pair of spur gears is shown in Figure. The input shaft (shaft 1) is connected to a motor that produces a torque T_1 at a speed ω_1, and drives the output shaft (shaft 2). One use of such a system is to increase the effective motor torque. The gear ratio N is defined as the ratio of the input rotation θ_1 to the output rotation θ_2. Thus, $N = \theta_1/\theta_2$. From geometry we can see that N is also the speed ratio $N = \omega_1/\omega_2$. Thus, the pair is a speed *reducer* if $N > 1$. The gear ratio is also the diameter ratio $N = D_2/D_1$, and the gear tooth ratio $N = n_2/n_1$, where n is the number of gear teeth.

EXAMPLE

Consider the spur gears shown in Figure. Derive the expression for the equivalent inertia I_e felt on the input shaft.



Solution

Let I_1 and I_2 be the total moments of inertia on the shafts. The kinetic energy of the system is then

$$KE = \frac{1}{2} I_1 \omega_1^2 + \frac{1}{2} I_2 \omega_2^2 = \frac{1}{2} I_1 \omega_1^2 + \frac{1}{2} I_2 \left(\frac{\omega_1}{N}\right)^2$$

or

$$\mathrm{KE} = \frac{1}{2}\left(I_1 + \frac{1}{N^2} I_2\right)\omega_1^2$$

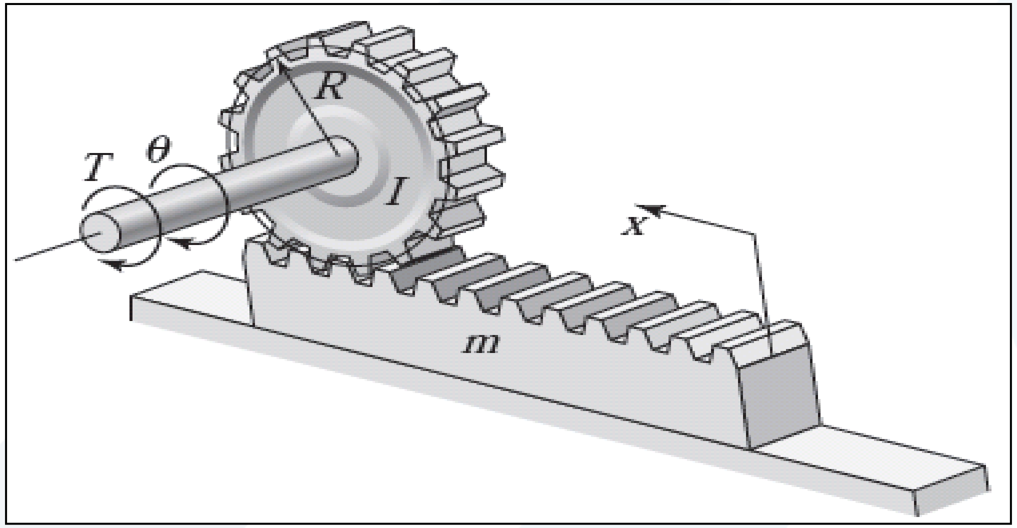
Therefore the equivalent inertia felt on the input shaft is

$$I_e = I_1 + \frac{I_2}{N^2} \qquad (1)$$

This means that the dynamics of the system can be described by the model $I_e\dot{\omega}_1 = T_1$.

EXAMPLE

A rack-and-pinion, shown in Figure, is used to convert rotation into translation. The input shaft rotates through the angle θ as a result of the torque T produced by a motor. The pinion rotates and causes the rack to translate. Derive the expression for the equivalent inertia I_e felt on the input shaft. The mass of the rack is m, the inertia of the pinion is I, and its mean radius is R.



Solution

The kinetic energy of the system is (neglecting the inertia of the shaft)

$$\text{KE} = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I\dot{\theta}^2$$

where $\dot{x}$ is the velocity of the rack and $\dot{\theta}$ is the angular velocity of the pinion and shaft. From geometry, $x = R\theta$, and thus $\dot{x} = R\dot{\theta}$. Substituting for $\dot{x}$ in the expression for KE, we obtain

$$\text{KE} = \frac{1}{2}m\left(R\dot{\theta}\right)^2 + \frac{1}{2}I\dot{\theta}^2 = \frac{1}{2}\left(mR^2 + I\right)\dot{\theta}^2$$

Thus the equivalent inertia felt on the shaft is

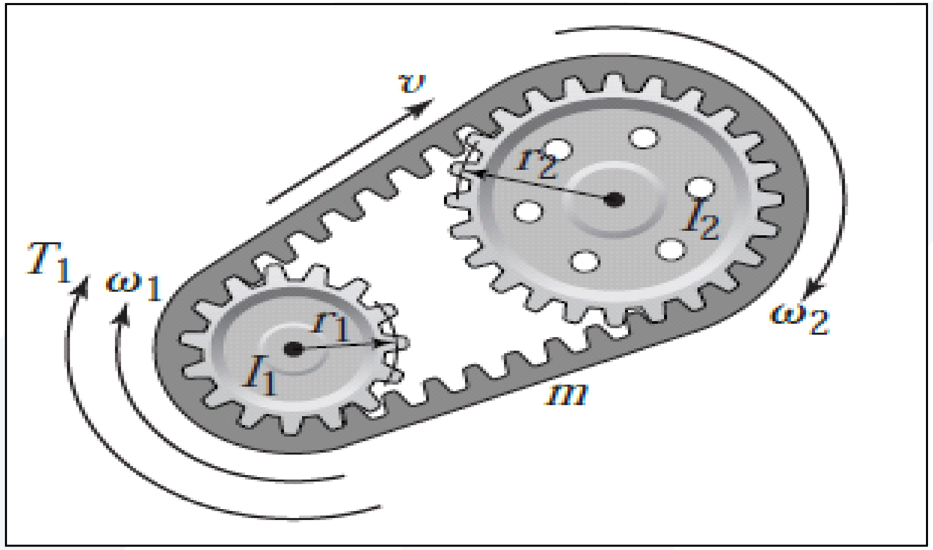
$$I_e = mR^2 + I \tag{1}$$

and the model of the system's dynamics is $I_e\ddot{\theta} = T$, which can be expressed in terms of x as $I_e\ddot{x} = RT$.

EXAMPLE

Belt drives and chain drives, like those used on bicycles, have similar characteristics and can be analyzed in a similar way. A belt drive is shown in Figure . The input shaft (shaft 1) is connected to a device (such as a bicycle crank) that produces a torque T_1 at a speed ω_1, and drives the output shaft (shaft 2). The mean sprocket radii are r_1 and r_2, and their inertias are I_1 and I_2. The belt mass is m.

Derive the expression for the equivalent inertia I_e felt on the input shaft.



Solution

The kinetic energy of the system is

$$\mathrm{KE} = \frac{1}{2}I_1\omega_1^2 + \frac{1}{2}I_2\omega_2^2 + \frac{1}{2}mv^2$$

If the belt does not stretch, the translational speed of the belt is $v = r_1\omega_1 = r_2\omega_2$. Thus we can express KE as

$$\mathrm{KE} = \frac{1}{2}I_1\omega_1^2 + \frac{1}{2}I_2\left(\frac{r_1\omega_1}{r_2}\right)^2 + \frac{1}{2}m(r_1\omega_1)^2 = \frac{1}{2}\left[I_1 + I_2\left(\frac{r_1}{r_2}\right)^2 + mr_1^2\right]\omega_1^2$$

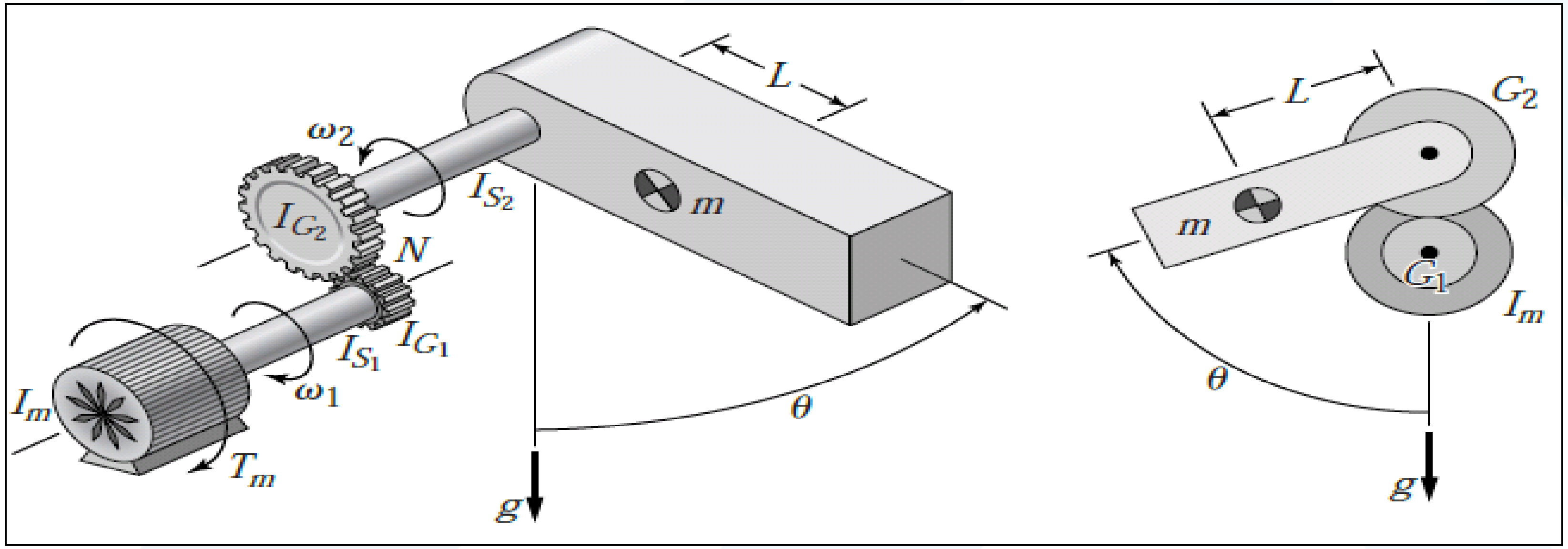
Therefore, the equivalent inertia felt on the input shaft is

$$I_e = I_1 + I_2\left(\frac{r_1}{r_2}\right)^2 + mr_1^2 \qquad (1)$$

This means that the dynamics of the system can be described by the model $I_e\dot{\omega}_1 = T_1$.

EXAMPLE

A single link of a robot arm is shown in Figure. The arm mass is m and its center of mass is located a distance L from the joint, which is driven by a motor torque T_m through a pair of spur gears. The values of m and L depend on the payload being carried in the hand and thus can be different for each application. The gear ratio is $N=2$ (the motor shaft has the greater speed). The motor and gear rotation axes are fixed by bearings.



To control the motion of the arm we need to have its equation of motion. Obtain this equation in terms of the angle θ. The given values for the motor, shaft, and gear inertias are

$$I_m = 0.05\ \text{kg}\cdot\text{m}^2 \qquad I_{G_1} = 0.025\ \text{kg}\cdot\text{m}^2 \qquad I_{S_1} = 0.01\ \text{kg}\cdot\text{m}^2$$

$$I_{G_2} = 0.1\ \text{kg}\cdot\text{m}^2 \qquad I_{S_2} = 0.02\ \text{kg}\cdot\text{m}^2$$

Solution

Our approach is to model the system as a single inertia rotating about the motor shaft with a speed ω_1. To find the equivalent inertia about this shaft we first obtain the expression for the kinetic energy of the total system and express it in terms of the shaft speed ω_1. Note that the mass m is translating with a speed $L\omega_2$.

$$\text{KE} = \frac{1}{2}(I_m + I_{S_1} + I_{G_1})\omega_1^2 + \frac{1}{2}(I_{S_2} + I_{G_2})\omega_2^2 + \frac{1}{2}m(L\omega_2)^2$$

But $\omega_2 = \omega_1/N = \omega_1/2$. Thus,

$$\text{KE} = \frac{1}{2}\left[I_m + I_{S_1} + I_{G_1} + \frac{1}{2^2}(I_{S_2} + I_{G_2} + mL^2)\right]\omega_1^2$$

Therefore, the equivalent inertia referenced to the motor shaft is

$$I_e = I_m + I_{S_1} + I_{G_1} + \frac{1}{2^2}\left(I_{S_2} + I_{G_2} + mL^2\right) = 0.115 + 0.25mL^2$$

The equation of motion for this equivalent inertia can be obtained in the same way as that of a pendulum, by noting that the gravity moment $mgL\sin\theta$, which acts on shaft 2, is also felt on the motor shaft, but *reduced* by a factor of N due to the gear pair. Thus,

$$I_e\dot{\omega}_1 = T_m - \frac{1}{N}mgL\sin\theta$$

But $\omega_1 = N\omega_2 = N\dot{\theta}$. Thus

$$I_e N\ddot{\theta} = T_m - \frac{1}{N}mgL\sin\theta$$

Substituting the given values, we have

$$2(0.115 + 0.25mL^2)\ddot{\theta} = T_m - \frac{9.8}{2}mL\sin\theta$$

or

$$(0.23 + 0.5mL^2)\ddot{\theta} = T_m - 4.9mL\sin\theta \qquad (1)$$

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