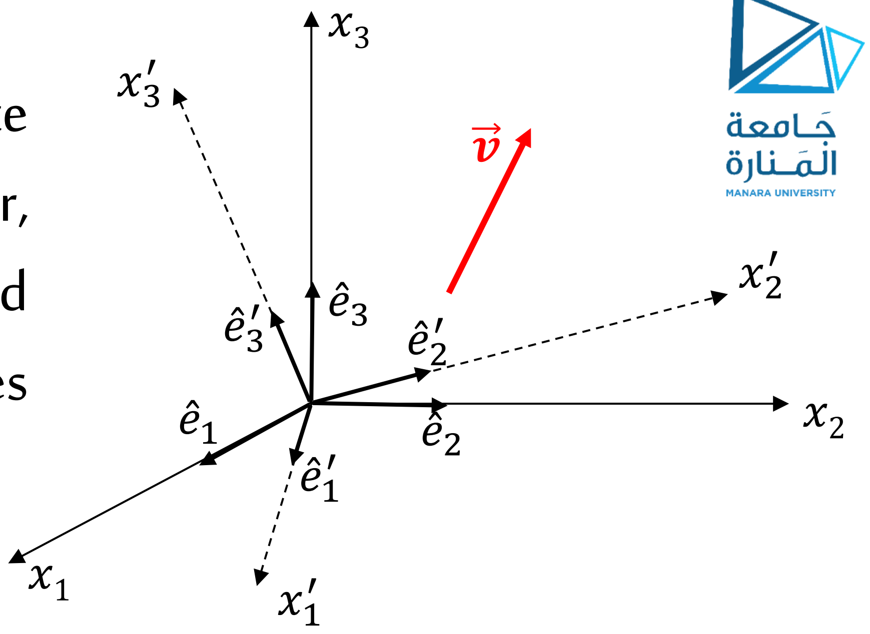
Coordinate Transformations

To express continuum mechanics variables in different coordinate systems requires development of transformation rules for scalar, vector, matrix and higher order variables – a concept connected with basic definitions of tensor variables. The two Cartesian frames (x_1, x_2, x_3) and (x'_1, x'_2, x'_3) differ only by orientation.



Using Rotation Matrix: $R_{ij} = \cos(x_i', x_j)$

$$\left.\begin{aligned} \hat{e}_1' &= R_{11}\hat{e}_1 + R_{12}\hat{e}_2 + R_{13}\hat{e}_3 \\ \hat{e}_2' &= R_{21}\hat{e}_1 + R_{22}\hat{e}_2 + R_{23}\hat{e}_3 \\ \hat{e}_3' &= R_{31}\hat{e}_1 + R_{32}\hat{e}_2 + R_{33}\hat{e}_3 \end{aligned}\right\} \Rightarrow \quad \hat{e}_i' = R_{ij}\hat{e}_j \quad \Rightarrow \quad \hat{e}_i = R_{ji}\hat{e}_j'$$

Transformation laws for Cartesian vector components

$$\vec{v} = v_1\hat{e}_1 + v_2\hat{e}_2 + v_3\hat{e}_3 = v_i\hat{e}_i = v'_1\hat{e}'_1 + v'_2\hat{e}'_2 + v'_3\hat{e}'_3 = v'_i\hat{e}'_i$$

$$\left.\begin{aligned}\vec{v} = v'_1\hat{e}'_1 + v'_2\hat{e}'_2 + v'_3\hat{e}'_3 = v_1(R_{11}\hat{e}'_1 + R_{21}\hat{e}'_2 + R_{31}\hat{e}'_2)\\ +v_2(R_{12}\hat{e}'_1 + R_{22}\hat{e}'_2 + R_{32}\hat{e}'_2)\\ +v_3(R_{13}\hat{e}'_1 + R_{23}\hat{e}'_2 + R_3\hat{e}'_2)\end{aligned}\right\} \Rightarrow \left\{\begin{aligned}v'_1 &= R_{11}v_1 + R_{12}v_2 + R_{13}v_3\\ v'_2 &= R_{21}v_1 + R_{22}v_2 + R_{23}v_3\\ v'_3 &= R_{31}v_1 + R_{32}v_2 + R_{33}v_3\end{aligned}\right\} \Rightarrow v'_i = R_{ij}v_j$$

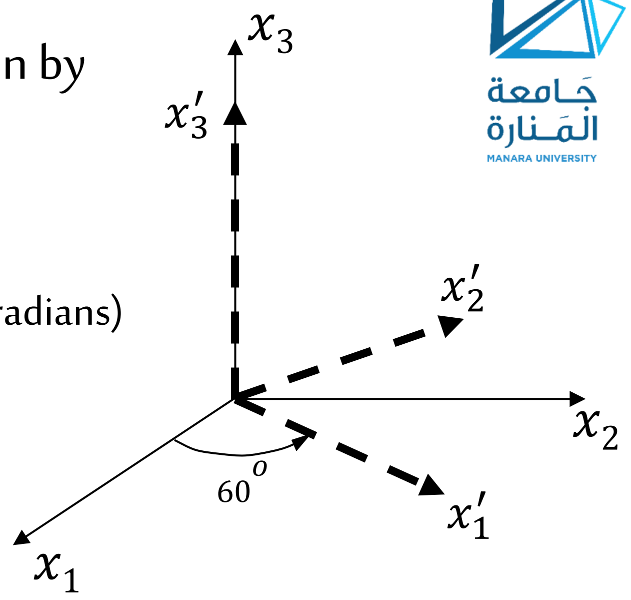
$$\left.\begin{aligned}\vec{v} = v_1\hat{e}_1 + v_2\hat{e}_2 + v_3\hat{e}_3 = v'_1(R_{11}\hat{e}_1 + R_{12}\hat{e}_2 + R_{13}\hat{e}_3)\\ +v'_2(R_{21}\hat{e}_1 + R_{22}\hat{e}_2 + R_{23}\hat{e}_3)\\ +v'_3(R_{31}\hat{e}_1 + R_{32}\hat{e}_2 + R_{33}\hat{e}_3)\end{aligned}\right\} \Rightarrow \left\{\begin{aligned}v_1 &= R_{11}v'_1 + R_{21}v'_2 + R_{31}v'_3\\ v_2 &= R_{12}v'_1 + R_{22}v'_2 + R_{32}v'_3\\ v_3 &= R_{13}v'_1 + R_{23}v'_2 + R_{33}v'_3\end{aligned}\right\} \Rightarrow v_i = R_{ji}v'_j$$

المقدار وترميزه	**Scalars** φ **سلميات**	**Vectors** $\vec{v}$ **الأشعة**	**Tensors** (2d order) $\overleftrightarrow{T}$ **موترات**	**...Higher order**
المرتبة الموترية	$m = 0$	$m = 1$	$m = 2$	$m = 3,4,..$
عدد المركبات	$n^m = 3^0 = 1$	$n^m = 3^1 = 3, v_i$	$n^m = 3^2 = 9, T_{ij}$	$n^m = 27, 81,..$
علاقات التحويل	$\varphi = \varphi'$	$v'_i = R_{ik}v_k$	$T'_{ij} = R_{ik}R_{jl}T_{kl}$	-------------------

Example 1 Transformation Examples

The components of a first and second order tensor in a particular coordinate frame are given by

$$a_i = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix},\; a_{ij} = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 2 \\ 3 & 2 & 4 \end{bmatrix}$$



Determine the components of each tensor in a new coordinate system found through a rotation of 60° (π/6 radians) about the x_3-axis. Choose a counterclockwise rotation when viewing down the negative x_3-axis.

The solution starts by determining the rotation matrix for this case

$$R_{ij} = \begin{bmatrix} \cos 60° & \cos 30° & \cos 90° \\ \cos 150° & \cos 60° & \cos 90° \\ \cos 90° & \cos 90° & \cos 0° \end{bmatrix} = \begin{bmatrix} 1/2 & \sqrt{3}/2 & 0 \\ -\sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$a_i' = R_{ij}a_j = \begin{bmatrix} 1/2 & \sqrt{3}/2 & 0 \\ -\sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 1/2+2\sqrt{3} \\ 2-\sqrt{3}/2 \\ 2 \end{bmatrix}$$

$$a_{ij}' = R_{ip}R_{jq}a_{pq} = \begin{bmatrix} 1/2 & \sqrt{3}/2 & 0 \\ -\sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 2 \\ 3 & 2 & 4 \end{bmatrix} \begin{bmatrix} 1/2 & \sqrt{3}/2 & 0 \\ -\sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}^T = \begin{bmatrix} 7/4 & \sqrt{3}/4 & 3/2+\sqrt{3} \\ \sqrt{3}/4 & 5/4 & 1-3\sqrt{3}/2 \\ 3/2+\sqrt{3} & 1-3\sqrt{3}/2 & 4 \end{bmatrix}$$

Principal Values and Directions for Symmetric Second Order Tensors

The direction determined by unit vector $\hat{n}$ is said to be a *principal direction* or *eigenvector* of the symmetric second order tensor T_{ij}, if there exists a parameter λ (*principal value* or *eigenvalue*) such that

$$T_{ij}n_j = \lambda n_i \quad \Rightarrow \quad (T_{ij}n_j - \lambda\delta_{ij})n_j = 0$$

This is a homogeneous system of three linear algebraic equations in the unknowns n_1, n_2, n_3. The system possesses nontrivial solution if and only if determinant of coefficient matrix vanishes

$$\det[T_{ij}n_j - \lambda\delta_{ij}] = -\lambda^3 + I_T\lambda^2 - II_T\lambda + III_T = 0$$

scalars $I_T, II_T and\ III_T$, are called the *fundamental invariants* of the tensor T_{ij} .

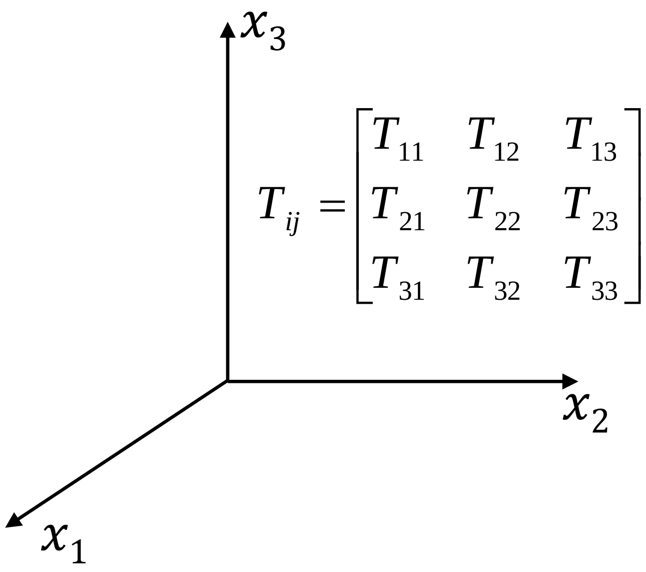
$$I_T = T_{ii} = T_{11} + T_{22} + T_{33}$$

$$II_T = \frac{1}{2}(T_{ii}T_{jj} - T_{ij}T_{ji}) = \begin{vmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{vmatrix} + \begin{vmatrix} T_{22} & T_{23} \\ T_{32} & T_{33} \end{vmatrix} + \begin{vmatrix} T_{33} & T_{31} \\ T_{13} & T_{11} \end{vmatrix}$$

$$III_T = \det[T_{ij}]$$

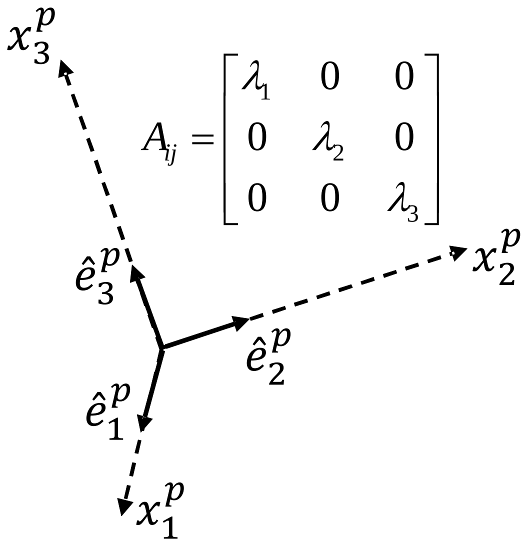
Principal Axes of Second Order Tensors

It is always possible to identify a right-handed Cartesian coordinate system such that each axes lie along principal directions of any given symmetric second order tensor. Such axes are called the *principal axes* of the tensor, and the basis vectors are the principal directions $\{\hat{e}_1^p, \hat{e}_2^p, \hat{e}_3^p\}$



$$T_{ij} = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}$$

Original Given Axes



$$A_{ij} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

Principal Axes

Example 2.11.1 Principal Value Problem

Determine the invariants, and principal values and directions of: $T_{ij} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$