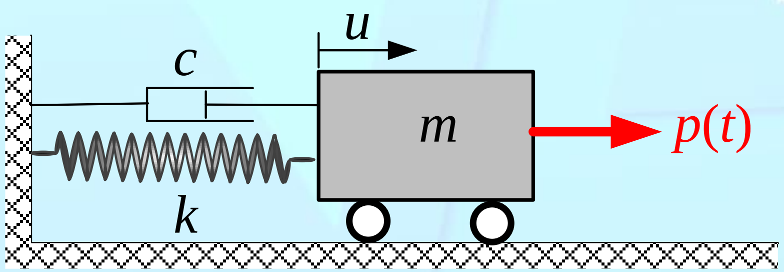
Free Response of a SDOF system

The equation of motion of a SDOF system has been written as follows:

$$m\ddot{u}(t)+c\dot{u}(t)+ku(t)=p(t)$$



which is a non-homogeneous second-order linear ordinary differential equation with constant coefficients.

The solution of this equation depends on the dynamic loading $p(t)$ and on the initial conditions.

Forced response is the solution of the equation, with $p(t) \neq 0$.

Free response is the solution of the homogeneous equation, with $p(t) = 0$.

It describes the motion of a SDOF oscillator with non-zero initial conditions.

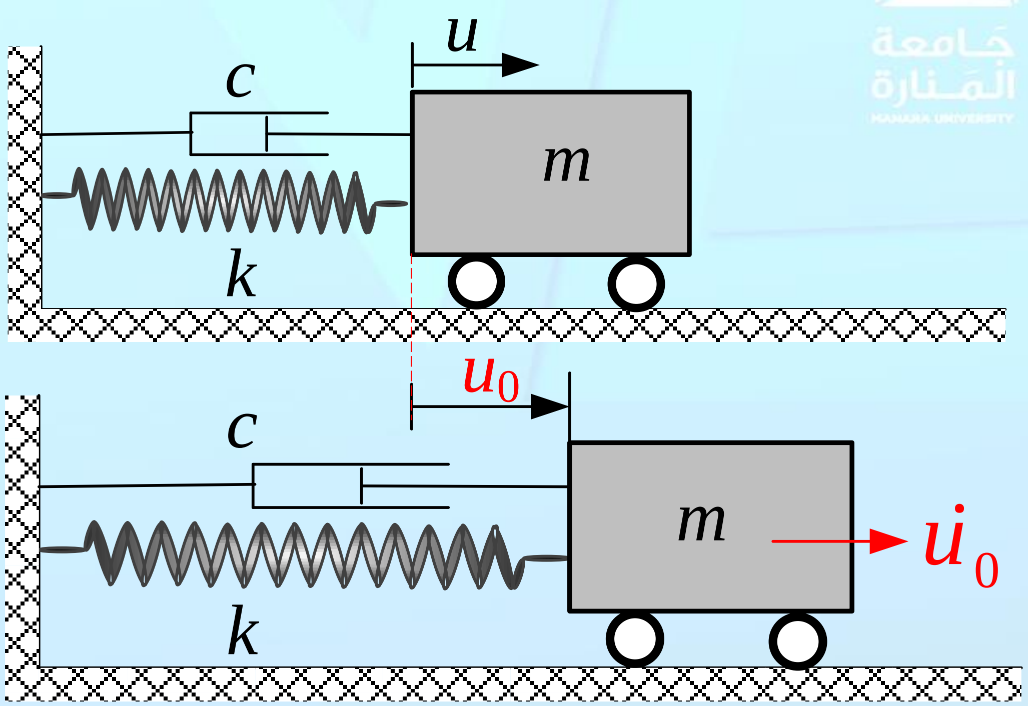
The *free response* is the solution of the homogeneous differential equation, with $p(t) = 0$.

$$m\ddot{u}(t)+c\dot{u}(t)+ku(t)=0$$

It describes the motion of a SDOF oscillator

with non-zero initial conditions.

$$u_0 = u(0) \text{ and } \dot{u}_0 = \dot{u}(0)$$



First of all we write the equation of

motion in its *canonical form.*

$$\ddot{u}(t)+\frac{c}{m}\dot{u}(t)+\frac{k}{m}u(t)=0$$

Putting:

$$\omega^2 = \frac{k}{m} \text{ and } 2\xi\omega = \frac{c}{m}$$

The canonical form of the equation of motion becomes

$$\ddot{u}(t)+2\xi\omega\dot{u}(t)+\omega^2 u(t)=0$$

ω : is the angular frequency [rad/sec]. التردد أو التواتر الزاويّ

c : damping factor [N/m/sec ≡ N.sec/m ≡ kg/sec] معامل التخميد : damping ration [?] نسبة التخميد

$$\ddot{u}(t)+2\xi\omega\dot{u}(t)+\omega^2 u(t)=0, \text{ with } u_0=u(0) \text{ and } \dot{u}_0=\dot{u}(0)$$

Characteristic equation

A particular solution for equation can be found considering that the variables $u, \dot{u}$ & $\ddot{u}$, are in some way linearly dependent for their sum to be zero. The exponential function has precisely that property and a solution could therefore be

$$u(t)=Ce^{st}$$

The constant C has dimension [L] and st has no dimension. Hence, constant s has dimension T^{-1}.

Substituting this solution into the canonical equation we have

$$\left(s^2+2\xi\omega s+\omega^2\right)Ce^{st}=0$$

This equation is valid for all values of t, if

$$s^2+2\xi\omega s+\omega^2=0$$

which is known as the *characteristic equation.* It is an Algebraic Equation

Resolving the second degree Algebraic characteristic equation $s^2 + 2\xi\omega s + \omega^2 = 0$ gives two roots:

$$s_1 = -\xi\omega + \omega\sqrt{\xi^2 - 1}, \text{ and } s_2 = -\xi\omega - \omega\sqrt{\xi^2 - 1}.$$

We have seen in the preceding lectures that harmonic motion results when $\xi = 0$.

In that case, the roots s_1 and s_2 were the imaginary numbers: $\pm i\omega$.

For non-zero damping, three types of motion are possible depending on the amount of damping present in the system or depending on the value of $(\xi^2 - 1)$ under the radical. The motions are:

1. Oscillatory when: $0<\xi<1$. Then s_1 and s_2 are complex conjugates;
2. Non-oscillatory when: $\xi=1$. Then s_1 and s_2 are real and equal;
3. Non-oscillatory when: $\xi>1$. Then s_1 and s_2 are real and distinct.

Bifurcation from oscillatory to non-oscillatory motion takes place when $\xi = 1$.

For this reason, the damping coefficient C corresponding to this case is called: *critical damping:* C_{cr}.

Its expression is obtained from equation: $\frac{c}{m} = 2\xi\omega$, putting $\xi = 1$, to get: $C_{cr} = 2m\omega = 2k/\omega = 2\sqrt{km}$

The ratio: $\xi = C/C_{cr}$, is called *ratio of critical damping,* or shortly, *damping ratio.*

Free Damped SDOF Systems - Underdamped Case

When $\xi < 1$ the systems is underdamped.

The two roots of the Characteristic equation can be written as

$$s_1 = -\xi\omega_n + i\,\omega_d \text{, and } s_2 = -\xi\omega_n - i\,\omega_d$$

Where ω_d denoting the damped circular natural frequency, التردد الزاوي المخمد , is given by

$$\omega_d = \omega_n\sqrt{1-\xi^2} \qquad \omega_d \le \omega_n$$

Like ω_d , this is expressed in (rad/sec). The corresponding damped period الدور المخمد , is

$$T_d = \frac{2\pi}{\omega_d} = T_n/\sqrt{1-\xi^2} \qquad T_d \ge T_n$$

With the help of these definitions and Euler's formula, the general solution of the motion equation can be expressed as:

$$u(t) = e^{-\xi\omega_n t}(A_1\cos\omega_d t + A_2\sin\omega_d t)$$

$$\dot{u}(t) = -\xi\omega_n e^{-\xi\omega_n t}(A_1\cos\omega_d t + A_2\sin\omega_d t) + \omega_d e^{-\xi\omega_n t}(-A_1\sin\omega_d t + A_2\cos\omega_d t)$$

Free Damped SDOF Systems
Underdamped Case

As in the undamped case, the two coefficients A_1 & A_2, of the general solution can be determined by the two initial conditions

$$u(0) = u_0 \quad \text{and} \quad \dot{u}(0) = \dot{u}_0$$

To get

$$u(t) = e^{-\xi\omega_n t}\left(u_0 \cos\omega_d t + \frac{\dot{u}_0 + \xi\omega_n u_0}{\omega_d}\sin\omega_d t\right)$$

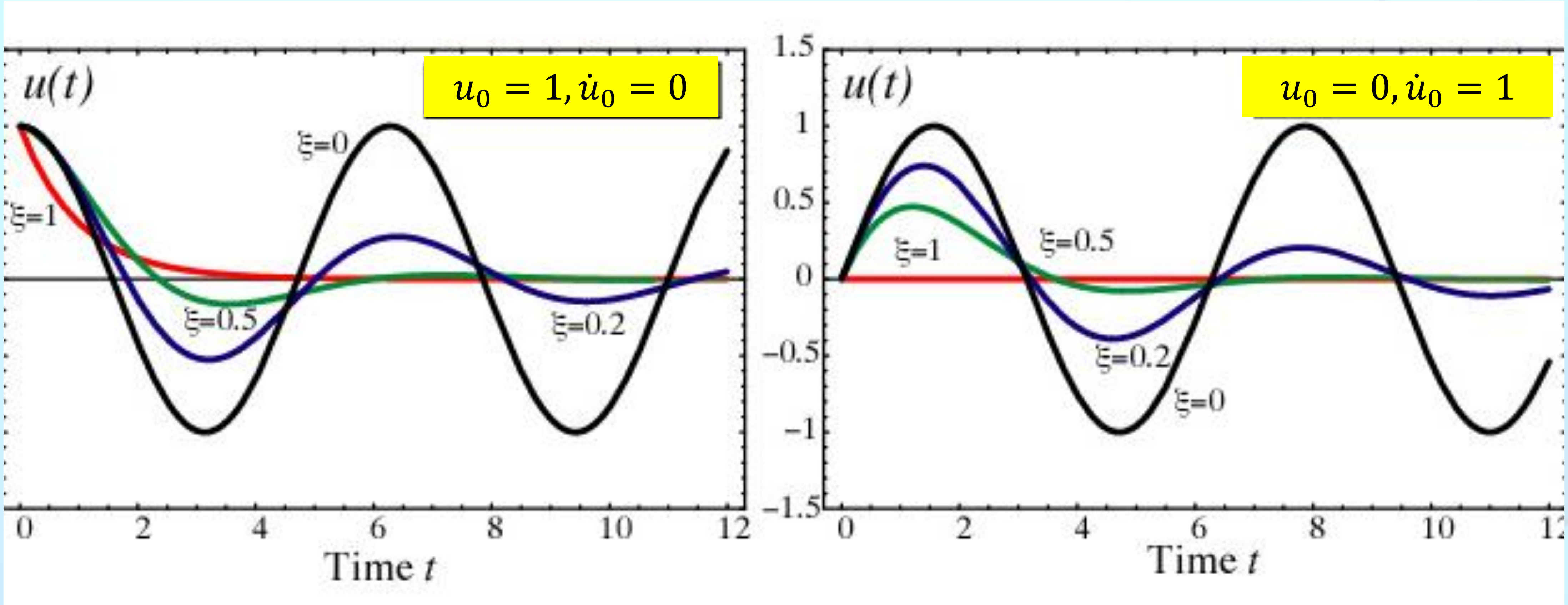
This equation in turn can be rewritten in the phase form

$$u(t) = U e^{-\xi\omega_n t}\cos(\omega_d t - \theta)$$

Where

$$U = \sqrt{u_0^2 + \left(\frac{\dot{u}_0 + \xi\omega_n u_0}{\omega_d}\right)^2} \quad \text{and} \quad \tan\theta = \frac{\dot{u}_0 + \xi\omega_n u_0}{u_0\omega_d}$$

Free Damped SDOF Oscillator
Underdamped Response Plots



$$u(t) = e^{-\xi\omega_n t}\left(u_0 \cos\omega_d t + \frac{\dot{u}_0 + \xi\omega_n u_0}{\omega_d}\sin\omega_d t\right)$$

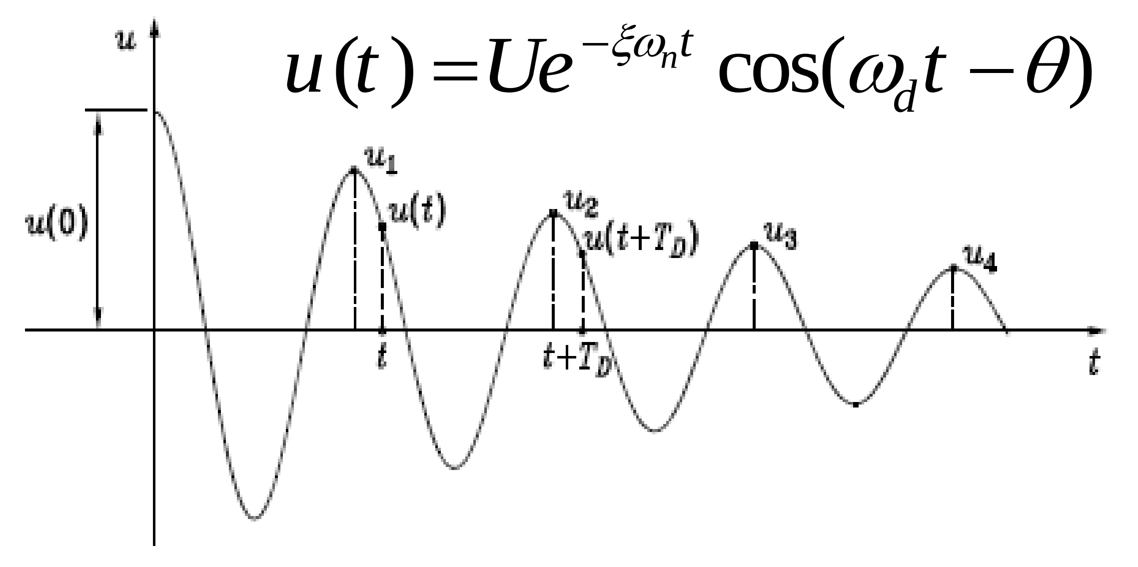
$$u(t) = Ue^{-\xi\omega_n t}\cos(\omega_d t - \theta)$$

Logarithmic decrement

Damping in real structures is generally not of the simple viscous type as analyzed above. But it is often expressed as **equivalent viscous damping obtained from free vibration response tests**.

A measure of damping is the reduction of the amplitude of the response after one cycle of free response.

Figure below shows displacements $u(t)$ and $u(t+TD)$ measured at the ends of a one-cycle interval during free vibration.



$$u(t)=Ue^{-\xi\omega_n t}\cos(\omega_d t-\theta)$$

The ratio of these two displacements gives

$$\frac{u(t)}{u(t+T_d)}=\frac{Ue^{-\xi\omega_n t}\cos(\omega_d t-\theta)}{Ue^{-\xi\omega_n(t+T_d)}\cos[\omega_d(t+T_d)-\theta]}$$

but this can be simplified because

$$\cos[\omega_d(t+T_d)-\theta]=\cos(\omega_d t-\theta+\omega_d T_d)$$

$$=\cos(\omega_d t-\theta+2\pi)=\cos(\omega_d t-\theta)$$

Then the ratio of these two displacements becomes

$$\frac{u(t)}{u(t+T_d)}=\frac{e^{-\xi\omega_n t}}{e^{-\xi\omega_n(t+T_d)}}=e^{\xi\omega_n T_d}$$

Logarithmic decrement

$$\frac{u(t)}{u(t+T_d)} = e^{\xi\omega_n T_d}$$

Taking the logarithm of both sides of we obtain

$$\delta \equiv \ln\frac{u(t)}{u(t+T_d)} = \xi\omega_n T_d = 2\pi\frac{\omega_n}{\omega_d} = \frac{2\pi\xi}{\sqrt{1-\xi^2}}$$

where the quantity δ is called ***logarithmic decrement.***

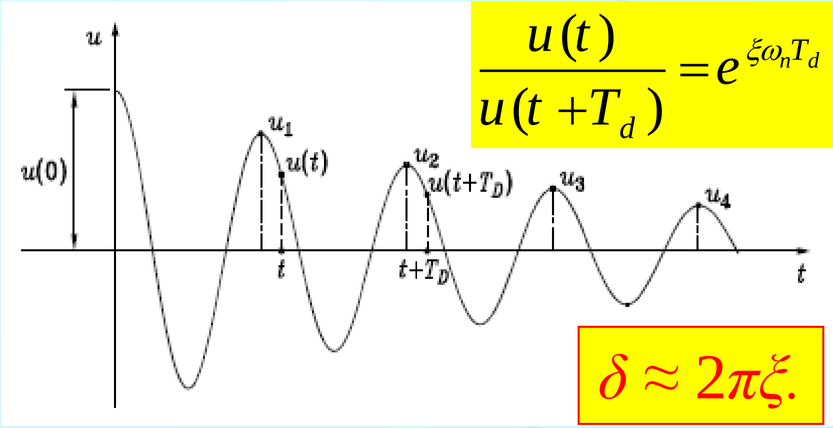
For small values of damping, ξ^2 is negligible compared to unity and an approximation of δ is given by

$$\delta \approx 2\pi\xi.$$

This is used to estimate global damping of a structure from two successive measured displacements after the structure is released from an initial displaced configuration.

Generally, $\xi << 1$ and the displacement $u(t)$ is very close to $u(t+T_D)$ **making the estimate very imprecise**.

To increase the precision of this estimate, the two maximums are taken j cycles apart.



$$\frac{u(t)}{u(t+T_d)} = e^{\xi \omega_n T_d}$$

$$\delta \approx 2\pi\xi.$$

Let u_i and u_{i+j} be the amplitudes at time t_i and $t_i + jT_D$, j being an integer. The ratio (u_i/u_{i+j}) can be expressed as

$$\frac{u_i}{u_{i+j}} = \frac{u_i}{u_{i+1}} \frac{u_{i+1}}{u_{i+2}} \frac{u_{i+2}}{u_{i+3}} \cdots \frac{u_{i+j-1}}{u_{i+j}} = \left(e^{\xi \omega_n T_d}\right)^j = e^{j \xi \omega_n T_d} = e^{j\delta}$$

which, accounting for above results, becomes

$$\frac{u_i}{u_{i+j}} = e^{j\delta}$$

Taking the logarithm of both sides:

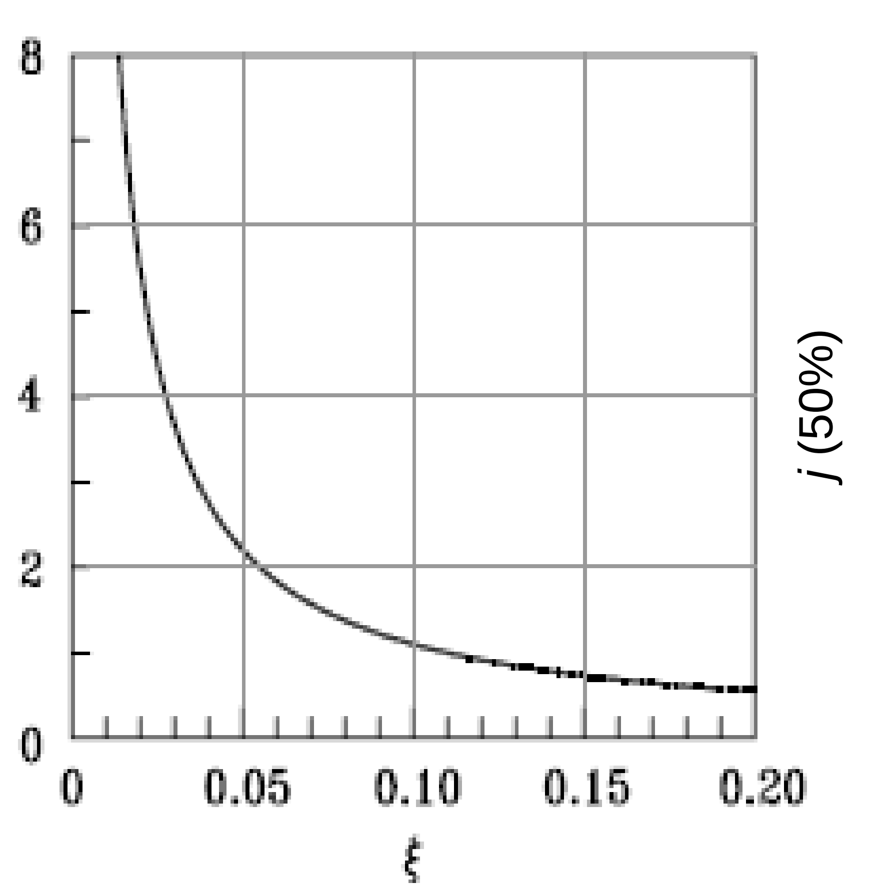
$$\delta = \frac{1}{j} \ln \frac{u_i}{u_{i+j}}$$

Finally the approximation value of the viscous damping ratio is given by

$$\xi \cong \frac{\delta}{2\pi} = \frac{1}{2j\pi} \ln \frac{u_i}{u_{i+j}}$$

$$\xi \approx \frac{\delta}{2\pi} = \frac{1}{2j\pi} \ln \frac{u_i}{u_{i+j}}$$

The number of cycles necessary to reduce the amplitude by 50% is obtained by writing: $\frac{u_i}{u_{i+j}} = 2,$ in this equation to get $j\,(50\%) = \frac{0.11}{\xi}$ This relationship is plotted here.

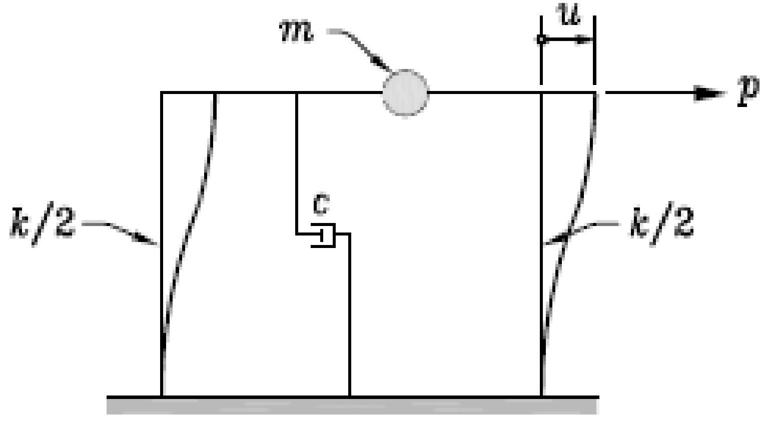


Note that, a useful approximation is: $j(50\%) \approx 10/\xi(\%)$.

During a free vibration test, it is easy to count the number of cycles it takes to reduce the displacement amplitude by 50% and obtain the percent damping ratio.

Hence, if it takes two cycles to reduce the displacement amplitude by 50%, the damping ratio ξ is equal to $10/2 = 5\%$. The damping ratio ξ would be approximately equal to $10/1 = 10\%$, if it takes one cycle to reduce the displacement amplitude by 50% .

EXAMPLE. A one-story structure being tested in a laboratory can be idealized by an infinitely rigid beam supported by two columns. The columns can be considered flexible laterally but rigid axially. The mass of the columns is negligible when compared to the total mass m =1941kg , which is concentrated at the level of the roof. To determine the dynamic properties of the structure, a free vibration test is performed by moving the roof by 20 mm with a cable and a winch. The cable is suddenly cut to set the structure in free vibration. The maximum displacement is 15 mm after one complete cycle which takes place in 0.2 s. Compute the damping ratio ξ, the damping coefficient c, the lateral stiffness of the structure, and the amplitude of the motion after 10 cycles.



Damping ratio ξ :

$$\xi = \frac{1}{2\pi} \ln \frac{20}{15} = 0.0458$$

Damping coefficient C :

$$T_D = 2\pi/\omega_D = 0.2\ s \Rightarrow \omega_D = 2\pi/0.2$$

$$\omega = \frac{\omega_d}{\sqrt{1-\xi^2}} = \frac{2\pi/0.2}{\sqrt{1-0.0458^2}} = 31.45\ rad/s$$

$$C = 2\xi m\omega = 2 \times 0.0458 \times 1941 \times 31.45 = 5592 Ns/m$$

lateral stiffness

$$K = m\omega^2 = 1941 \times 31.45^2 = 1920\ kN/m$$

amplitude after 10 cycles

$$u_{10} = u_0 \left(\frac{u_1}{u_0}\right)^{10} = (20)\left(\frac{15}{20}\right)^{10} = 1.13\ mm$$

Ex. The roof of the building shown in the figure is displaced by 6mm by applying a force of 90 kN and released instantaneously in a free vibration test.

The maximum displacement is 4.8mm after one complete cycle which takes place in 1.2 s.

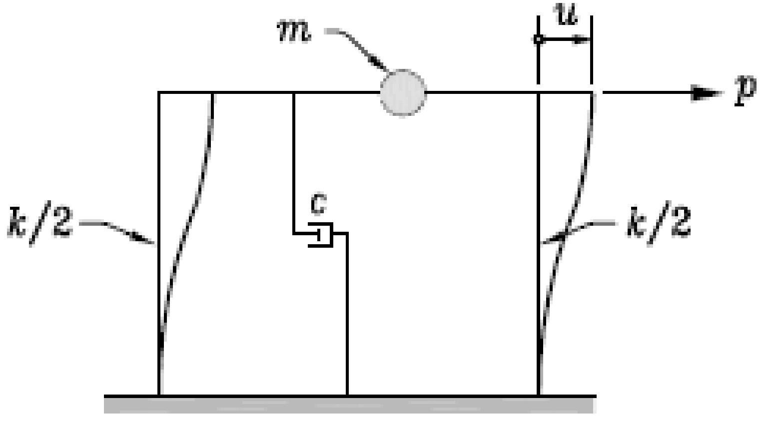
Compute

(a) the damping ratio

(b) the effective mass of the structure,

(c) the angular frequency, and

(d) the number of cycles necessary to reduce the displacement to 5% of the maximum displacement at the initial release.



Ex. Consider the transverse vibration of a bridge structure. For the fundamental frequency it can be considered as a single degree of freedom system. The bridge is deflected at midspan (by winching the bridge down) and suddenly released. After the initial disturbance the vibration was found to decay exponentially from an amplitude of 10 mm to 5.8 mm in three cycles with a frequency of 1.62 Hz. The test was repeated with a vehicle of mass 40 000 kg at mid-span, and the frequency of free vibration was measured to be 1.54 Hz. Find the damping ratio of the structure, the effective mass, and the effective stiffness, and the damping ratio of the structure.

Ex. A free vibration test is conducted on an empty elevated water tank such as the one in figure. A cable attached to the tank applies a lateral (horizontal) force of 75 KN and pulls the tank horizontally by 5 cm. The cable is suddenly cut and the resulting free vibration is recorded.
At the end of four complete cycles, the time is 2.0 sec and the amplitude is 2.5 cm. From these data compute the following:

(a) The damping ratio, ξ;
(b) The natural period of undamped vibration, T_n;
(c) The stiffness, k;
(d) The effective mass of the empty tank, m;
(e) The damping coefficient, c; and
(f) The number of cycles, j, required for the displacement amplitude to decrease to 0.5 cm.