

Structural Mechanics (1)

Lecture No-05

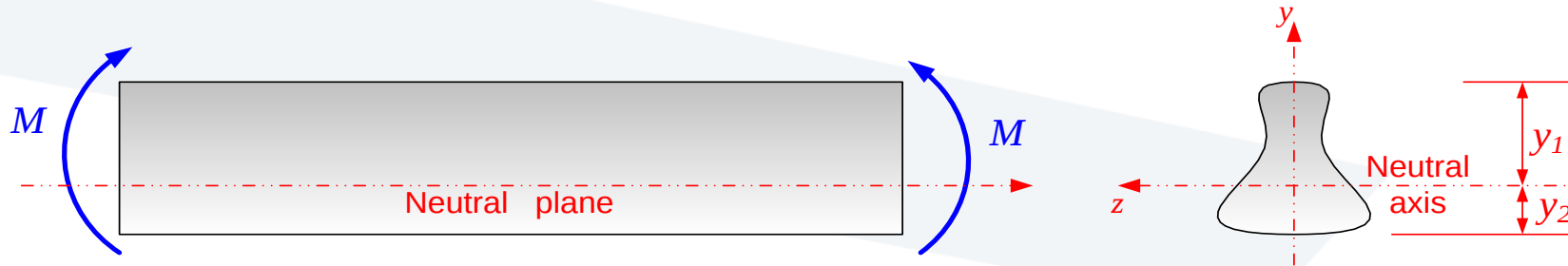
Deflection in Determinate Structures

Deflections of Trusses, Beams, & Frames: Work-Energy Methods

- Deflection of trusses by Work & Strain energy principle
- Principle of Virtual Work
- Deflections of Trusses by the V. W. M.
- Deflections of Beams by the V. W. M.
- Deflections of Frames by the V. W. M.

Strain energy in a beam element

Bending Strain energy in a beam element



$$U = \iiint_V \frac{1}{2} \sigma_x \epsilon_x dV = \iiint_V \frac{\sigma_x^2}{2E} dV = \iiint_V \frac{1}{2E} \left(\frac{M^2}{I^2} y^2 \right) dx dA = \int_0^L \frac{1}{2E} dx \left(\frac{M^2}{I^2} \right) \iint_A y^2 dA = \int_0^L \frac{M^2}{2EI} dx$$

Shear Strain energy in a beam element,

$$U = \iiint_V \frac{1}{2} \tau \gamma dV = \iiint_V \frac{\tau^2}{2G} dV = k \int_0^L \frac{S^2}{2GA} dx$$

$$k = \begin{cases} 1.2 & \text{for a rectangle} \\ 1.1 & \text{for a circle} \\ 1.2 & \text{for a thin circular} \end{cases}$$

Comparison of bending and shear strain energies in a simple beam

$$U_b = \int_0^L \frac{M^2}{2EI} dx$$

$$M(x) = -\frac{1}{2}wx^2 + \frac{1}{2}wLx$$

$$U_s = 1.2 \int_0^L \frac{S^2}{2GA} dx$$

$$S(x) = -wx + \frac{1}{2}wL$$

$$U_b = \int_0^L \frac{M^2}{2EI} dx = \frac{6}{Ebh^3} \int_0^L \left(-\frac{1}{2}wx^2 + \frac{1}{2}wLx\right)^2 dx$$

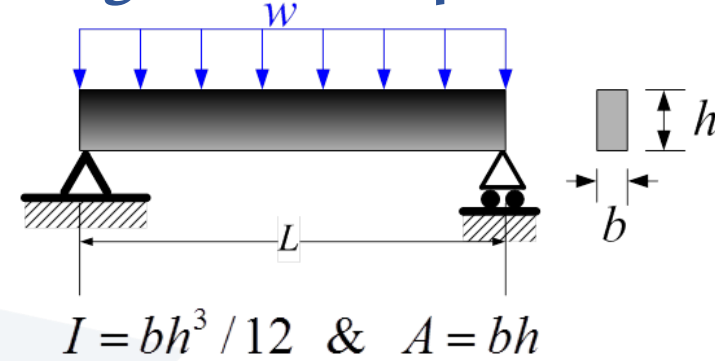
$$= \frac{6w^2}{Ebh^3} \int_0^L \left(\frac{1}{4}x^4 - \frac{1}{2}Lx^3 + \frac{1}{4}L^2x^2\right) dx$$

$$= \frac{6w^2}{Ebh^3} \left[\frac{1}{20}x^5 - \frac{1}{8}Lx^4 + \frac{1}{12}L^2x^3 \right]_0^L = \frac{0.05w^2L^5}{Ebh^3}$$

$$U_s = 1.2 \int_0^L \frac{S^2}{2GA} dx = \frac{0.6}{Gbh} \int_0^L \left(wx - \frac{1}{2}wL\right)^2 dx$$

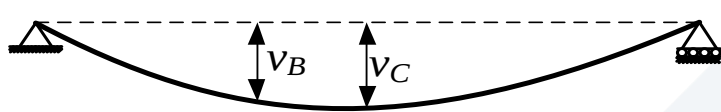
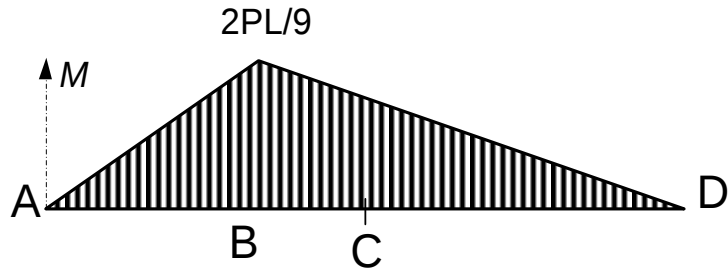
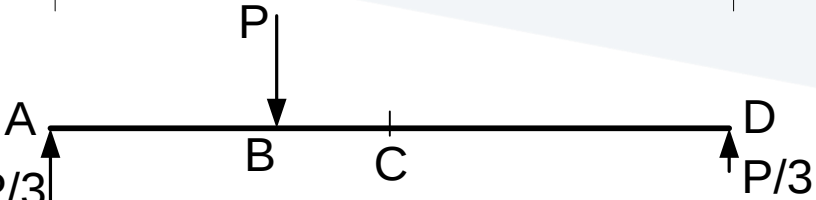
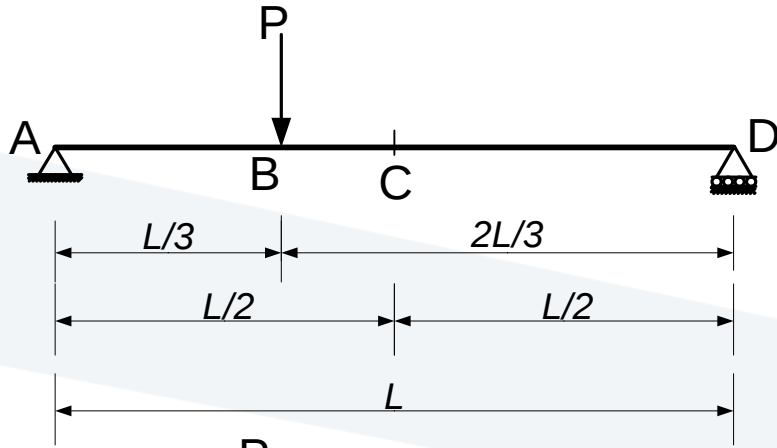
$$= \frac{0.6w^2}{Gbh} \int_0^L \left(x^2 - Lx + \frac{1}{4}L^2\right) dx$$

$$= \frac{0.6w^2}{Gbh} \left[\frac{1}{3}x^3 - \frac{1}{2}Lx^2 + \frac{1}{4}L^2x \right]_0^L = \frac{0.05w^2L^3}{Gbh}$$

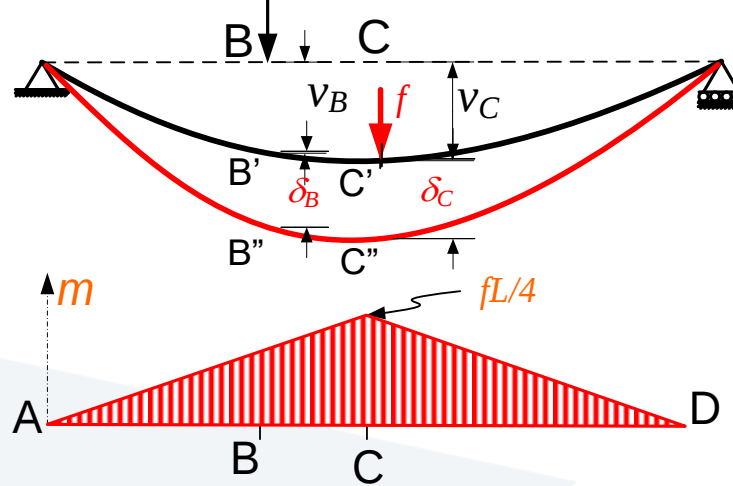


$$\begin{aligned} U_s / U_b &= \left(\frac{0.05w^2L^3}{Gbh} \right) / \left(\frac{0.05w^2L^5}{Ebh^3} \right) \\ &= (E / G) (h^2 / L^2) \approx 2h^2 / L^2 \ll 1. \end{aligned}$$

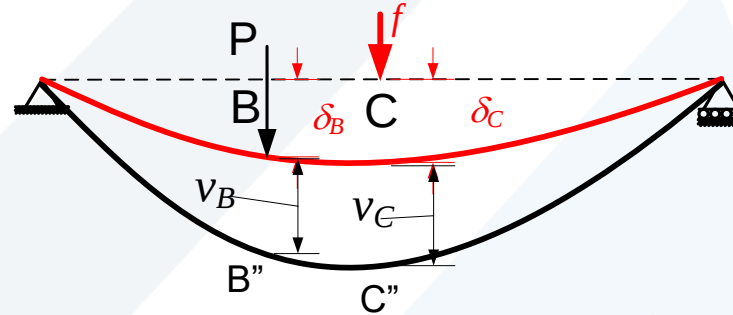
Shear and Axial Strain energies are negligible in comparison with the bending moment energy



$$\frac{1}{2} P v_B = \int_0^L \frac{M^2}{2EI} dx$$



$$\frac{1}{2} P v_B + P \delta_B + \frac{1}{2} f \delta_C = \int_0^L \frac{(M + m)^2}{2EI} dx$$

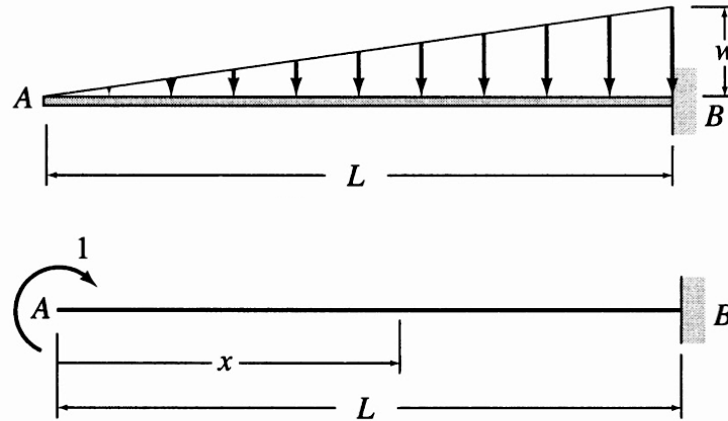


$$\frac{1}{2} f \delta_C + f v_C + \frac{1}{2} P v_B = \int_0^L \frac{(m + M)^2}{2EI} dx$$

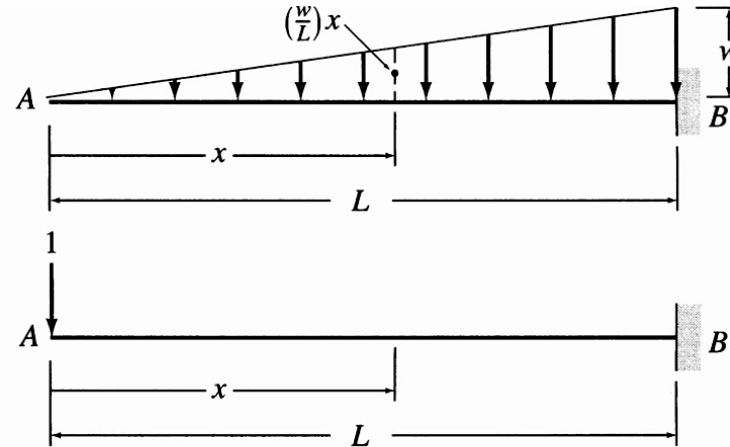
$$P \delta_B = f v_C = \int_0^L \frac{mM}{EI} dx$$

DEFLECTIONS OF BEAMS BY THE V. W. M.

Example-01. Determine the slope and deflection at point A of the beam shown in the figure, by the virtual work method



(c) Virtual System for Determining θ_A — M_{v1}



(d) Virtual System for Determining Δ_A — M_{v2}

Solution

Real System: The real B. M. is:

$$\text{for, } 0 < x < L, \quad M(x) = -\frac{1}{2}(x)\left(\frac{wx}{L}\right)\left(\frac{x}{3}\right) = -\frac{wx^3}{6L}$$

Virtual System 1: The V. B. M. is:

$$\text{for, } 0 < x < L, \quad M_{v1}(x) = 1$$

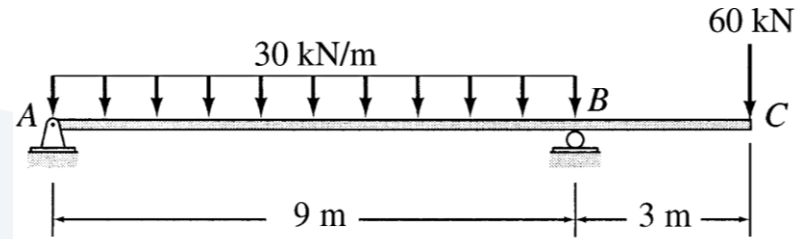
Virtual System 2: The V. B. M. is:

$$\text{for, } 0 < x < L, \quad M_{v2}(x) = -1(x) = -x$$

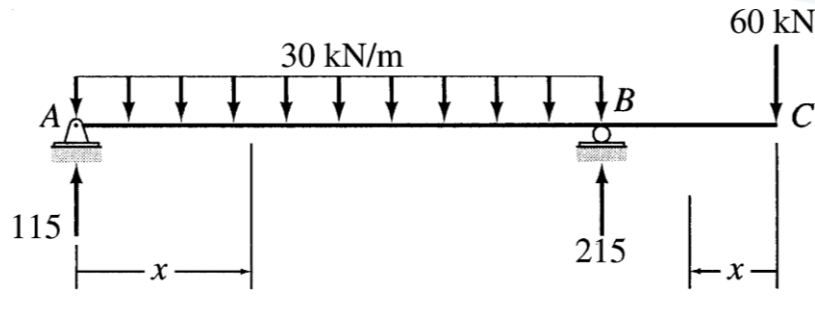
$$\theta_A = \int_0^L \frac{M_{v1}M}{EI} dx = \int_0^L 1\left(-\frac{wx^3}{6LEI}\right) dx = -\frac{wL^3}{24EI} \quad \left| \quad \Delta_A = \int_0^L \frac{M_{v2}M}{EI} dx = \int_0^L (-x)\left(-\frac{wx^3}{6LEI}\right) dx = \frac{wL^4}{30EI}$$

$$\theta_A = \frac{wL^3}{24EI} (\downarrow \uparrow) \quad \Delta_A = \frac{wL^4}{30EI} (\downarrow)$$

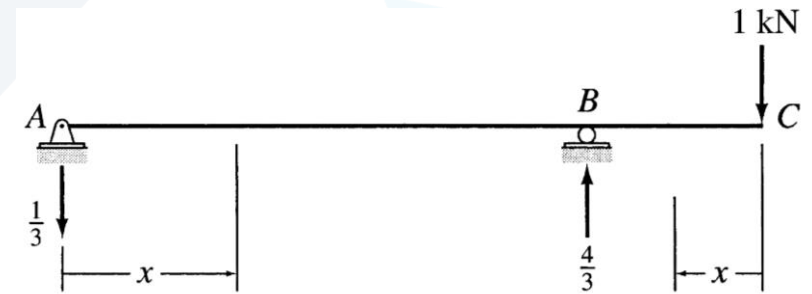
Example-02. Determine the deflection at point C of the beam shown in the figure, by the virtual work method. $EI = \text{const.}$ $E = 200 \text{ GPa}$, $I = 800(10^6) \text{ mm}^4$



Real System



Virtual System



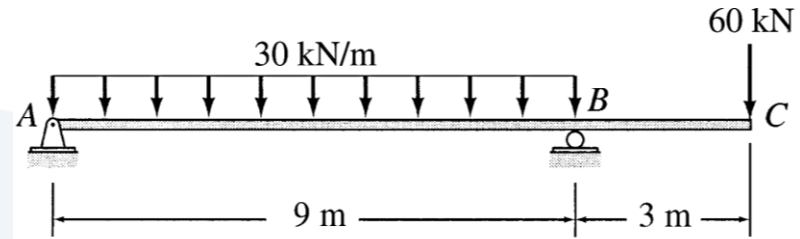
Segment AB: $0 < x < 9, M(x) = 115x - 15x^2$

Segment BC: $0 < x < 3, M(x) = -60x$

Segment AB: $0 < x < 9, m(x) = -\frac{x}{3}$

Segment BC: $0 < x < 3, m(x) = -x$

Example-02. Determine the deflection at point C of the beam shown in the figure, by the virtual work method. $EI = \text{const.}$ $E = 200 \text{ GPa}$, $I = 800(10^6) \text{ mm}^4$



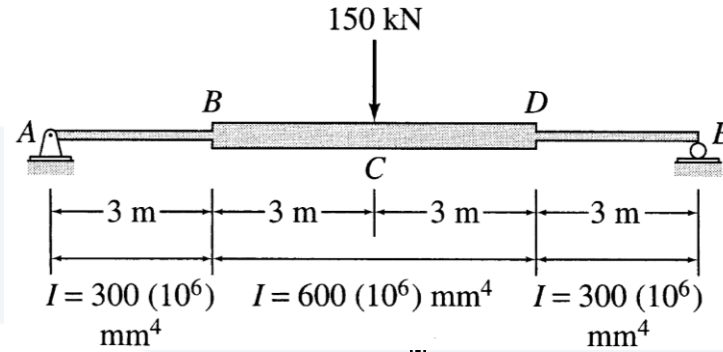
Segment	x Coordinate		M (kN-m)	M_v (kN-m)
	Origin	Limits (m)		
AB	A	0–9	$(115x - 15x^2)$	$-\frac{x}{3}$
CB	C	0–3	$-60x$	$-x$

$$\Delta_C = \int_0^L \frac{M(x) \cdot m(x)}{EI} dx = \frac{1}{EI} \int_0^9 (115x - 15x^2) \left(-\frac{x}{3}\right) dx + \frac{1}{EI} \int_0^3 (-60x)(-x) dx = -\frac{573.75}{EI}$$

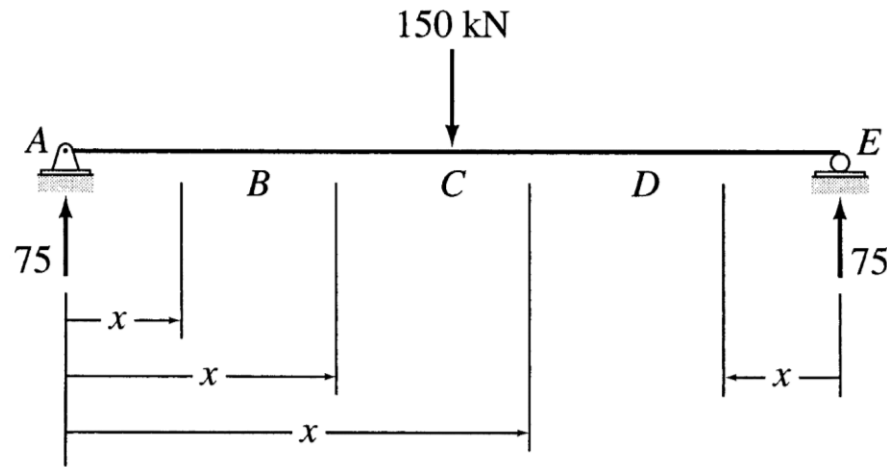
$$\Delta_C = -\frac{573.75}{EI} = -\frac{573.75}{200 (10^6) 800 (10^{-6})} = -0.00358 \text{ m}$$

$$\Delta_C = 3.58 \text{ mm} \uparrow$$

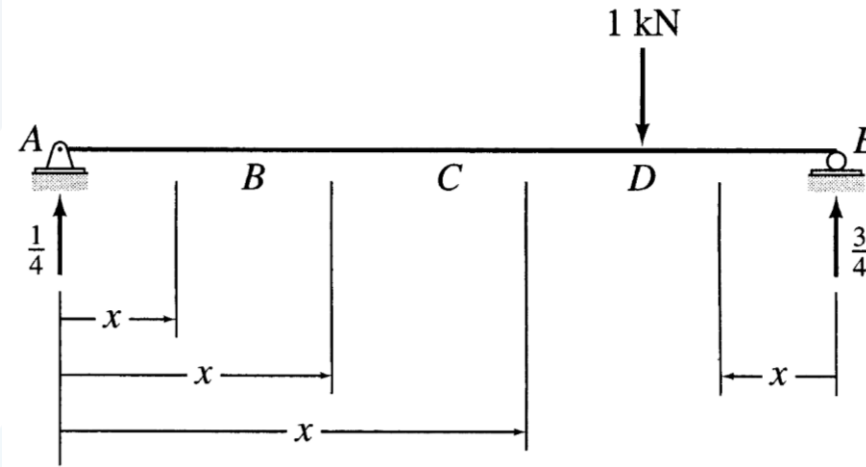
Example-03. Determine the deflection at point D of the beam shown in the figure, by the virtual work method. $E=200$ GPa.



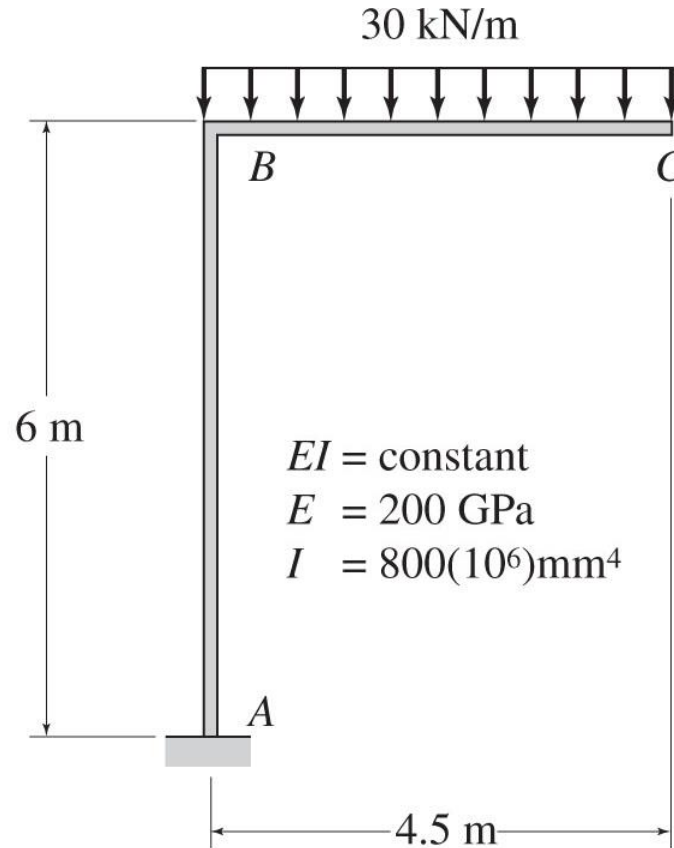
Real System



Virtual System

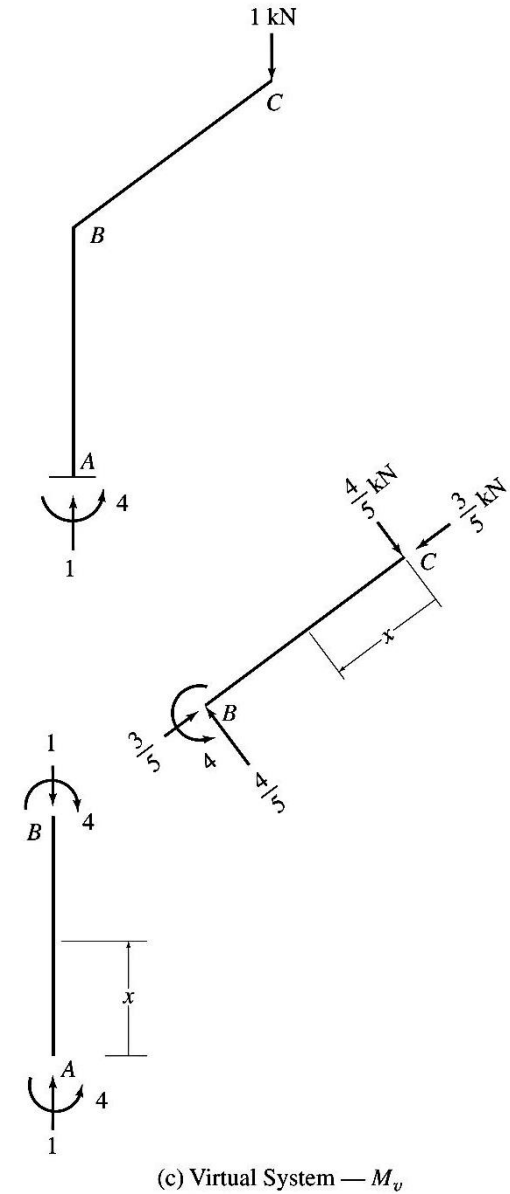
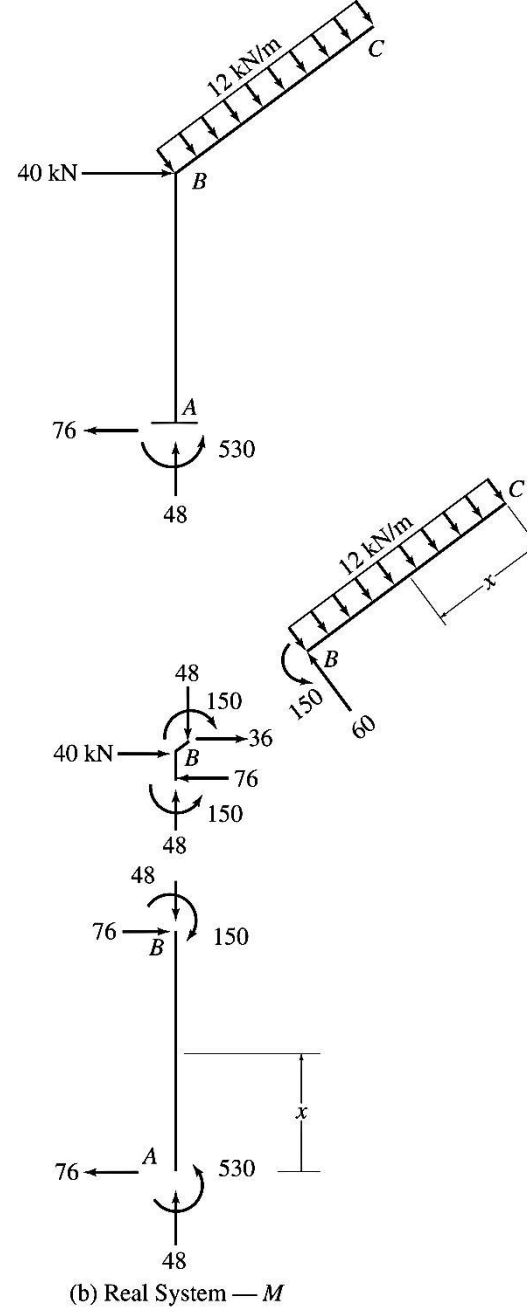
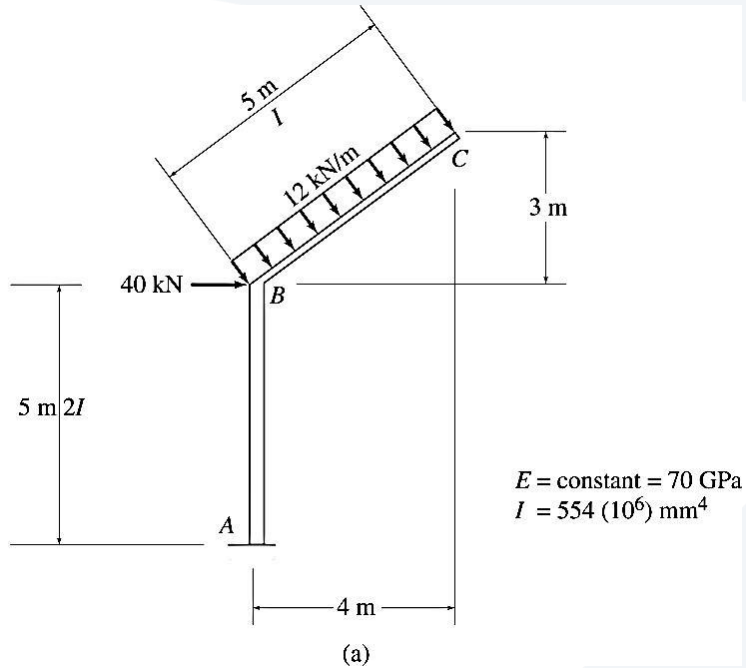


Example-04. Use the virtual work method to determine the deflection at joint C of the following frame.



$$\Delta_C = 60.9 \text{ mm} \downarrow$$

Ex.6. Compute the vertical deflection at joint C of the shown frame

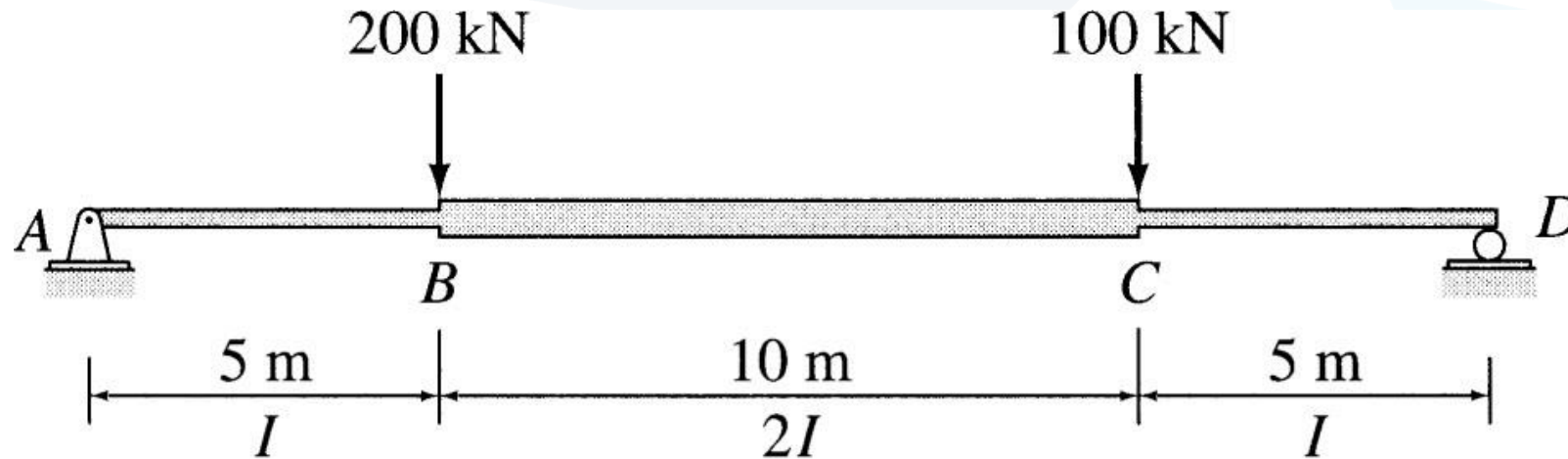


Segment	x Coordinate		EI ($I = 554 \times 10^6 \text{ mm}^4$)	M (kN–m)	M_v (kN–m)
	Origin	Limits (m)			
AB	A	0–5	$2EI$	$76x - 530$	-4
CB	C	0–5	EI	$-12\frac{x^2}{2}$	$-\frac{4}{5}x$

$$\Delta_C = 107 \text{ mm} \downarrow$$

Homework

Problem.01: Use the virtual work method to determine the deflection at joint C of the following beam.

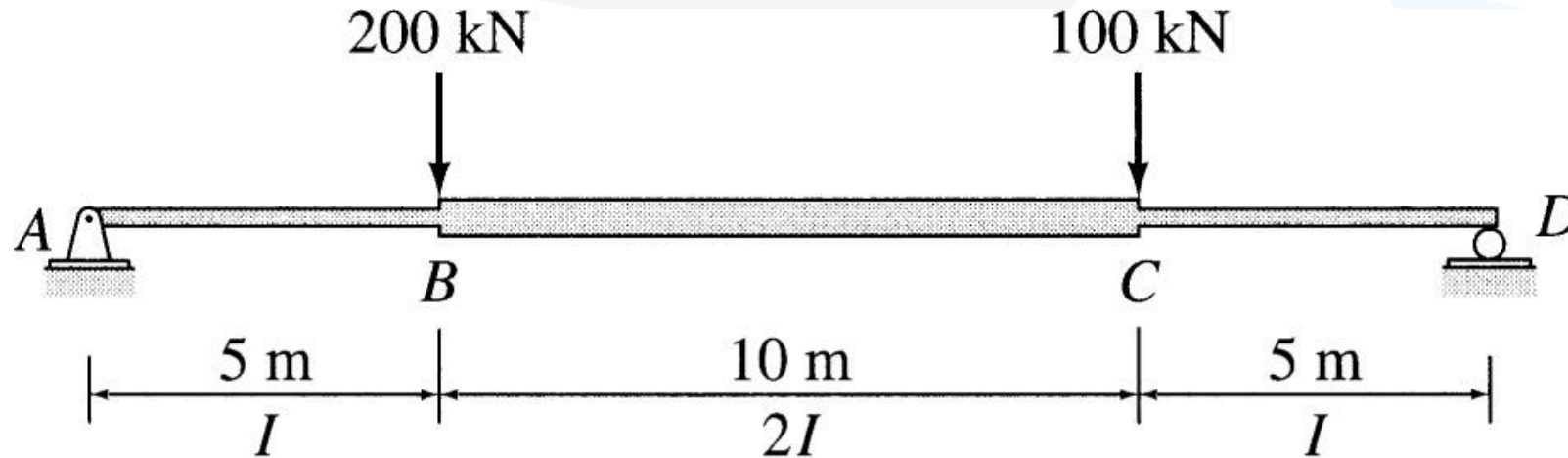


$$E = \text{constant} = 250 \text{ GPa}$$

$$I = 600(10^6) \text{ mm}^4$$

Homework

Problem.01: Use the virtual work method to determine the deflection at joint C of the following beam.

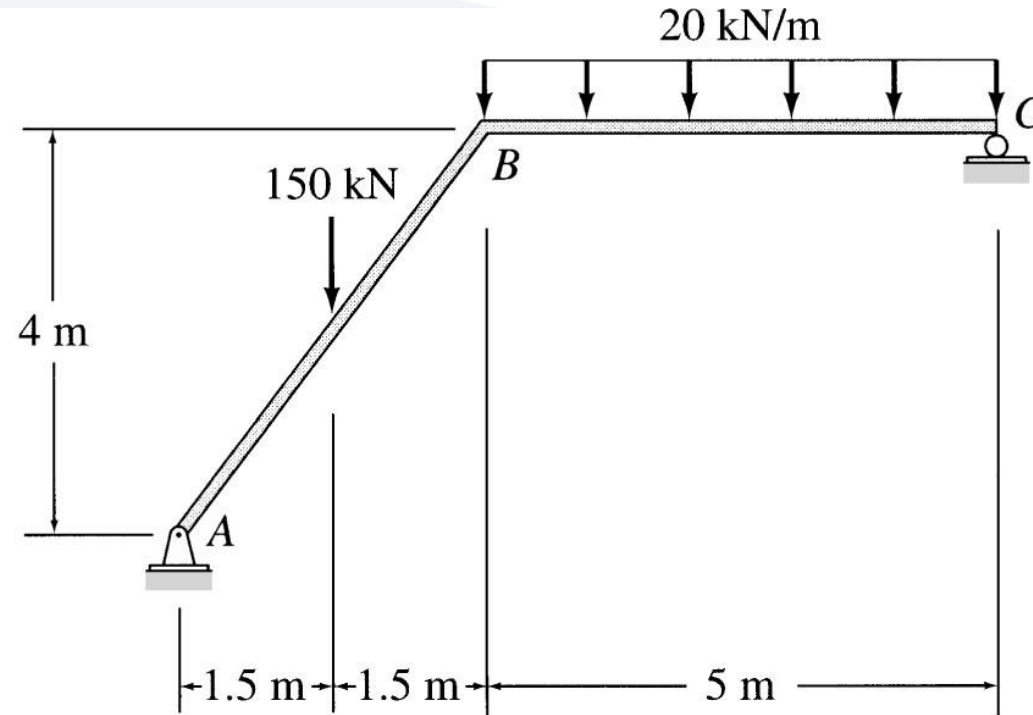


$$E = \text{constant} = 250 \text{ GPa}$$

$$I = 600(10^6) \text{ mm}^4$$

Homework

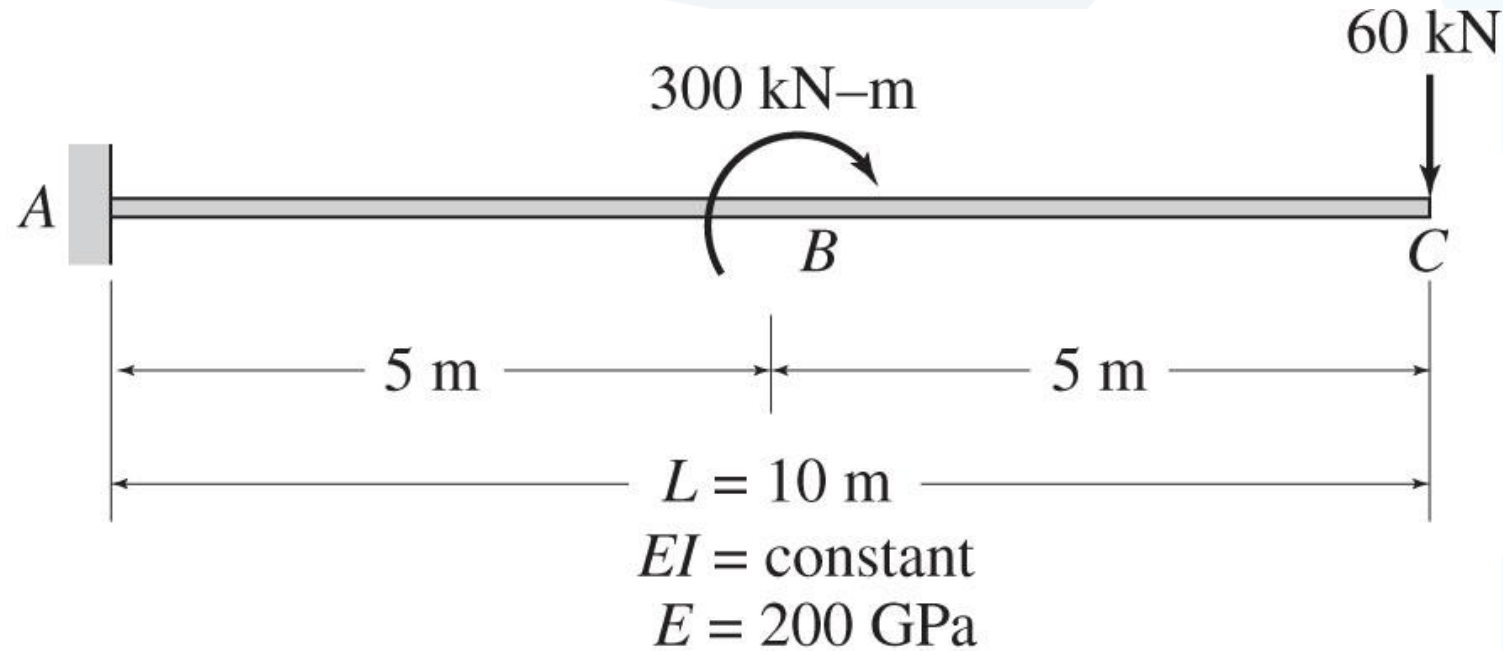
Problem.02: Use the virtual work method to determine the deflection at joint B of the following frame.



$$\begin{aligned}EI &= \text{constant} \\ E &= 200 \text{ GPa} \\ I &= 500(10^6) \text{ mm}^4\end{aligned}$$

Homework

Problem.03: Determine the smallest moment of inertia (I) required for the beam shown, so that its maximum deflection does not exceed the limit of $1/360$ of the span length (i.e., $\Delta_{\max} \leq L/360$). Use the method of virtual work.



Homework

Problem.04: Determine the smallest moment of inertia (I) required for the members of the frame shown, so that the horizontal deflection at joint C does not exceed 20 mm. Use the method of virtual work.

