جامعة المنارة

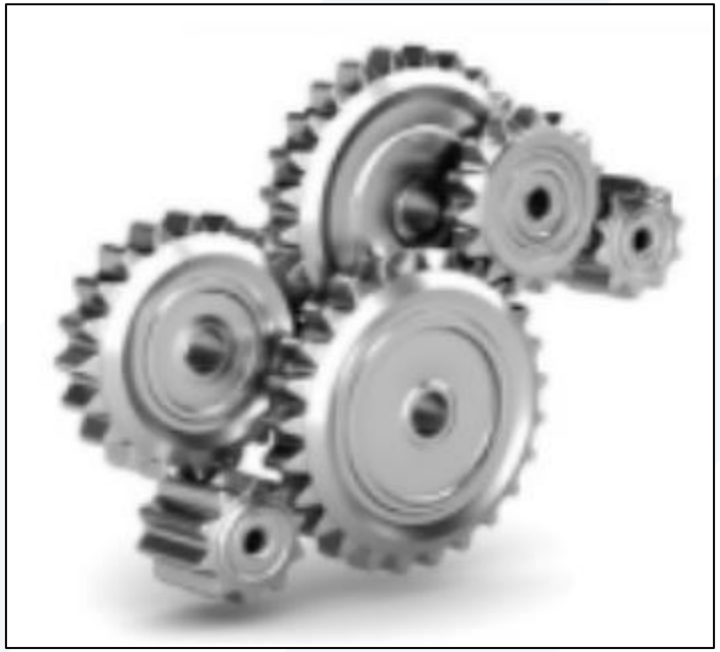
كلية الهندسة

قسم الروبوت و الأنظمة الذكية

مقرر ديناميك الآلات و الاهتزازات الميكانيكية

Planar Kinetics of a Rigid Body

Mass Moment of Inertia



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Mass Moment of Inertia

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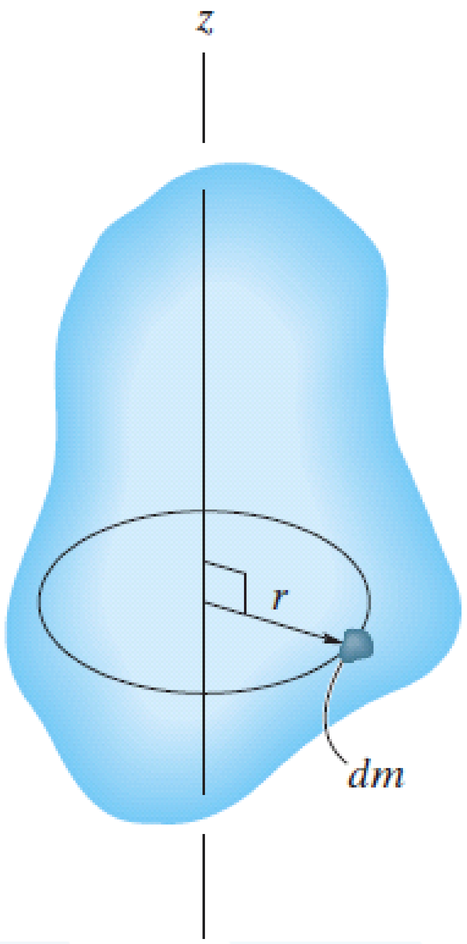
Since a body has a definite size and shape, an applied nonconcurrent force system can cause the body to both translate and rotate. The translational aspects of the motion were studied and are governed by the equation $\mathbf{F} = m\mathbf{a}$. It will be shown in the next section that the rotational aspects, caused by a moment $\mathbf{M}$, are governed by an equation of the form $\mathbf{M} = I\boldsymbol{\alpha}$. The symbol I in this equation is termed the mass moment of inertia. By comparison, the *moment of inertia* is a measure of the resistance of a body to *angular acceleration* ($\mathbf{M} = I\boldsymbol{\alpha}$) in the same way that *mass* is a measure of the body's resistance to *acceleration* ($\mathbf{F} = m\mathbf{a}$).

We define the *moment of inertia* as the integral of the "second moment" about an axis of all the elements of mass dm which compose the body. For example, the body's moment of inertia about the z axis is

$$I = \int_m r^2 \, dm$$

Here the "moment arm" r is the perpendicular distance from the z axis to the arbitrary element dm. Since the formulation involves r, the value of I is different for each axis about which it is computed. In the study of planar kinetics, the axis chosen for analysis generally passes through the body's mass center G and is always perpendicular to the plane of motion. The moment of inertia about this axis will be denoted as I_G. Since r is squared, the mass moment of inertia is always a *positive* quantity.

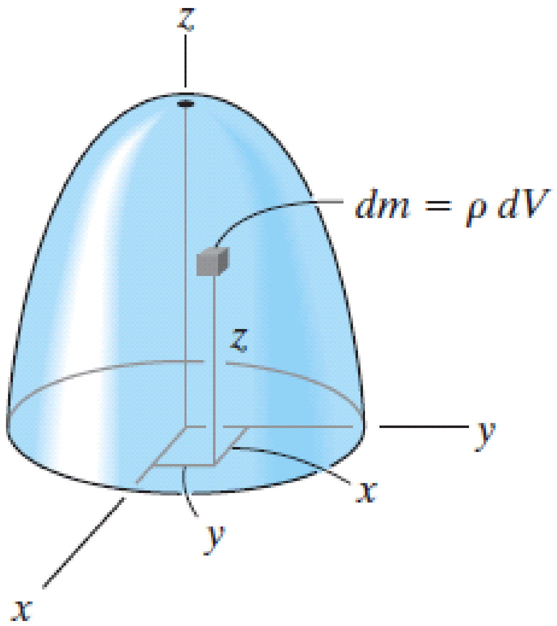
Common units used for its measurement are $kg \cdot m^2$ or $slug \cdot ft^2$.



If the body consists of material having a variable density, $\rho = \rho(x, y, z)$, the elemental mass dm of the body can be expressed in terms of its density and volume as $dm = \rho\, dV$. Substituting dm, the body's moment of inertia is then computed using *volume elements* for integration; i.e.,

$$I = \int_V r^2 \rho\, dV$$

When the volume element chosen for integration has infinitesimal dimensions in all three directions, the moment of inertia of the body must be determined using "triple integration." The integration process can, however, be simplified to a *single integration* provided the chosen volume element has a differential size or thickness in only *one direction*. Shell or disk elements are often used for this purpose.



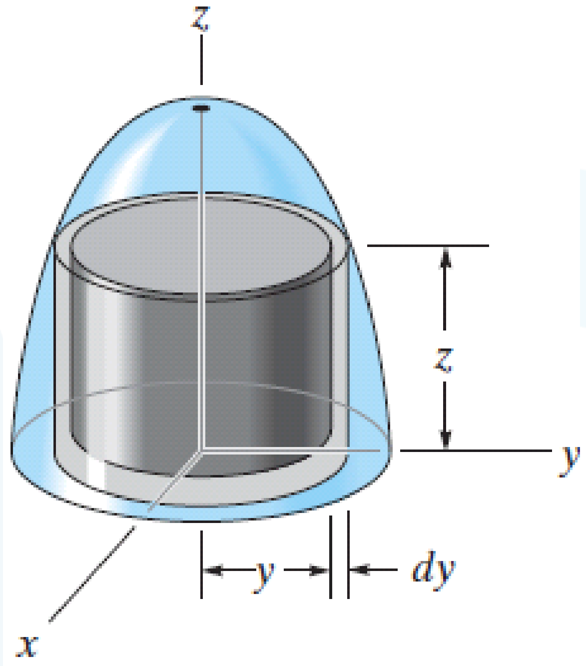
Procedure for Analysis

To obtain the moment of inertia by integration, we will consider only symmetric bodies having volumes which are generated by revolving a curve about an axis. An example of such a body is shown Two types of differential elements can be chosen.

Shell Element.

- If a *shell element* having a height z, radius $r = y$, and thickness dy is chosen for integration, then the volume is $dV = (2\pi y)(z)dy$.

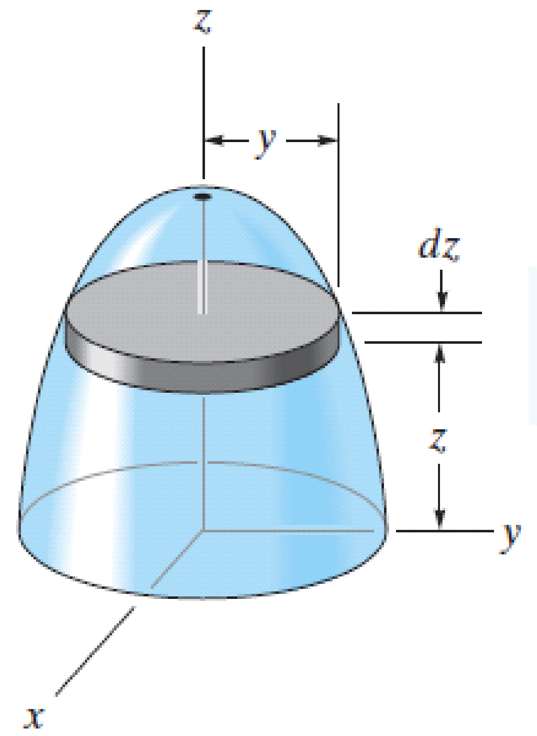
- This element may be used for determining the moment of inertia I_z of the body about the z axis, since the *entire element*, due to its "thinness," lies at the *same* perpendicular distance $r = y$ from the z axis



Disk Element.

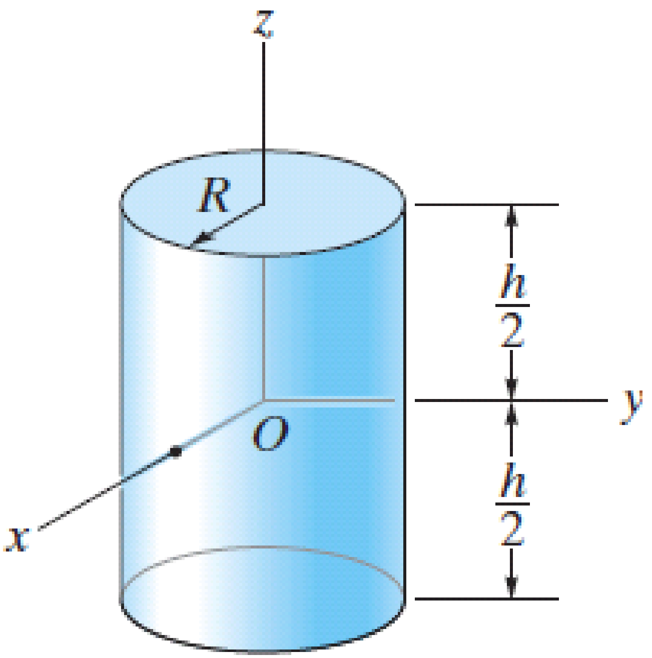
- If a disk element having a radius y and a thickness dz is chosen for integration, then the volume is $dV = (\pi y^2)dz$.

- This element is *finite* in the radial direction, and consequently its parts *do not* all lie at the *same radial distance r* from the z axis. Instead, to perform the integration it is first necessary to determine the moment of inertia *of the element* about the z axis and then integrate this result



EXAMPLE

Determine the moment of inertia of the cylinder shown about the z axis. The density of the material, ρ, is constant.



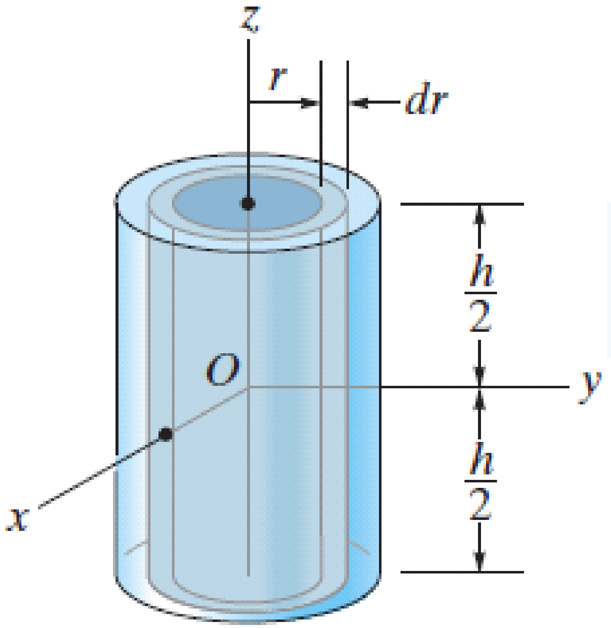
SOLUTION

Shell Element. This problem can be solved using the *shell element* and a single integration. The volume of the element is $dV = (2\pi r)(h)\,dr$, so that its mass is $dm = \rho dV = \rho(2\pi hr\,dr)$. Since the *entire element* lies at the same distance r from the z axis, the moment of inertia *of the element* is

$$dI_z = r^2 dm = \rho 2\pi hr^3\,dr$$

Integrating over the entire region of the cylinder yields

$$I_z = \int_m r^2\,dm = \rho 2\pi h \int_0^R r^3\,dr = \frac{\rho\pi}{2}R^4 h$$



The mass of the cylinder is

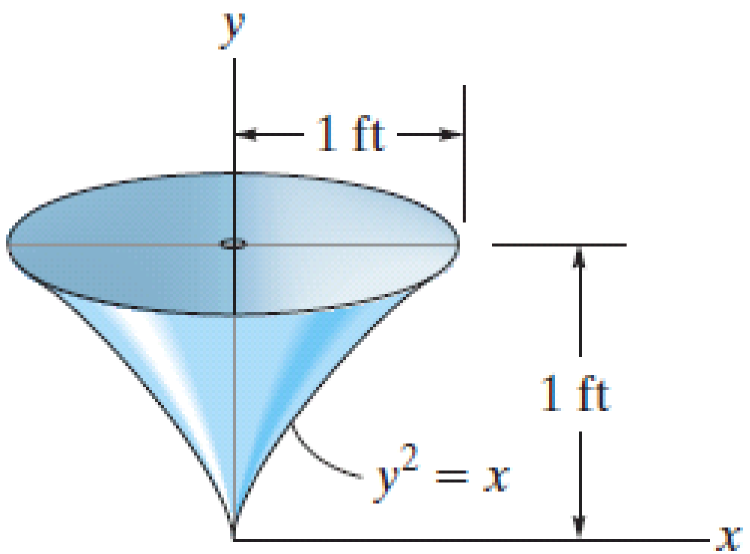
$$m = \int_m dm = \rho 2\pi h \int_0^R r\,dr = \rho\pi h R^2$$

so that

$$I_z = \frac{1}{2} m R^2 \qquad \textit{Ans.}$$

EXAMPLE

If the density of the material is 5 slug/ft^3, determine the moment of inertia of the solid about the y axis.

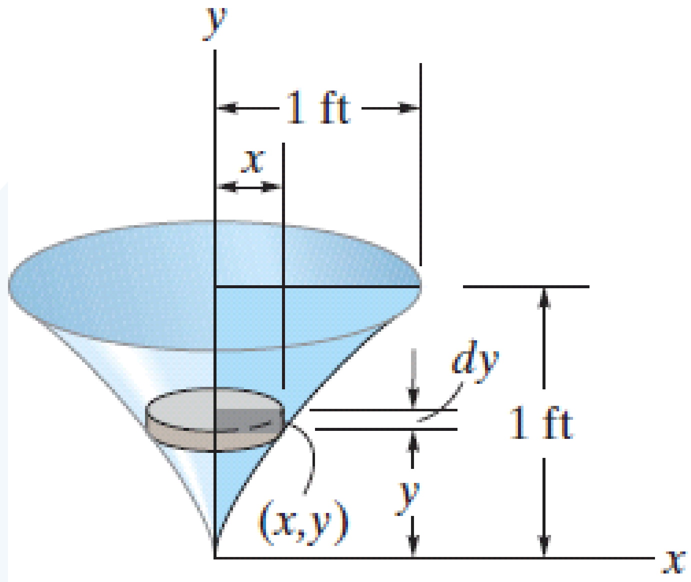


SOLUTION

Disk Element. The moment of inertia will be found using a *disk element*, as shown. Here the element intersects the curve at the arbitrary point (x,y) and has a mass

$$dm = \rho\, dV = \rho(\pi x^2)\, dy$$

Although all portions of the element are *not* located at the same distance from the y axis, it is still possible to determine the moment of inertia dI_y *of the element* about the y axis. In the preceding example it was shown that the moment of inertia of a cylinder about its longitudinal axis is $I = \frac{1}{2}mR^2$, where m and R are the mass and radius of the cylinder. Since the height is not involved in this formula, the disk itself can be thought of as a cylinder. Thus, for the disk element



$$dI_y = \frac{1}{2}(dm)x^2 = \frac{1}{2}[\rho(\pi x^2)\, dy]x^2$$

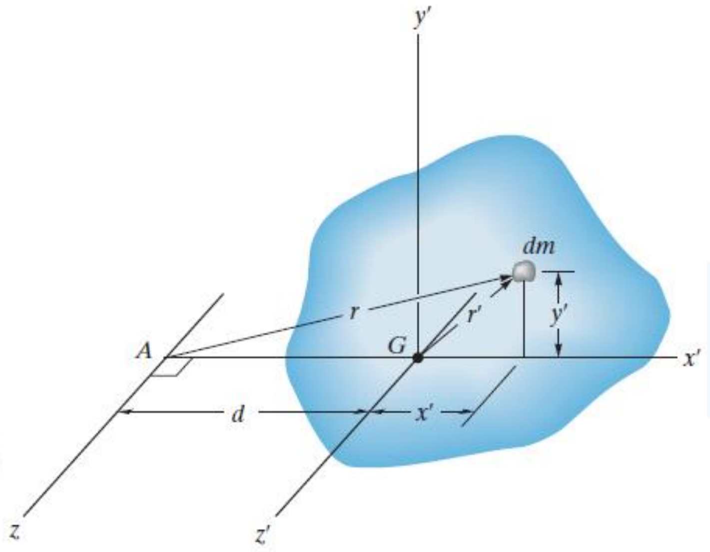
Substituting $x = y^2$, $\rho = 5\ \text{slug/ft}^3$, and integrating with respect to y, from $y = 0$ to $y = 1$ ft, yields the moment of inertia for the entire solid.

$$I_y = \frac{\pi(5\ \text{slug/ft}^3)}{2}\int_0^{1\ \text{ft}} x^4\,dy = \frac{\pi(5)}{2}\int_0^{1\ \text{ft}} y^8\,dy = 0.873\ \text{slug}\cdot\text{ft}^2 \quad \textit{Ans.}$$

Parallel-Axis Theorem.

If the moment of inertia of the body about an axis passing through the body's mass center is known, then the moment of inertia about any other *parallel axis* can be determined by using the *parallel-axis theorem*. This theorem can be derived by considering the body shown

Here the z' axis passes through the mass center G, whereas the corresponding *parallel z axis* lies at a constant distance d away. Selecting the differential element of mass dm, which is located at point (x', y'), and using the Pythagorean theorem, $r^2 = (d + x')^2 + y'^2$, we can express the moment of inertia of the body about the z axis as



$$I = \int_m r^2\, dm = \int_m [(d + x')^2 + y'^2]\, dm$$

$$= \int_m (x'^2 + y'^2)\, dm + 2d \int_m x'\, dm + d^2 \int_m dm$$

Since $r'^2 = x'^2 + y'^2$, the first integral represents I_G. The second integral equals *zero*, since the z' axis passes through the body's mass center, i.e., $\int x' dm = \bar{x}' m = 0$ since $\bar{x}' = 0$. Finally, the third integral represents the total mass m of the body. Hence, the moment of inertia about the z axis can be written as

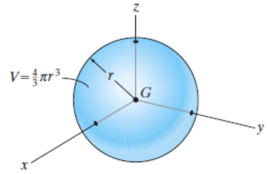
$$I = I_G + md^2$$

where

I_G = moment of inertia about the z' axis passing through the mass center G
m = mass of the body
d = perpendicular distance between the parallel z and z' axes

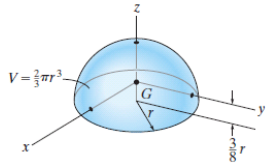
Composite Bodies. If a body consists of a number of simple shapes such as disks, spheres, and rods, the moment of inertia of the body about any axis can be determined by adding algebraically the moments of inertia of all the composite shapes computed about the axis. Algebraic addition is necessary since a composite part must be considered as a negative quantity if it has already been counted as a piece of another part—for example, a "hole" subtracted from a solid plate. The parallel-axis theorem is needed for the calculations if the center of mass of each composite part does not lie on the axis. For the calculation, then, $I = \Sigma(I_G + md^2)$. Here I_G for each of the composite parts is determined by integration

Center of Gravity and Mass Moment of Inertia of Homogeneous Solids



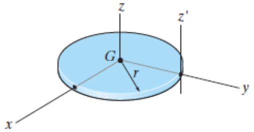
Sphere

$I_{xx} = I_{yy} = I_{zz} = \frac{2}{5}mr^2$



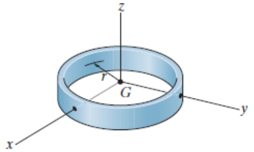
Hemisphere

$I_{xx} = I_{yy} = 0.259\,mr^2 \quad I_{zz} = \frac{2}{5}mr^2$



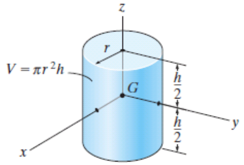
Thin Circular disk

$I_{xx} = I_{yy} = \frac{1}{4}mr^2 \quad I_{zz} = \frac{1}{2}mr^2 \quad I_{z'z'} = \frac{3}{2}mr^2$



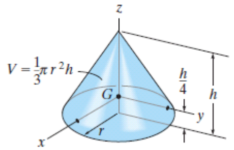
Thin ring

$I_{xx} = I_{yy} = \frac{1}{2}mr^2 \quad I_{zz} = mr^2$



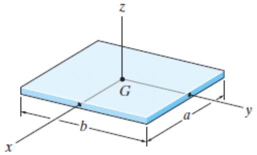
Cylinder

$I_{xx} = I_{yy} = \frac{1}{12}m(3r^2 + h^2) \quad I_{zz} = \frac{1}{2}mr^2$



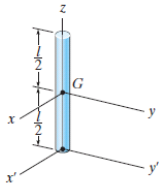
Cone

$I_{xx} = I_{yy} = \frac{3}{80}m(4r^2 + h^2) \quad I_{zz} = \frac{3}{10}mr^2$



Thin plate

$I_{xx} = \frac{1}{12}mb^2 \quad I_{yy} = \frac{1}{12}ma^2 \quad I_{zz} = \frac{1}{12}m(a^2 + b^2)$

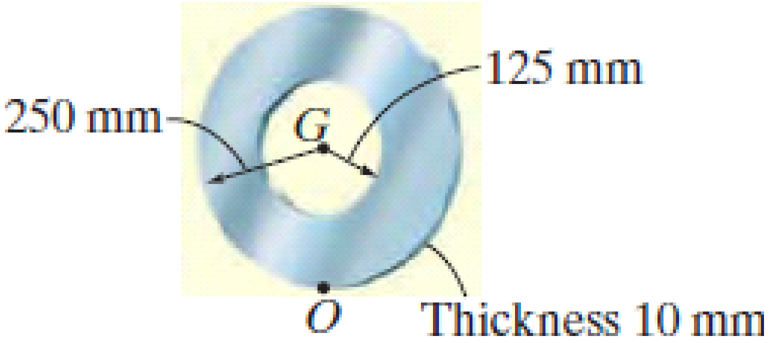


Slender Rod

$I_{xx} = I_{yy} = \frac{1}{12}ml^2 \quad I_{x'x'} = I_{y'y'} = \frac{1}{3}ml^2 \quad I_{z'z'} = 0$

EXAMPLE

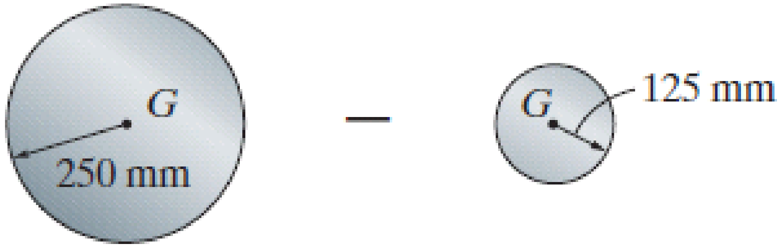
If the plate shown has a density of 8000 kg/m^3 and a thickness of 10 mm, determine its moment of inertia about an axis directed perpendicular to the page and passing through point O.



SOLUTION

The plate consists of two composite parts, the 250-mm-radius disk *minus* a 125-mm-radius disk. The moment of inertia about O can be determined by computing the moment of inertia of each of these parts about O and then adding the results *algebraically*. The calculations are performed by using the parallel-axis theorem in conjunction with the data listed in the table on the inside back cover.

Disk. The moment of inertia of a disk about the centroidal axis perpendicular to the plane of the disk is $I_G = \frac{1}{2}mr^2$. The mass center of the disk is located at a distance of 0.25 m from point O. Thus,



$$m_d = \rho_d V_d = 8000 \text{ kg/m}^3 \, [\pi(0.25 \text{ m})^2(0.01 \text{ m})] = 15.71 \text{ kg}$$

$$(I_d)_O = \tfrac{1}{2} m_d r_d^2 + m_d d^2$$

$$= \frac{1}{2}(15.71 \text{ kg})(0.25 \text{ m})^2 + (15.71 \text{ kg})(0.25 \text{ m})^2$$

$$= 1.473 \text{ kg} \cdot \text{m}^2$$

Hole. For the 125-mm-radius disk (hole), we have

$$m_h = \rho_h V_h = 8000 \text{ kg/m}^3 \, [\pi(0.125 \text{ m})^2(0.01 \text{ m})] = 3.927 \text{ kg}$$

$$(I_h)_O = \tfrac{1}{2} m_h r_h^2 + m_h d^2$$

$$= \frac{1}{2}(3.927 \text{ kg})(0.125 \text{ m})^2 + (3.927 \text{ kg})(0.25 \text{ m})^2$$

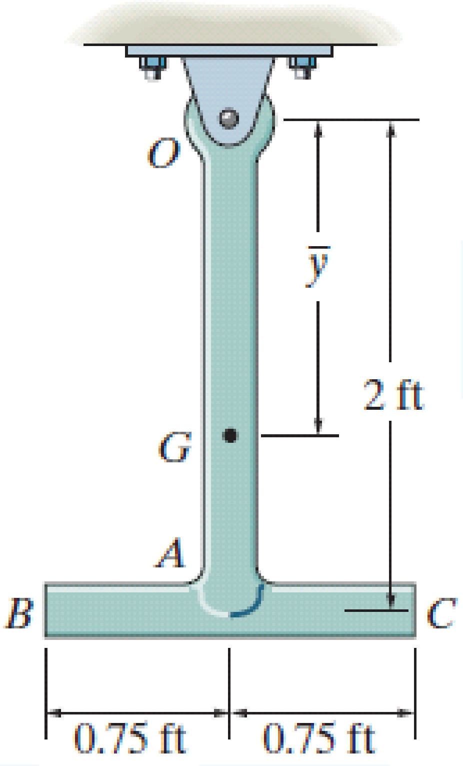
$$= 0.276 \text{ kg} \cdot \text{m}^2$$

The moment of inertia of the plate about point O is therefore

$$I_O = (I_d)_O - (I_h)_O$$
$$= 1.473 \text{ kg} \cdot \text{m}^2 - 0.276 \text{ kg} \cdot \text{m}^2$$
$$= 1.20 \text{ kg} \cdot \text{m}^2 \qquad \textit{Ans.}$$

EXAMPLE

The pendulum is suspended from the pin at O and consists of two thin rods. Rod OA weighs 10 lb, and BC weighs 8 lb. Determine the moment of inertia of the pendulum about an axis passing through (a) point O, and (b) the mass center G of the pendulum.



SOLUTION

Part (a). Using the table, the moment of inertia of rod OA about an axis perpendicular to the page and passing through point O of the rod is $I_O = \frac{1}{3}ml^2$. Hence,

$$(I_{OA})_O = \frac{1}{3}ml^2 = \frac{1}{3}\left(\frac{10 \text{ lb}}{32.2 \text{ ft/s}^2}\right)(2 \text{ ft})^2 = 0.414 \text{ slug} \cdot \text{ft}^2$$

This same value can be obtained using $I_G = \frac{1}{12}ml^2$ and the parallel-axis theorem.

$$(I_{OA})_O = \frac{1}{12}ml^2 + md^2 = \frac{1}{12}\left(\frac{10 \text{ lb}}{32.2 \text{ ft/s}^2}\right)(2 \text{ ft})^2 + \left(\frac{10 \text{ lb}}{32.2 \text{ ft/s}^2}\right)(1 \text{ ft})^2$$

$$= 0.414 \text{ slug} \cdot \text{ft}^2$$

For rod BC we have

$$(I_{BC})_O = \frac{1}{12}ml^2 + md^2 = \frac{1}{12}\left(\frac{8\text{ lb}}{32.2\text{ ft/s}^2}\right)(1.5\text{ ft})^2 + \left(\frac{8\text{ lb}}{32.2\text{ ft/s}^2}\right)(2\text{ ft})^2$$

$$= 1.040\text{ slug}\cdot\text{ft}^2$$

The moment of inertia of the pendulum about O is therefore

$$I_O = 0.414 + 1.040 = 1.454 = 1.45\text{ slug}\cdot\text{ft}^2 \qquad \textit{Ans.}$$

Part (b). The mass center G will be located relative to point O. Assuming this distance to be $\bar{y}$, and using the formula for determining the mass center, we have

$$\bar{y} = \frac{\Sigma\tilde{y}m}{\Sigma m} = \frac{1(10/32.2) + 2(8/32.2)}{(10/32.2) + (8/32.2)} = 1.444\text{ ft}$$

The moment of inertia I_G may be found in the same manner as I_O, which requires successive applications of the parallel-axis theorem to transfer the moments of inertia of rods *OA* and *BC* to *G*. A more direct solution, however, involves using the result for I_O, i.e.,

$$I_O = I_G + md^2; \quad 1.454 \text{ slug} \cdot \text{ft}^2 = I_G + \left(\frac{18 \text{ lb}}{32.2 \text{ ft/s}^2}\right)(1.444 \text{ ft})^2$$

$$I_G = 0.288 \text{ slug} \cdot \text{ft}^2 \qquad \textit{Ans.}$$

EXAMPLE

Determine the moment of inertia of a homogeneous solid sphere of mass ***m*** and radius ***r*** about a diameter

EXAMPLE

Determine the moments of inertia of the homogeneous rectangular parallelepiped of mass *m about the centroidal **x**- and **z**-axes*

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