جامعة المنارة

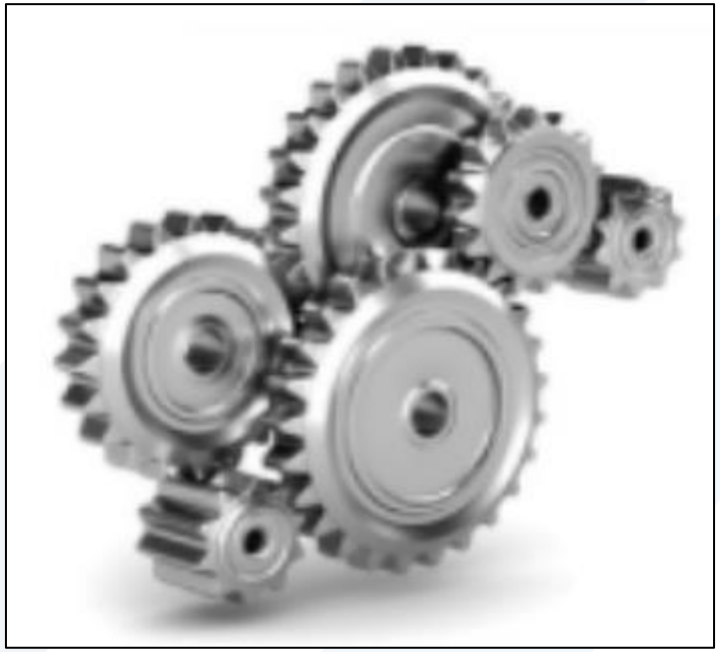
كلية الهندسة

قسم الروبوت و الأنظمة الذكية

مقرر ديناميك الآلات و الاهتزازات الميكانيكية

Planar Kinetics of a Rigid Body

Force and Acceleration



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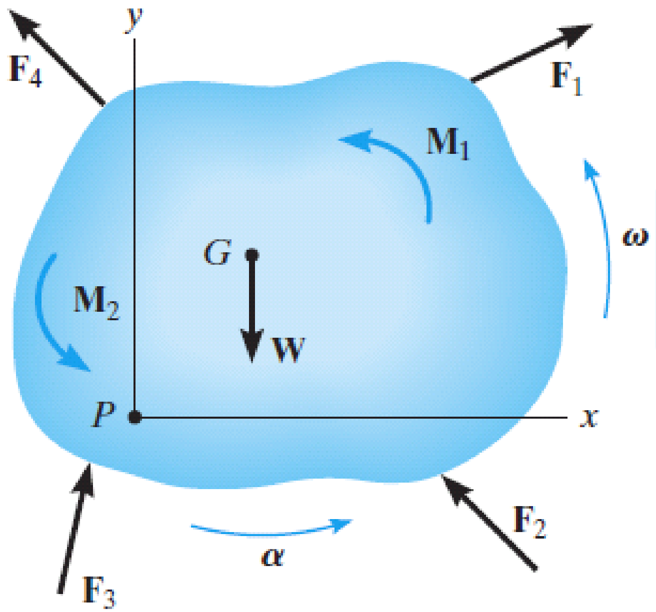
Contents

Planar Kinetic Equations of Motion

In the following analysis we will limit our study of planar kinetics to rigid bodies which, along with their loadings, are considered to be *symmetrical* with respect to a fixed reference plane. Since the motion of the body can be viewed within the reference plane, all the forces (and couple moments) acting on the body can then be projected onto the plane. An example of an arbitrary body of this type is shown. *Here the inertial frame of reference x, y, z has its origin coincident with the arbitrary point P* in the body.



Equation of Translational Motion. The external forces acting on the body represent the effect of gravitational, electrical, magnetic, or contact forces between adjacent bodies.

$$\Sigma \mathbf{F} = m\mathbf{a}_G$$

This equation is referred to as the *translational equation of motion* for the mass center of a rigid body. It states that *the sum of all the external forces acting on the body is equal to the body's mass times the acceleration of its mass center G.*

For motion of the body in the x–y plane, the translational equation of motion may be written in the form of two independent scalar equations, namely,

$$\Sigma F_x = m(a_G)_x$$

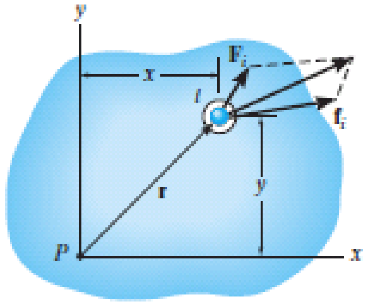
$$\Sigma F_y = m(a_G)_y$$

Equation of Rotational Motion. We will now determine the effects caused by the moments of the external force system computed about an axis perpendicular to the plane of motion (the z axis) and passing through point P. As shown on the free-body diagram of the ith particle, $\mathbf{F}_i$ represents the *resultant external force* acting on the particle, and $\mathbf{f}_i$ is the *resultant of the internal forces* caused by interactions with adjacent particles. If the particle has a mass m_i and its acceleration is $\mathbf{a}_i$, then its kinetic diagram is shown. Summing moments about point P, we require

$$\mathbf{r} \times \mathbf{F}_i + \mathbf{r} \times \mathbf{f}_i = \mathbf{r} \times m_i\mathbf{a}_i$$

or

$$(\mathbf{M}_P)_i = \mathbf{r} \times m_i\mathbf{a}_i$$



Particle free-body diagram

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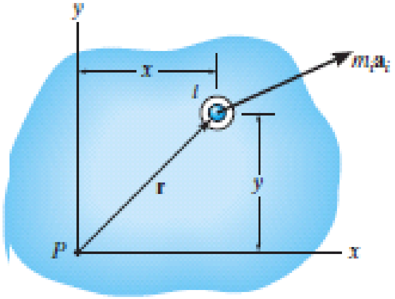
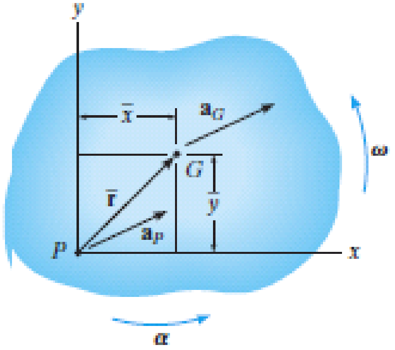


Particle kinetic diagram



The moments about P can also be expressed in terms of the acceleration of point P. If the body has an angular acceleration $\boldsymbol{\alpha}$ and angular velocity $\boldsymbol{\omega}$, then we have

$$(\mathbf{M}_P)_i = m_i \mathbf{r} \times (\mathbf{a}_P + \boldsymbol{\alpha} \times \mathbf{r} - \omega^2 \mathbf{r})$$
$$= m_i[\mathbf{r} \times \mathbf{a}_P + \mathbf{r} \times (\boldsymbol{\alpha} \times \mathbf{r}) - \omega^2(\mathbf{r} \times \mathbf{r})]$$

The last term is zero, since $\mathbf{r} \times \mathbf{r} = \mathbf{0}$. Expressing the vectors with Cartesian components and carrying out the cross-product operations yields

$$(M_P)_i\mathbf{k} = m_i\{(x\mathbf{i} + y\mathbf{j}) \times [(a_P)_x\mathbf{i} + (a_P)_y\mathbf{j}] + (x\mathbf{i} + y\mathbf{j}) \times [\alpha\mathbf{k} \times (x\mathbf{i} + y\mathbf{j})]\}$$

$$(M_P)_i\mathbf{k} = m_i[-y(a_P)_x + x(a_P)_y + \alpha x^2 + \alpha y^2]\mathbf{k}$$

$$\circlearrowleft (M_P)_i = m_i[-y(a_P)_x + x(a_P)_y + \alpha r^2]$$

Letting $m_i \rightarrow dm$ and integrating with respect to the entire mass m of the body, we obtain the resultant moment equation

$$\circlearrowleft \Sigma M_P = -\left(\int_m y\, dm\right)(a_P)_x + \left(\int_m x\, dm\right)(a_P)_y + \left(\int_m r^2 dm\right)\alpha$$

Here ΣM_P represents only the moment of the *external forces* acting on the body about point P. The resultant moment of the internal forces is zero, since for the entire body these forces occur in equal and opposite collinear pairs and thus the moment of each pair of forces about P cancels. The integrals in the first and second terms on the right are used to locate the body's center of mass G with respect to P, since $\bar{y}m = \int y\, dm$ and $\bar{x}m = \int x\, dm$. Also, the last integral represents the body's moment of inertia about the z axis, i.e., $I_P = \int r^2 dm$. Thus,

$$\circlearrowleft \Sigma M_P = -\bar{y}m(a_P)_x + \bar{x}m(a_P)_y + I_P\alpha$$

It is possible to reduce this equation to a simpler form if point P coincides with the mass center G for the body. If this is the case, then $\bar{x} = \bar{y} = 0$, and therefore*

$$\Sigma M_G = I_G \alpha$$

This rotational equation of motion states that the sum of the moments of all the external forces about the body's mass center G is equal to the product of the moment of inertia of the body about an axis passing through G and the body's angular acceleration.

If point G is located at $(\bar{x}, \bar{y})$, then by the parallel-axis theorem,

$$I_P = I_G + m(\bar{x}^2 + \bar{y}^2).$$

$$\circlearrowleft \Sigma M_P = \bar{y}m[-(a_P)_x + \bar{y}\alpha] + \bar{x}m[(a_P)_y + \bar{x}\alpha] + I_G\alpha$$

$$\mathbf{a}_G = \mathbf{a}_P + \boldsymbol{\alpha} \times \bar{\mathbf{r}} - \omega^2 \bar{\mathbf{r}}$$

$$(a_G)_x\mathbf{i} + (a_G)_y\mathbf{j} = (a_P)_x\mathbf{i} + (a_P)_y\mathbf{j} + \alpha\mathbf{k} \times (\bar{x}\mathbf{i} + \bar{y}\mathbf{j}) - \omega^2(\bar{x}\mathbf{i} + \bar{y}\mathbf{j})$$

Carrying out the cross product and equating the respective **i** and **j** components yields the two scalar equations

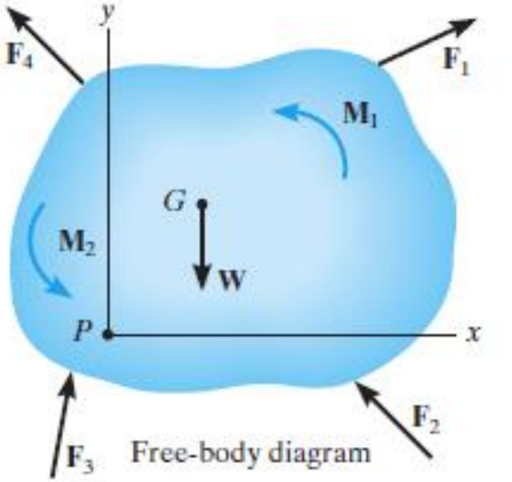
$$(a_G)_x = (a_P)_x - \bar{y}\alpha - \bar{x}\omega^2$$

$$(a_G)_y = (a_P)_y + \bar{x}\alpha - \bar{y}\omega^2$$

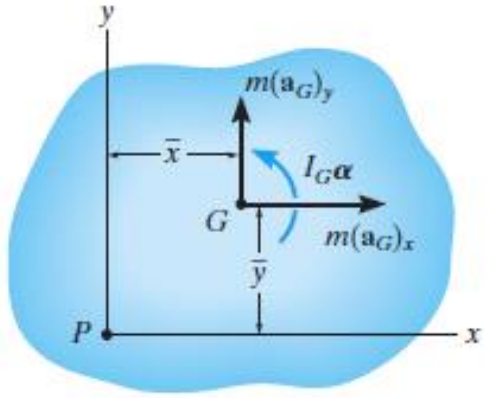
From these equations, $[-(a_P)_x + \bar{y}\alpha] = [-(a_G)_x - \bar{x}\omega^2]$ and $[(a_P)_y + \bar{x}\alpha] = [(a_G)_y + \bar{y}\omega^2]$.

$$\circlearrowleft \Sigma M_P = -\bar{y}m(a_G)_x + \bar{x}m(a_G)_y + I_G\alpha$$

This important result indicates that when moments of the external forces shown on the free-body diagram are summed about point P, they are equivalent to the sum of the “kinetic moments” of the components of $m\mathbf{a}_G$ about P plus the “kinetic moment” of $I_G\boldsymbol{\alpha}$.



Free-body diagram



Kinetic diagram

General Application of the Equations of Motion.

To summarize this analysis, *three* independent scalar equations can be written to describe the general plane motion of a symmetrical rigid body.

$$\Sigma F_x = m(a_G)_x$$

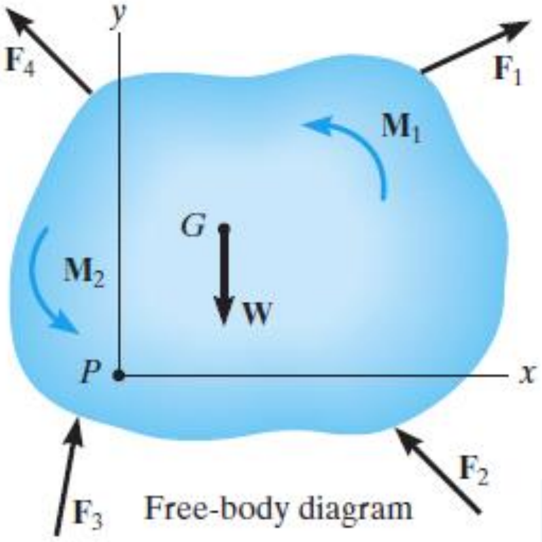
$$\Sigma F_y = m(a_G)_y$$

$$\Sigma M_G = I_G \alpha$$

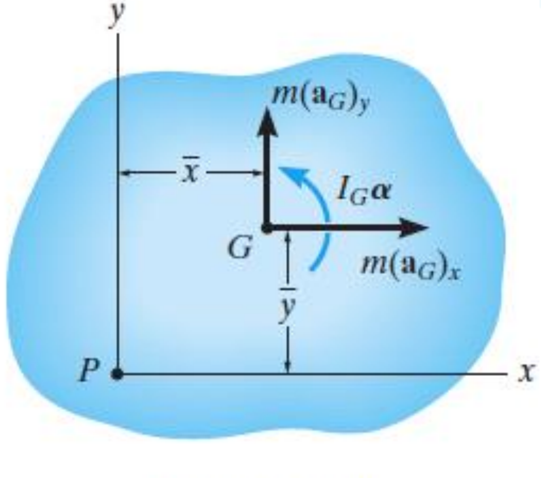
or

$$\Sigma M_P = \Sigma(\mathcal{M}_k)_P$$

When applying these equations, one should *always* draw a free-body diagram, in order to account for the terms involved in $\Sigma F_x, \Sigma F_y, \Sigma M_G$, or ΣM_P. In some problems it may also be helpful to draw the *kinetic diagram* for the body. This diagram graphically accounts for the terms $m(\mathbf{a}_G)_x$, $m(\mathbf{a}_G)_y$, and $I_G \boldsymbol{\alpha}$. It is especially convenient when used to determine the components of $m\mathbf{a}_G$ and the moment of these components in $\Sigma(\mathcal{M}_k)_P$.



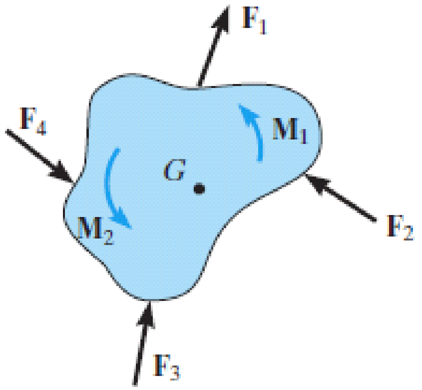
Free-body diagram



Kinetic diagram

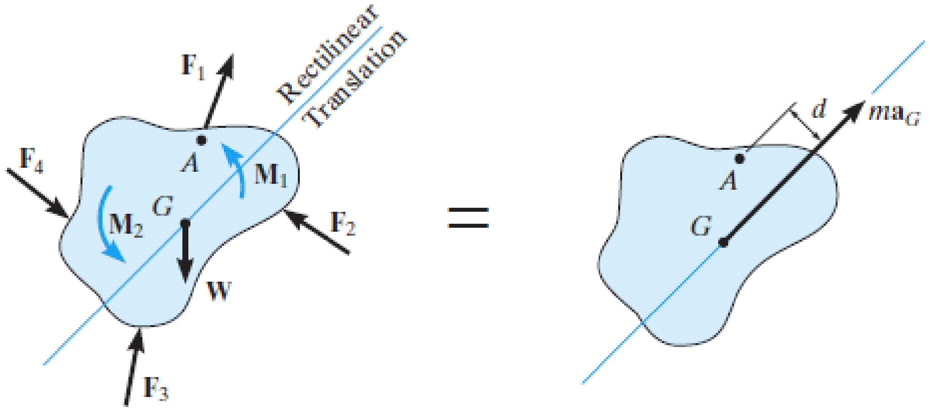
Equations of Motion: Translation

When the rigid body undergoes a *translation*, all the particles of the body have the *same acceleration*. Furthermore, $\boldsymbol{\alpha} = \mathbf{0}$, in which case the rotational equation of motion applied at point G reduces to a simplified form, namely, $\Sigma M_G = 0$. Application of this and the force equations of motion will now be discussed for each of the two types of translation.



Rectilinear Translation. When a body is subjected to *rectilinear translation*, all the particles of the body (slab) travel along parallel straight-line paths. The free-body and kinetic diagrams are shown .
Since $I_G\boldsymbol{\alpha} = \mathbf{0}$, only $m\mathbf{a}_G$ is shown on the kinetic diagram. Hence, the equations of motion which apply in this case become

$$\Sigma F_x = m(a_G)_x$$
$$\Sigma F_y = m(a_G)_y$$
$$\Sigma M_G = 0$$



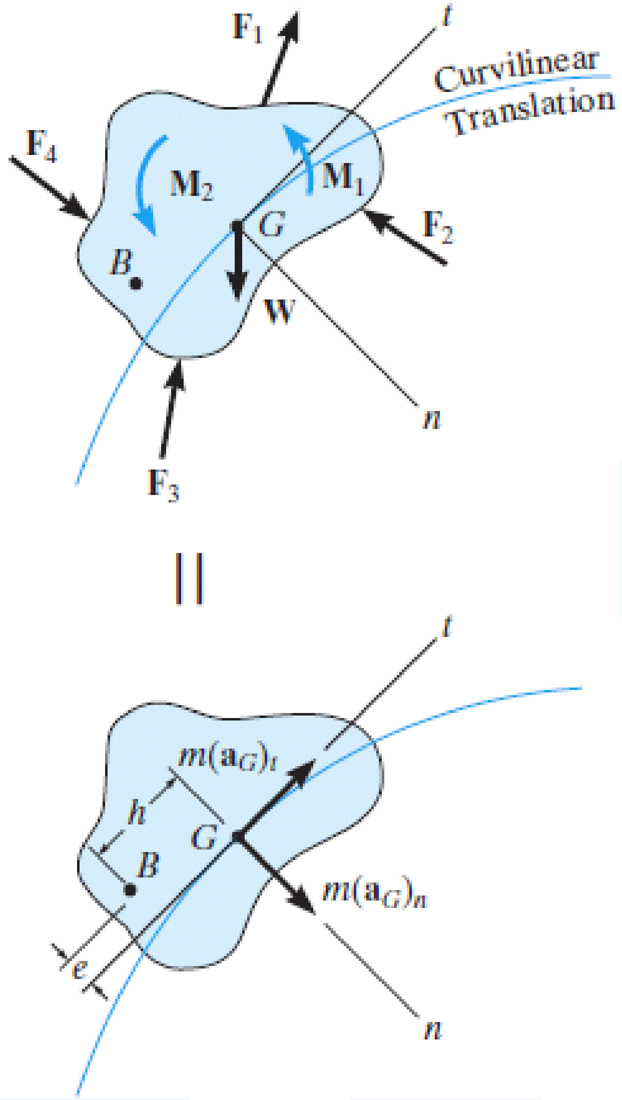
It is also possible to sum moments about other points on or off the body, in which case the moment of $m\mathbf{a}_G$ must be taken into account. For example, if point A is chosen, which lies at a perpendicular distance d from the line of action of $m\mathbf{a}_G$, the following moment equation applies:

$$\circlearrowleft + \Sigma M_A = \Sigma(\mathcal{M}_k)_A; \qquad \Sigma M_A = (ma_G)d$$

Here the sum of moments of the external forces and couple moments about A (ΣM_A, free-body diagram) equals the moment of $m\mathbf{a}_G$ about A ($\Sigma(\mathcal{M}_k)_A$, kinetic diagram).

Curvilinear Translation. When a rigid body is subjected to *curvilinear translation*, all the particles of the body have the same accelerations as they travel along *curved paths* as noted . For analysis, it is often convenient to use an inertial coordinate system having an origin which coincides with the body's mass center at the instant considered, and axes which are oriented in the normal and tangential directions to the path of motion. The three scalar equations of motion are then

$$\Sigma F_n = m(a_G)_n$$
$$\Sigma F_t = m(a_G)_t$$
$$\Sigma M_G = 0$$



If moments are summed about the arbitrary point B, then it is necessary to account for the moments, $\Sigma(\mathcal{M}_k)_B$, of the two components $m(\mathbf{a}_G)_n$ and $m(\mathbf{a}_G)_t$ about this point. From the kinetic diagram, h and e represent the perpendicular distances (or "moment arms") from B to the lines of action of the components. The required moment equation therefore becomes

$$\circlearrowleft + \Sigma M_B = \Sigma(\mathcal{M}_k)_B; \qquad \Sigma M_B = e[m(a_G)_t] - h[m(a_G)_n]$$

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Procedure for Analysis

Kinetic problems involving rigid-body *translation* can be solved using the following procedure.

Free-Body Diagram.

- Establish the x, y or n, t inertial coordinate system and draw the free-body diagram in order to account for all the external forces and couple moments that act on the body.
- The direction and sense of the acceleration of the body's mass center $\mathbf{a}_G$ should be established.
- Identify the unknowns in the problem.
- If it is decided that the rotational equation of motion $\Sigma M_P = \Sigma(\mathcal{M}_k)_P$ is to be used in the solution, then consider drawing the kinetic diagram, since it graphically accounts for the components $m(\mathbf{a}_G)_x$, $m(\mathbf{a}_G)_y$ or $m(\mathbf{a}_G)_t$, $m(\mathbf{a}_G)_n$ and is therefore convenient for "visualizing" the terms needed in the moment sum $\Sigma(\mathcal{M}_k)_P$.

Equations of Motion.

- Apply the three equations of motion in accordance with the established sign convention.
- To simplify the analysis, the moment equation $\Sigma M_G = 0$ can be replaced by the more general equation $\Sigma M_P = \Sigma(\mathcal{M}_k)_P$, where point P is usually located at the intersection of the lines of action of as many unknown forces as possible.
- If the body is in contact with a *rough surface* and slipping occurs, use the friction equation $F = \mu_k N$. Remember, **F** always acts on the body so as to oppose the motion of the body relative to the surface it contacts.

Kinematics.

- Use kinematics to determine the velocity and position of the body.
- For rectilinear translation with *variable acceleration*

$$a_G = dv_G/dt \quad a_G ds_G = v_G dv_G$$

- For rectilinear translation with *constant acceleration*

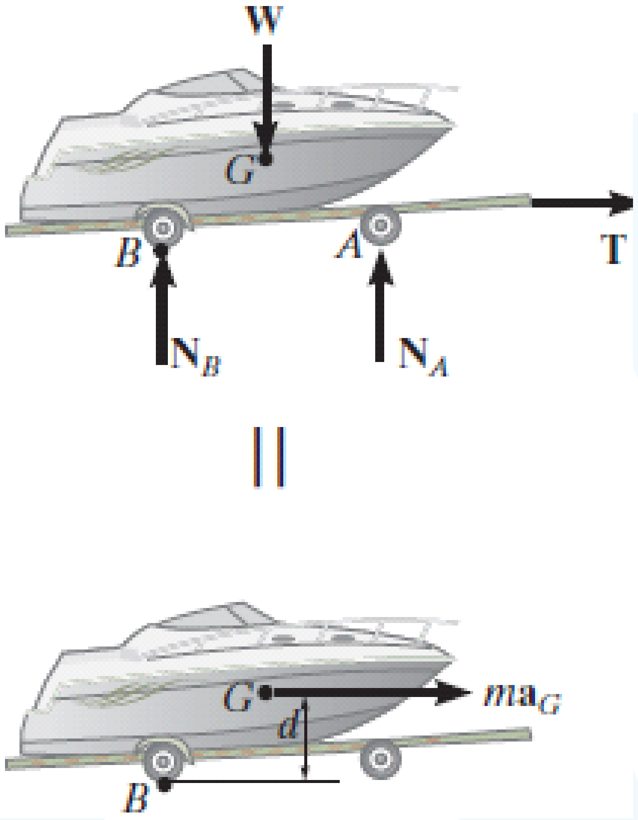
$$v_G = (v_G)_0 + a_G t \quad v_G^2 = (v_G)_0^2 + 2a_G[s_G - (s_G)_0]$$

$$s_G = (s_G)_0 + (v_G)_0 t + \frac{1}{2} a_G t^2$$

- For curvilinear translation

$$(a_G)_n = v_G^2/\rho$$

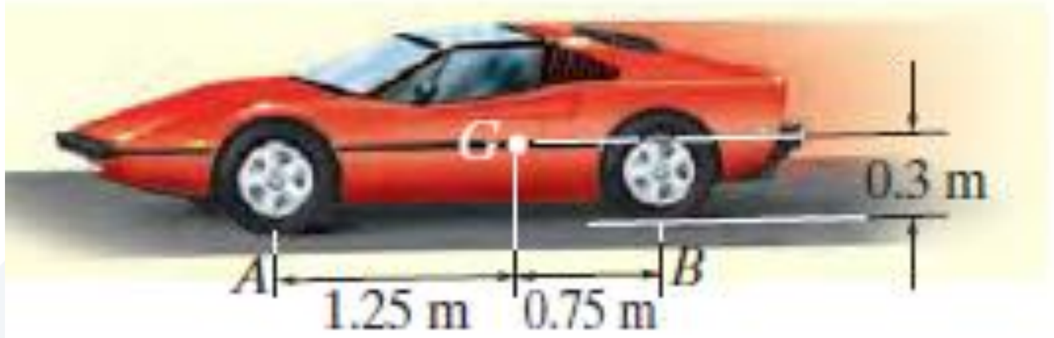
$$(a_G)_t = dv_G/dt \quad (a_G)_t \, ds_G = v_G \, dv_G$$



The free-body and kinetic diagrams for this boat and trailer are drawn first in order to apply the equations of motion. Here the forces on the free-body diagram cause the effect shown on the kinetic diagram. If moments are summed about the mass center, G, then $\Sigma M_G = 0$. However, if moments are summed about point B then $\circlearrowleft +\Sigma M_B = ma_G(d)$.

EXAMPLE

The car shown has a mass of 2 Mg and a center of mass at G. Determine the acceleration if the rear "driving" wheels are always slipping, whereas the front wheels are free to rotate. Neglect the mass of the wheels. The coefficient of kinetic friction between the wheels and the road is $\mu_k = 0.25$.



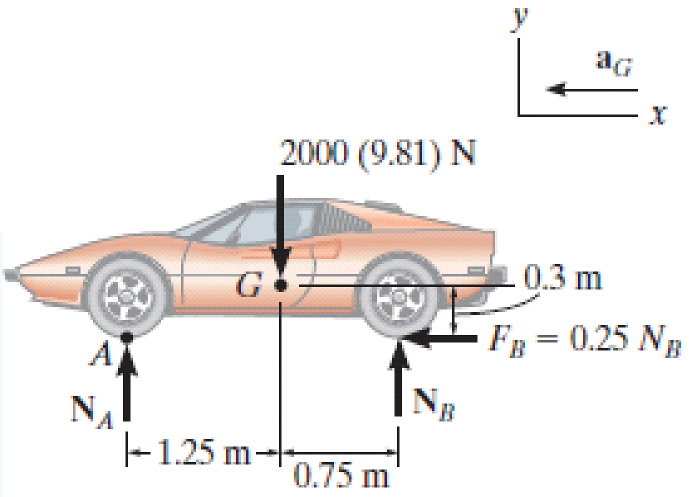
SOLUTION I

Free-Body Diagram. As shown, the rear-wheel frictional force $\mathbf{F}_B$ pushes the car forward, and since *slipping occurs*, $F_B = 0.25N_B$.

There are three unknowns in the problem, N_A, N_B, and a_G.

Here we will sum moments about the mass center.

The car (point G) accelerates to the left, i.e., in the negative x direction.



Equations of Motion.

$$\xrightarrow{+} \Sigma F_x = m(a_G)_x; \qquad -0.25N_B = -(2000\text{ kg})a_G \qquad (1)$$

$$+\uparrow \Sigma F_y = m(a_G)_y; \qquad N_A + N_B - 2000(9.81)\text{ N} = 0 \qquad (2)$$

$$\circlearrowleft + \Sigma M_G = 0; \quad -N_A(1.25\text{ m}) - 0.25N_B(0.3\text{ m}) + N_B(0.75\text{ m}) = 0 \quad (3)$$

Solving,

$$a_G = 1.59\text{ m/s}^2 \leftarrow \qquad \textit{Ans.}$$

$$N_A = 6.88\text{ kN}$$

$$N_B = 12.7\text{ kN}$$

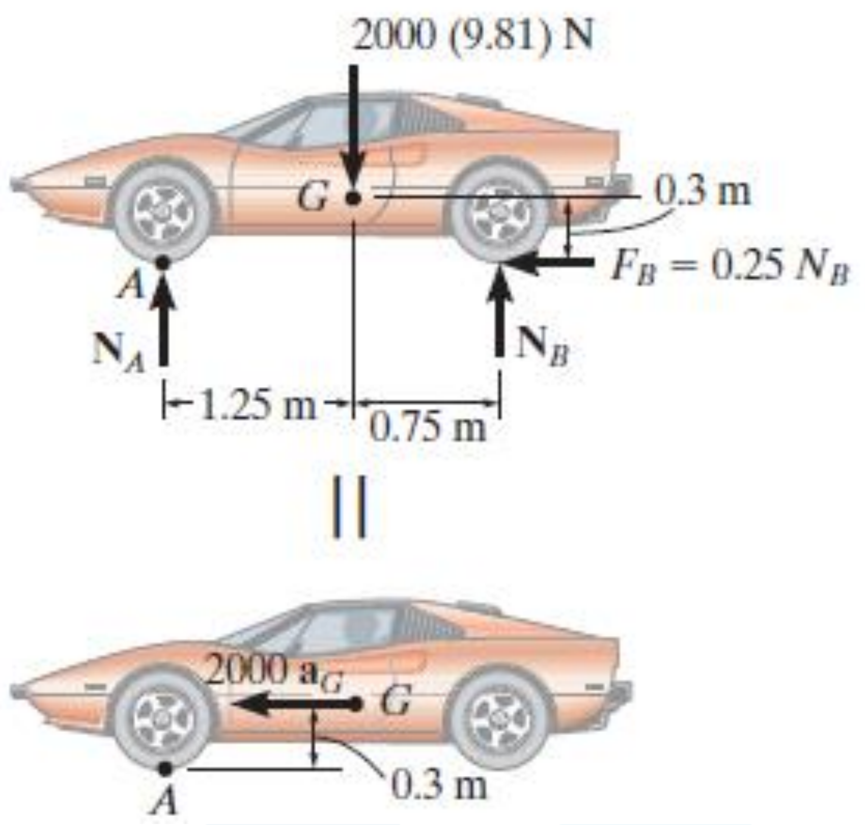
SOLUTION II

Free-Body and Kinetic Diagrams. If the "moment" equation is applied about point A, then the unknown N_A will be eliminated from the equation. To "visualize" the moment of ma_G about A, we will include the kinetic diagram as part of the analysis.

Equation of Motion.

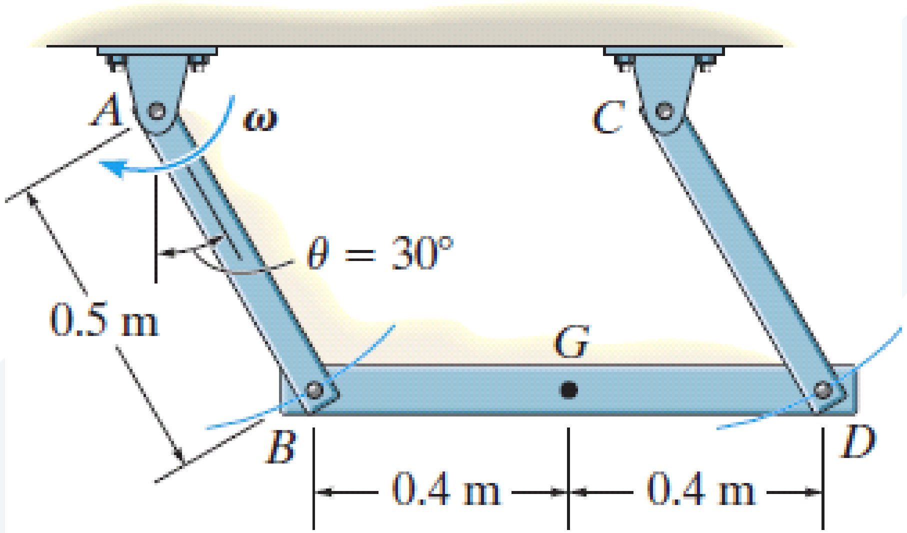
$$\circlearrowleft +\Sigma M_A = \Sigma(\mathcal{M}_k)_A; \qquad N_B(2\text{ m}) - [2000(9.81)\text{ N}](1.25\text{ m}) = (2000\text{ kg})a_G(0.3\text{ m})$$

Solving this and Eq. 1 for a_G leads to a simpler solution than that obtained from Eqs. 1 to 3.



EXAMPLE

The 100-kg beam BD shown is supported by two rods having negligible mass. Determine the force developed in each rod if at the instant $\theta = 30°, \omega = 6\ \text{rad/s}$.

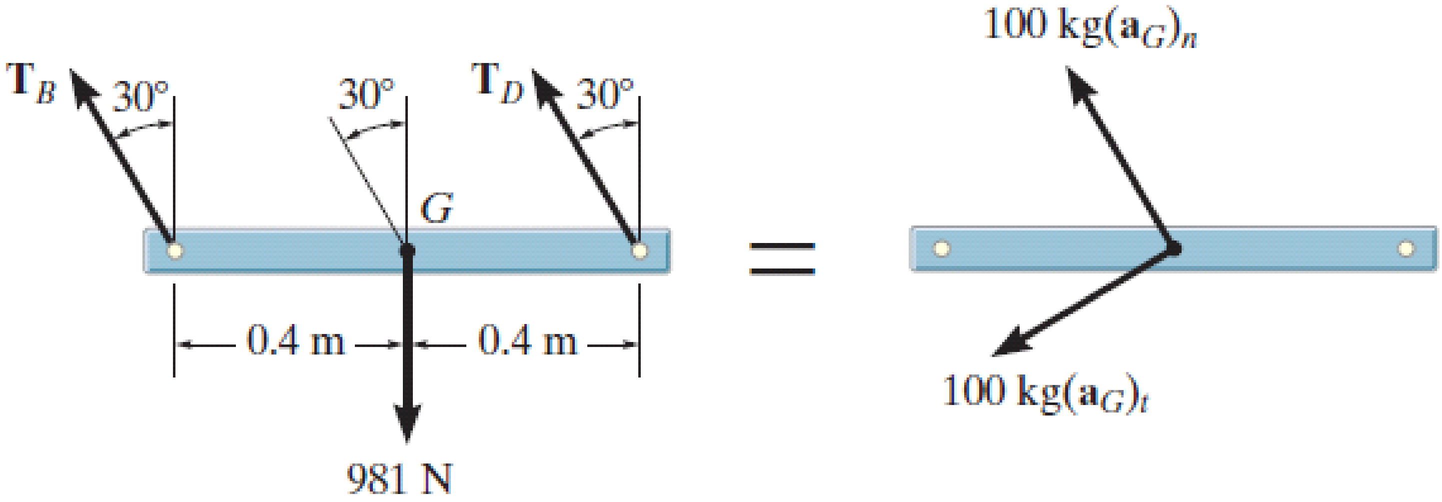


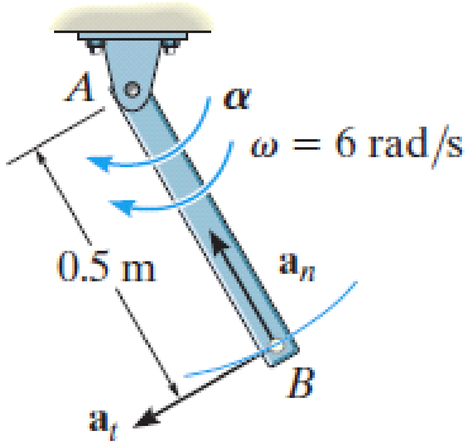
SOLUTION

Free-Body and Kinetic Diagrams. The beam moves with *curvilinear translation* since all points on the beam move along circular paths, each path having the same radius of 0.5 m, but different centers of curvature. Using normal and tangential coordinates, the free-body and kinetic diagrams for the beam are shown. Because of the *translation*, G has the *same* motion as the pin at B, which is connected to both the rod and the beam. Note that the tangential component of acceleration acts downward to the left due to the clockwise direction of $\boldsymbol{\alpha}$. Furthermore, the normal component of acceleration is *always* directed toward the center of curvature (toward point A for rod AB). Since the angular velocity of AB is 6 rad/s when $\theta = 30°$, then

$$(a_G)_n = \omega^2 r = (6\ \text{rad/s})^2(0.5\ \text{m}) = 18\ \text{m/s}^2$$

The three unknowns are T_B, T_D, and $(a_G)_t$.


T_B
30°
30°
T_D
30°
G
0.4 m
0.4 m
981 N
=
100 kg$(a_G)_n$
100 kg$(a_G)_t$


A
α
ω = 6 rad/s
0.5 m
a_n
B
a_t

Equations of Motion.

$$+\nwarrow \Sigma F_n = m(a_G)_n; \quad T_B + T_D - 981 \cos 30^\circ \text{ N} = 100 \text{ kg}(18 \text{ m/s}^2) \qquad (1)$$

$$+\swarrow \Sigma F_t = m(a_G)_t; \quad 981 \sin 30^\circ = 100 \text{ kg}(a_G)_t \qquad (2)$$

$$\circlearrowleft + \Sigma M_G = 0; \quad -(T_B \cos 30^\circ)(0.4 \text{ m}) + (T_D \cos 30^\circ)(0.4 \text{ m}) = 0 \qquad (3)$$

Simultaneous solution of these three equations gives

$$T_B = T_D = 1.32 \text{ kN} \qquad \textit{Ans.}$$

$$(a_G)_t = 4.905 \text{ m/s}^2$$

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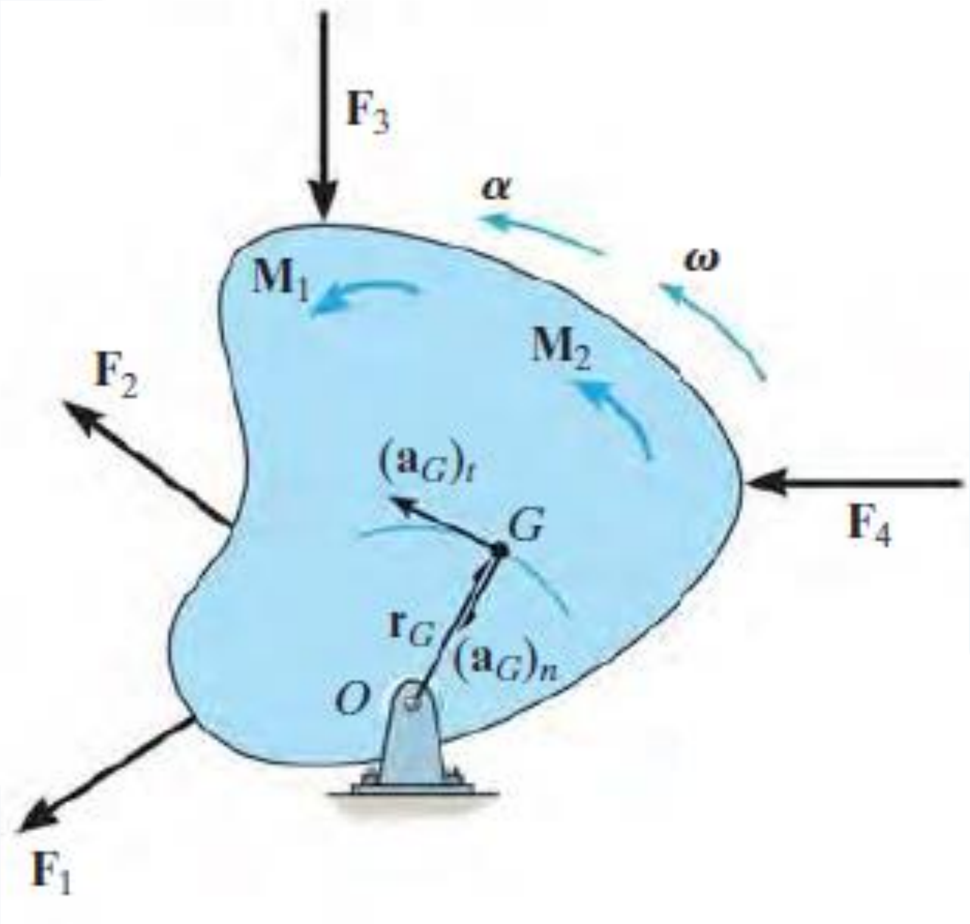
Equations of Motion: Rotation about a Fixed Axis

Consider the rigid body (or slab), which is constrained to rotate in the vertical plane about a fixed axis perpendicular to the page and passing through the pin at *O*. The angular velocity and angular acceleration are caused by the external force and couple moment system acting on the body. Because the body's center of mass *G* moves around a *circular path*, the acceleration of this point is best represented by its tangential and normal components. The *tangential component of acceleration* has a *magnitude* of $(a_G)_t = \alpha r_G$ and must act in a *direction* which is *consistent* with the body's angular acceleration α.

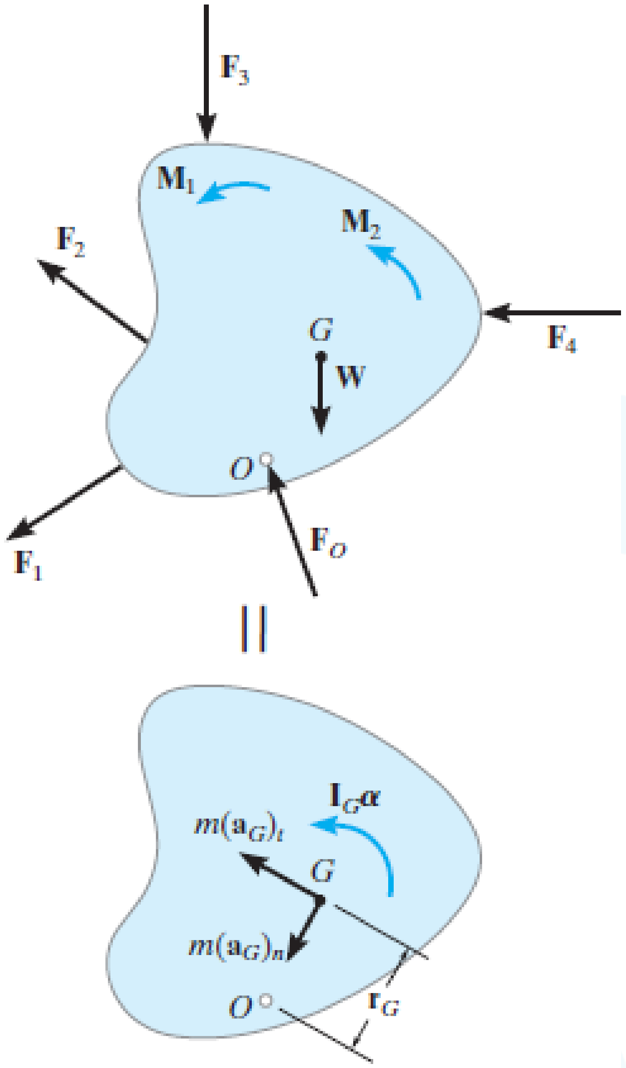
The *magnitude* of the *normal component of acceleration* is $(a_G)_n = \omega^2 r_G$

This component is *always directed* from point *G* to *O*, regardless of the rotational sense of w.


F3
α
ω
M1
M2
F2
(aG)t
G
F4
rG
(aG)n
O
F1

The free-body and kinetic diagrams for the body are shown .

The two components $m(\mathbf{a}_G)_t$ and $m(\mathbf{a}_G)_n$, shown on the kinetic diagram, are associated with the tangential and normal components of acceleration of the body's mass center. The $I_G\boldsymbol{\alpha}$ vector acts in the same *direction* as $\boldsymbol{\alpha}$ and has a *magnitude* of $I_G\alpha$, where I_G is the body's moment of inertia calculated about an axis which is perpendicular to the page and passes through G. From the derivation, the equations of motion which apply to the body can be written in the form



$$\Sigma F_n = m(a_G)_n = m\omega^2 r_G$$
$$\Sigma F_t = m(a_G)_t = m\alpha r_G$$
$$\Sigma M_G = I_G\alpha$$

The moment equation can be replaced by a moment summation about any arbitrary point P on or off the body provided one accounts for the moments $\Sigma(\mathcal{M}_k)_P$ produced by $I_G\boldsymbol{\alpha}$, $m(\mathbf{a}_G)_t$, and $m(\mathbf{a}_G)_n$ about the point.

Moment Equation About Point O. Often it is convenient to sum moments about the pin at O in order to eliminate the *unknown* force $\mathbf{F}_O$. From the kinetic diagram, this requires

$$\circlearrowleft+\Sigma M_O = \Sigma(\mathcal{M}_k)_O; \qquad \Sigma M_O = r_G m(a_G)_t + I_G\alpha$$

Note that the moment of $m(\mathbf{a}_G)_n$ is not included here since the line of action of this vector passes through O. Substituting $(a_G)_t = r_G\alpha$, we may rewrite the above equation as $\circlearrowleft+\Sigma M_O = (I_G + mr_G^2)\alpha$. From the parallel-axis theorem, $I_O = I_G + md^2$, and therefore the term in parentheses represents the *moment of inertia of the body about the fixed axis of rotation passing through O*. Consequently, we can write the three equations of motion for the body as

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$$\Sigma F_n = m(a_G)_n = m\omega^2 r_G$$
$$\Sigma F_t = m(a_G)_t = m\alpha r_G$$
$$\Sigma M_O = I_O\alpha$$

When using these equations, remember that "$I_O\alpha$" accounts for the "moment" of *both* $m(\mathbf{a}_G)_t$ *and* $I_G\boldsymbol{\alpha}$ about point O. In other words,

$$\Sigma M_O = \Sigma(\mathcal{M}_k)_O = I_O\alpha$$

Procedure for Analysis

Kinetic problems which involve the rotation of a body about a fixed axis can be solved using the following procedure.

Free-Body Diagram.

- Establish the inertial n, t coordinate system and specify the direction and sense of the accelerations $(\mathbf{a}_G)_n$ and $(\mathbf{a}_G)_t$ and the angular acceleration $\boldsymbol{\alpha}$ of the body. Recall that $(\mathbf{a}_G)_t$ must act in a direction which is in accordance with the rotational sense of $\boldsymbol{\alpha}$, whereas $(\mathbf{a}_G)_n$ always acts toward the axis of rotation, point O.
- Draw the free-body diagram to account for all the external forces and couple moments that act on the body.
- Determine the moment of inertia I_G or I_O.
- Identify the unknowns in the problem.

- If it is decided that the rotational equation of motion $\Sigma M_P = \Sigma(\mathcal{M}_k)_P$ is to be used, i.e., P is a point other than G or O, then consider drawing the kinetic diagram in order to help "visualize" the "moments" developed by the components $m(\mathbf{a}_G)_n$, $m(\mathbf{a}_G)_t$, and $I_G\boldsymbol{\alpha}$ when writing the terms for the moment sum $\Sigma(\mathcal{M}_k)_P$.

Equations of Motion.

- Apply the three equations of motion in accordance with the established sign convention.
- If moments are summed about the body's mass center, G, then $\Sigma M_G = I_G\alpha$, since $(m\mathbf{a}_G)_t$ and $(m\mathbf{a}_G)_n$ create no moment about G.
- If moments are summed about the pin support O on the axis of rotation, then $(m\mathbf{a}_G)_n$ creates no moment about O, and it can be shown that $\Sigma M_O = I_O\alpha$.

Kinematics.

- Use kinematics if a complete solution cannot be obtained strictly from the equations of motion.
- If the angular acceleration is variable, use

$$\alpha = \frac{d\omega}{dt} \quad \alpha\, d\theta = \omega\, d\omega \qquad \omega = \frac{d\theta}{dt}$$

- If the angular acceleration is constant, use

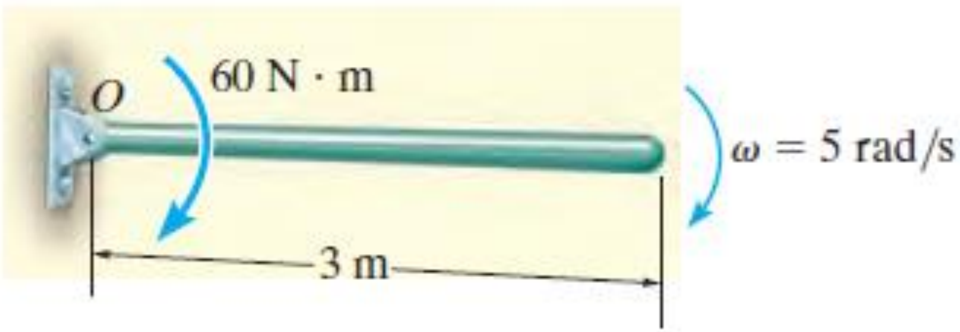
$$\omega = \omega_0 + \alpha_c t$$

$$\theta = \theta_0 + \omega_0 t + \tfrac{1}{2}\alpha_c t^2$$

$$\omega^2 = \omega_0^2 + 2\alpha_c(\theta - \theta_0)$$

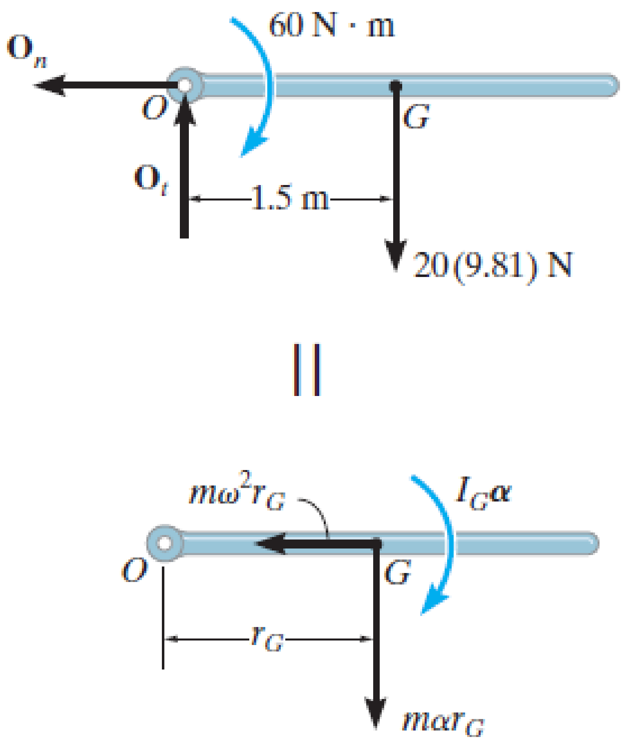
EXAMPLE

At the instant shown, the 20-kg slender rod has an angular velocity of $\omega = 5$ rad/s. Determine the angular acceleration and the horizontal and vertical components of reaction of the pin on the rod at this instant.



SOLUTION

Free-Body and Kinetic Diagrams. As shown on the kinetic diagram, point G moves around a circular path and so it has two components of acceleration. It is important that the tangential component $a_t = \alpha r_G$ act downward since it must be in accordance with the rotational sense of $\boldsymbol{\alpha}$. The three unknowns are O_n, O_t, and α.



Equation of Motion.

$$\overset{+}{\leftarrow} \Sigma F_n = m\omega^2 r_G; \qquad O_n = (20\text{ kg})(5\text{ rad/s})^2(1.5\text{ m})$$

$$+\downarrow \Sigma F_t = m\alpha r_G; \qquad -O_t + 20(9.81)\text{N} = (20\text{ kg})(\alpha)(1.5\text{ m})$$

$$\circlearrowright + \Sigma M_G = I_G\alpha; \qquad O_t(1.5\text{ m}) + 60\text{ N}\cdot\text{m} = \left[\tfrac{1}{12}(20\text{ kg})(3\text{ m})^2\right]\alpha$$

Solving

$$O_n = 750\text{ N} \quad O_t = 19.05\text{ N} \quad \alpha = 5.90\text{ rad/s}^2 \qquad \textit{Ans.}$$

A more direct solution to this problem would be to sum moments about point O to eliminate $\mathbf{O}_n$ and $\mathbf{O}_t$ and obtain a *direct solution* for α. Here,

$$\circlearrowleft +\Sigma M_O = \Sigma(\mathcal{M}_k)_O;\ \ 60\text{ N}\cdot\text{m} + 20(9.81)\text{ N}(1.5\text{ m}) =$$

$$\left[\tfrac{1}{12}(20\text{ kg})(3\text{ m})^2\right]\alpha + [20\text{ kg}(\alpha)(1.5\text{ m})](1.5\text{ m})$$

$$\alpha = 5.90\text{ rad/s}^2 \qquad \textit{Ans.}$$

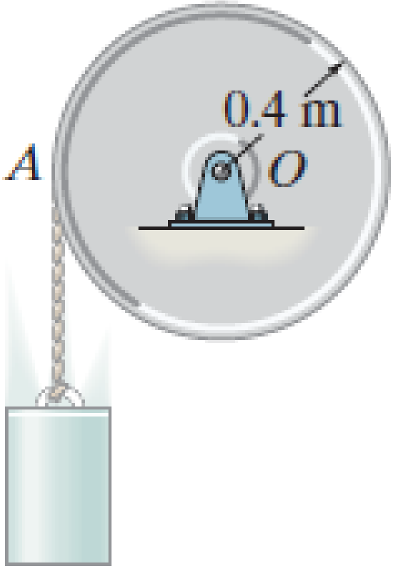
Also, since $I_O = \frac{1}{3}ml^2$ for a slender rod, we can apply

$$\circlearrowleft +\Sigma M_O = I_O\alpha;\ \ 60\text{ N}\cdot\text{m} + 20(9.81)\text{ N}(1.5\text{ m}) = \left[\tfrac{1}{3}(20\text{ kg})(3\text{ m})^2\right]\alpha$$

$$\alpha = 5.90\text{ rad/s}^2 \qquad \textit{Ans.}$$

EXAMPLE

The drum shown has a mass of 60 kg and a radius of gyration $k_O = 0.25$ m. A cord of negligible mass is wrapped around the periphery of the drum and attached to a block having a mass of 20 kg. If the block is released, determine the drum's angular acceleration.



SOLUTION I

Free-Body Diagram. Here we will consider the drum and block separately. Assuming the block accelerates *downward* at **a**, it creates a *counterclockwise* angular acceleration $\boldsymbol{\alpha}$ of the drum. The moment of inertia of the drum is

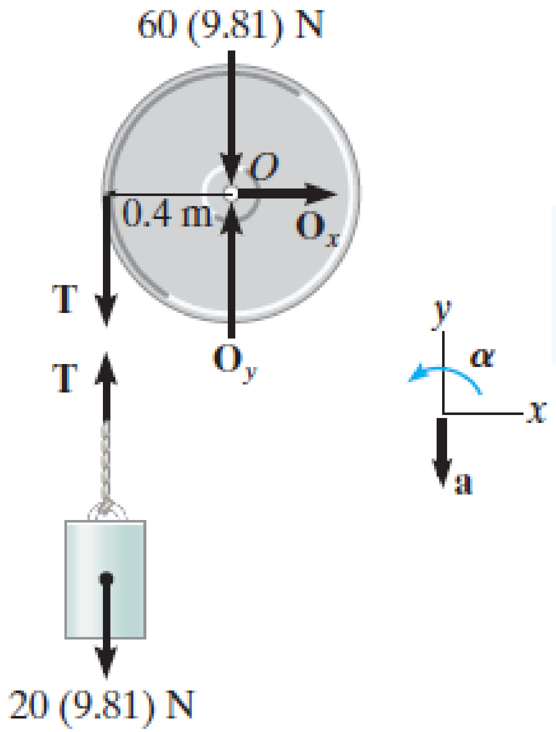
$$I_O = mk_O^2 = (60\text{ kg})(0.25\text{ m})^2 = 3.75\text{ kg}\cdot\text{m}^2$$

There are five unknowns, namely O_x, O_y, T, a, and α.

Equations of Motion. Applying the translational equations of motion $\Sigma F_x = m(a_G)_x$ and $\Sigma F_y = m(a_G)_y$ to the drum is of no consequence to the solution, since these equations involve the unknowns O_x and O_y. Thus, for the drum and block, respectively,

$$\circlearrowleft + \Sigma M_O = I_O\alpha; \qquad T(0.4 \text{ m}) = (3.75 \text{ kg} \cdot \text{m}^2)\alpha \qquad (1)$$

$$+\uparrow \Sigma F_y = m(a_G)_y; \qquad -20(9.81)\text{N} + T = -(20 \text{ kg})a \qquad (2)$$



Kinematics. Since the point of contact *A* between the cord and drum has a tangential component of acceleration **a**, then

$$\circlearrowleft + a = \alpha r; \qquad a = \alpha(0.4 \text{ m}) \qquad (3)$$

Solving the above equations,

$$T = 106 \text{ N} \quad a = 4.52 \text{ m/s}^2$$

$$\alpha = 11.3 \text{ rad/s}^2 \circlearrowleft \qquad \textit{Ans.}$$

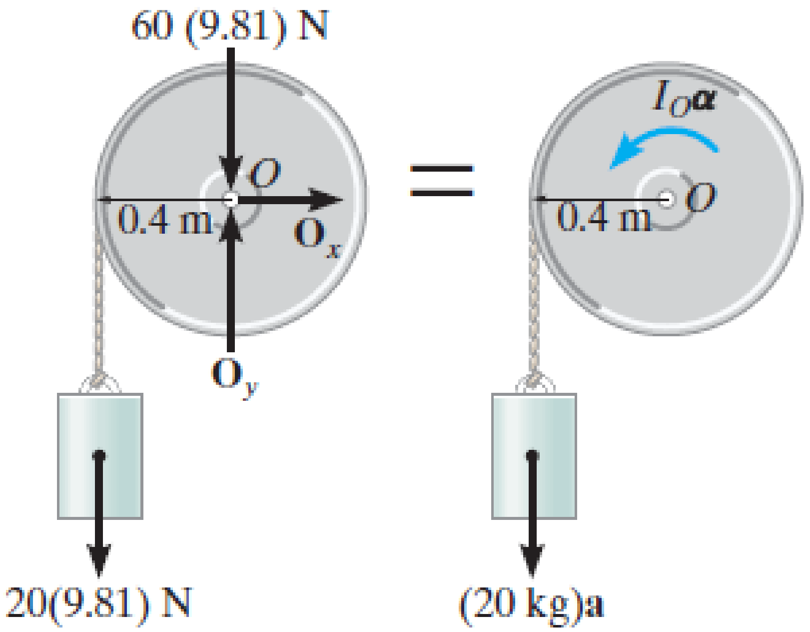
SOLUTION II

Free-Body and Kinetic Diagrams. The cable tension T can be eliminated from the analysis by considering the drum and block as a *single system*. The kinetic diagram is shown since moments will be summed about point O.

Equations of Motion. Using Eq. 3 and applying the moment equation about O to eliminate the unknowns O_x and O_y, we have

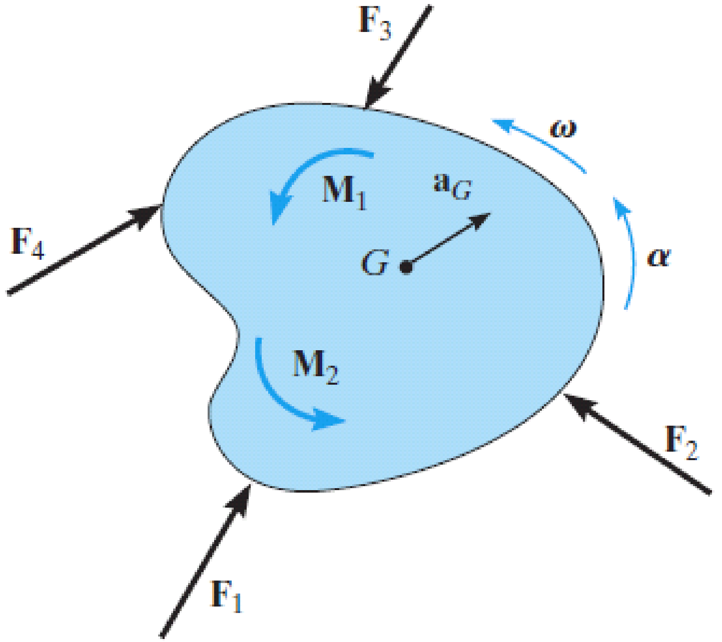
$$\circlearrowleft + \Sigma M_O = \Sigma(\mathcal{M}_k)_O; \qquad [20(9.81)\text{ N}]\,(0.4\text{ m}) = (3.75\text{ kg}\cdot\text{m}^2)\alpha + [20\text{ kg}(\alpha\, 0.4\text{ m})](0.4\text{ m})$$

$$\alpha = 11.3\text{ rad/s}^2 \qquad \textit{Ans.}$$



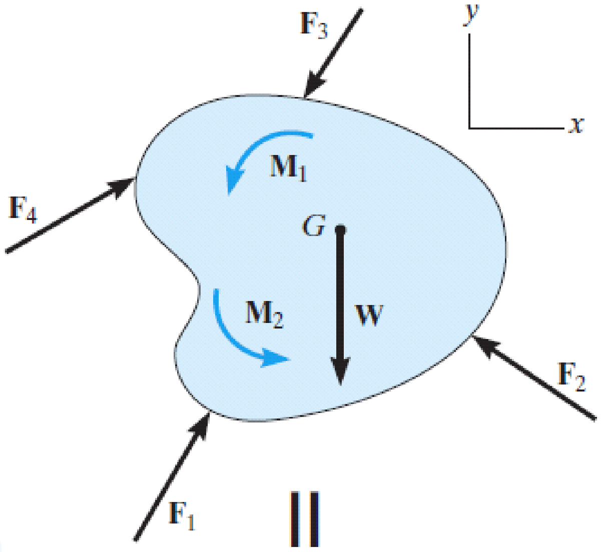
Equations of Motion: General Plane Motion

The rigid body (or slab) *is subjected to general plane* motion caused by the externally applied force and couple-moment system.



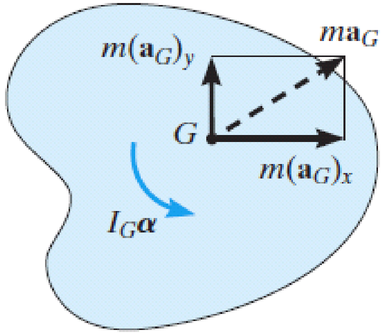
The free-body and kinetic diagrams for the body are shown. If an *x and y inertial coordinate system is established as shown, the three* equations of motion are

$$\Sigma F_x = m(a_G)_x$$
$$\Sigma F_y = m(a_G)_y$$
$$\Sigma M_G = I_G\alpha$$



In some problems it may be convenient to sum moments about a point *P* other than *G in order to eliminate as many unknown forces as possible* from the moment summation. When used in this more general case, the three equations of motion are

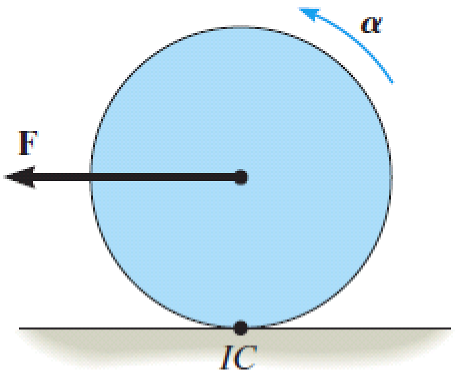
$$\Sigma F_x = m(a_G)_x$$
$$\Sigma F_y = m(a_G)_y$$
$$\Sigma M_P = \Sigma(\mathcal{M}_k)_P$$



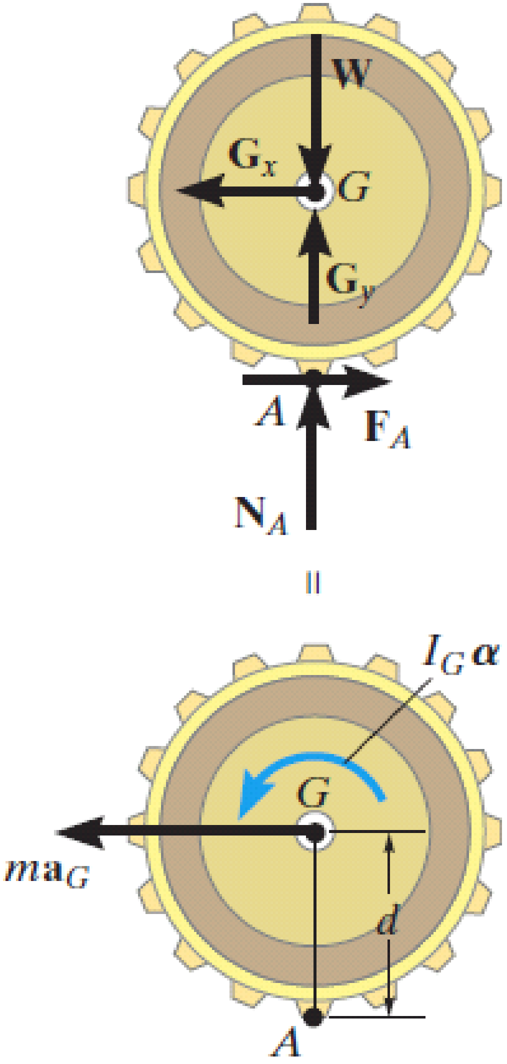
Moment Equation About the *IC*. There is a particular type of problem that involves a uniform disk, or body of circular shape, that rolls on a rough surface *without slipping*. If we sum the moments about the instantaneous center of zero velocity, then $\Sigma(\mathcal{M}_k)_{IC}$ becomes $I_{IC}\alpha$, so that

$$\Sigma M_{IC} = I_{IC}\alpha$$

This result compares with $\Sigma M_O = I_O\alpha$, which is used for a body pinned at point O



As the soil compactor, or "sheep's foot roller" moves forward, the roller has general plane motion. The forces shown on its free-body diagram cause the effects shown on the kinetic diagram. If moments are summed about the mass center, G, then $\Sigma M_G = I_G\alpha$. However, if moments are summed about point A (the IC) then $\circlearrowleft + \Sigma M_A = I_G\alpha + (ma_G)d = I_A\alpha$.



Procedure for Analysis

Kinetic problems involving general plane motion of a rigid body can be solved using the following procedure.

Free-Body Diagram.

- Establish the x, y inertial coordinate system and draw the free-body diagram for the body.
- Specify the direction and sense of the acceleration of the mass center, $\mathbf{a}_G$, and the angular acceleration $\boldsymbol{\alpha}$ of the body.
- Determine the moment of inertia I_G.
- Identify the unknowns in the problem.
- If it is decided that the rotational equation of motion $\Sigma M_P = \Sigma(\mathcal{M}_k)_P$ is to be used, then consider drawing the kinetic diagram in order to help "visualize" the "moments" developed by the components $m(\mathbf{a}_G)_x$, $m(\mathbf{a}_G)_y$, and $I_G\boldsymbol{\alpha}$ when writing the terms in the moment sum $\Sigma(\mathcal{M}_k)_P$.

Equations of Motion.

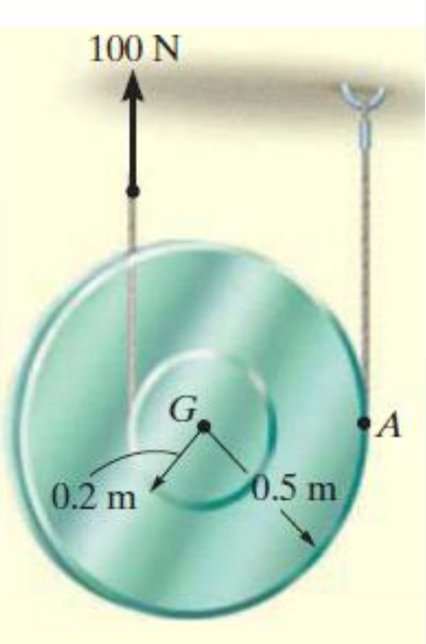
- Apply the three equations of motion in accordance with the established sign convention.
- When friction is present, there is the possibility for motion with no slipping or tipping. Each possibility for motion should be considered.

Kinematics.

- Use kinematics if a complete solution cannot be obtained strictly from the equations of motion.
- If the body's motion is *constrained* due to its supports, additional equations may be obtained by using $\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A}$, which relates the accelerations of any two points A and B on the body.
- When a wheel, disk, cylinder, or ball *rolls without slipping*, then $a_G = \alpha r$.

EXAMPLE

Determine the angular acceleration of the spool. The spool has a mass of 8 kg and a radius of gyration of $k_G = 0.35$ m. The cords of negligible mass are wrapped around its inner hub and outer rim.



SOLUTION I

Free-Body and Kinetic Diagrams. The 100-N force causes $\mathbf{a}_G$ to act upward. Also, $\boldsymbol{\alpha}$ acts clockwise, since the spool winds around the cord at A.

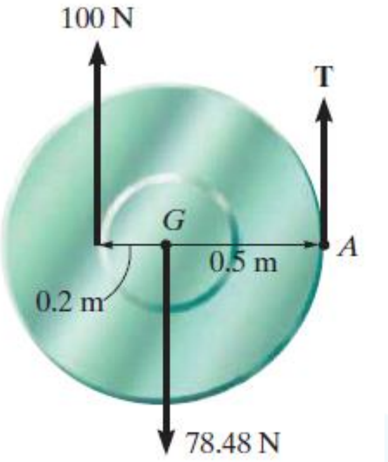
There are three unknowns T, a_G, and α. The moment of inertia of the spool about its mass center is

$$I_G = mk_G^2 = 8\text{ kg}(0.35\text{ m})^2 = 0.980\text{ kg}\cdot\text{m}^2$$

Equations of Motion.

$$+\uparrow\Sigma F_y = m(a_G)_y; \qquad T + 100\text{ N} - 78.48\text{ N} = (8\text{ kg})a_G \qquad (1)$$

$$\circlearrowright +\Sigma M_G = I_G\alpha; \quad 100\text{ N}(0.2\text{ m}) - T(0.5\text{ m}) = (0.980\text{ kg}\cdot\text{m}^2)\alpha \qquad (2)$$



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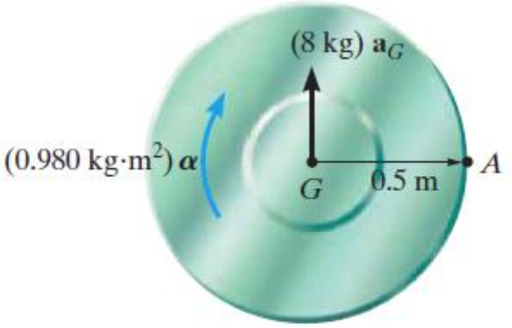


Kinematics. A complete solution is obtained if kinematics is used to relate a_G to α. In this case the spool "rolls without slipping" on the cord at A. Hence

$$(\circlearrowright +)\ a_G = \alpha r; \qquad a_G = \alpha\,(0.5\text{ m}) \qquad (3)$$

Solving Eqs. 1 to 3, we have

$$\alpha = 10.3\text{ rad/s}^2 \qquad \textit{Ans.}$$

$$a_G = 5.16\text{ m/s}^2$$

$$T = 19.8\text{ N}$$

SOLUTION II

Equations of Motion. We can eliminate the unknown T by summing moments about point A. From the free-body and kinetic diagrams

$$\circlearrowright + \Sigma M_A = \Sigma(\mathcal{M}_k)_A; \qquad 100\ \text{N}(0.7\ \text{m}) - 78.48\ \text{N}(0.5\ \text{m})$$

$$= (0.980\ \text{kg} \cdot \text{m}^2)\alpha + [(8\ \text{kg})a_G](0.5\ \text{m})$$

Using Eq. (3),

$$\alpha = 10.3\ \text{rad/s}^2 \qquad \textit{Ans.}$$

SOLUTION III

Equations of Motion. The simplest way to solve this problem is to realize that point A is the IC for the spool.

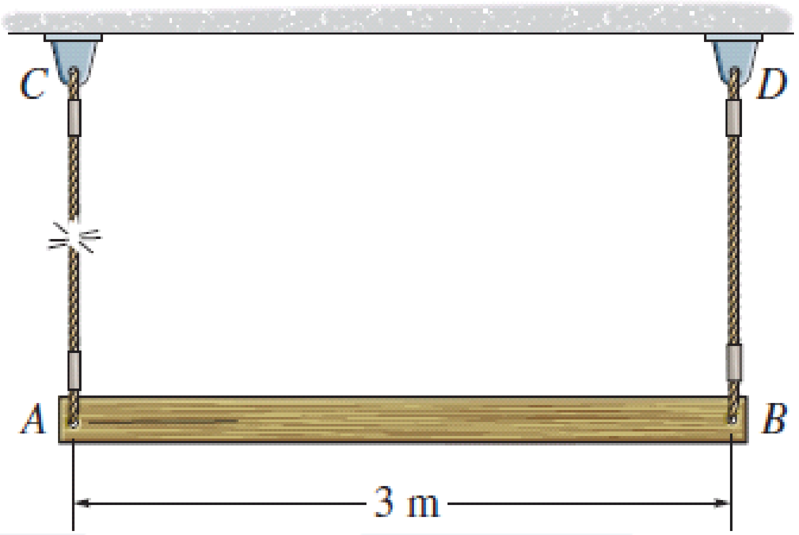
$$\circlearrowleft +\Sigma M_A = I_A\alpha; \quad (100\ \text{N})(0.7\ \text{m}) - (78.48\ \text{N})(0.5\ \text{m})$$

$$= [0.980\ \text{kg}\cdot\text{m}^2 + (8\ \text{kg})(0.5\ \text{m})^2]\alpha$$

$$\alpha = 10.3\ \text{rad/s}^2$$

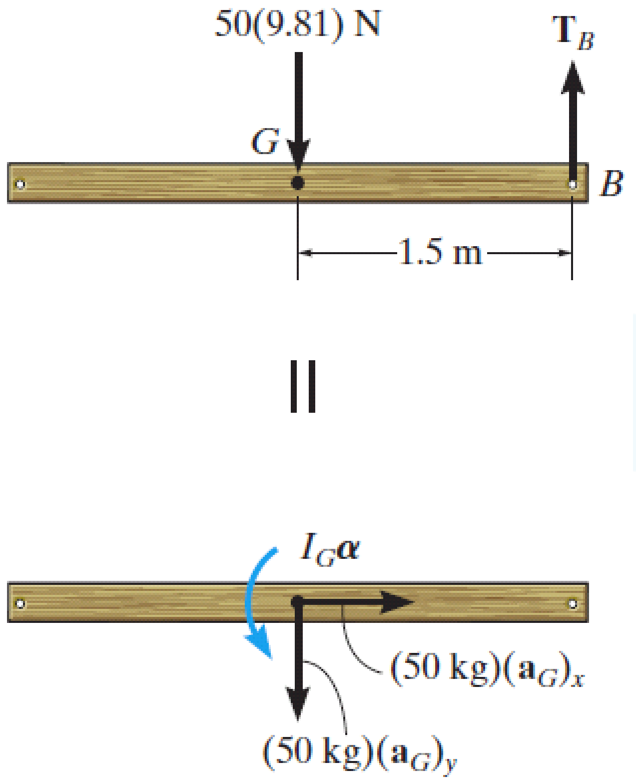
EXAMPLE

The uniform 50-kg bar is held in the equilibrium position by cords *AC* and *BD*. Determine the tension in *BD* and the angular acceleration of the bar immediately after *AC* is cut.



SOLUTION

Free-Body and Kinetic Diagrams. There are four unknowns, T_B, $(a_G)_x$, $(a_G)_y$, and α.



Equations of Motion.

$$\xrightarrow{+} \Sigma F_x = m(a_G)_x; \qquad 0 = 50\text{ kg }(a_G)_x$$

$$(a_G)_x = 0$$

$$+\uparrow \Sigma F_y = m(a_G)_y; \quad T_B - 50(9.81)\text{N} = -50\text{ kg }(a_G)_y \qquad (1)$$

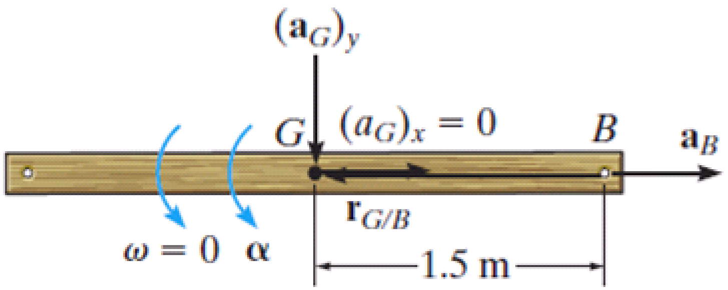
$$\circlearrowleft + \Sigma M_G = I_G\alpha; \quad T_B(1.5\text{ m}) = \left[\frac{1}{12}(50\text{ kg})(3\text{ m})^2\right]\alpha \qquad (2)$$

Kinematics. Since the bar is at rest just after the cable is cut, then its angular velocity and the velocity of point B at this instant are equal to zero. Thus $(a_B)_n = v_B^2/\rho_{BD} = 0$. Therefore, $\mathbf{a}_B$ only has a tangential component, which is directed along the x axis. Applying the relative acceleration equation to points G and B,

$$\mathbf{a}_G = \mathbf{a}_B + \boldsymbol{\alpha} \times \mathbf{r}_{G/B} - \omega^2 \mathbf{r}_{G/B}$$

$$-(a_G)_y\mathbf{j} = a_B\mathbf{i} + (\alpha\mathbf{k}) \times (-1.5\mathbf{i}) - \mathbf{0}$$

$$-(a_G)_y\mathbf{j} = a_B\mathbf{i} - 1.5\alpha\mathbf{j}$$



Equating the **i** and **j** components of both sides of this equation,

$$0 = a_B$$

$$(a_G)_y = 1.5\alpha \tag{3}$$

Solving Eqs. (1) through (3) yields

$$\alpha = 4.905 \text{ rad/s}^2 \qquad \textit{Ans.}$$

$$T_B = 123 \text{ N} \qquad \textit{Ans.}$$

$$(a_G)_y = 7.36 \text{ m/s}^2$$

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