Center of Gravity, Center of Mass, Centroids مركز الثقل، مركز الكتلة، المركز الجيومتري

1. Center of Forces.
2. Center of Gravity and Center of Mass.
3. Centroids: Center of a Volume, Center of an Area and Center of a Line.
4. Examples and Exercises.

1. Center of Forces: مركز القوى

It was shown that a system of forces that are not in equilibrium can be replaced by a single force, namely, the resultant R ($\vec{R}$), provided that the reduction does not lead to a couple.

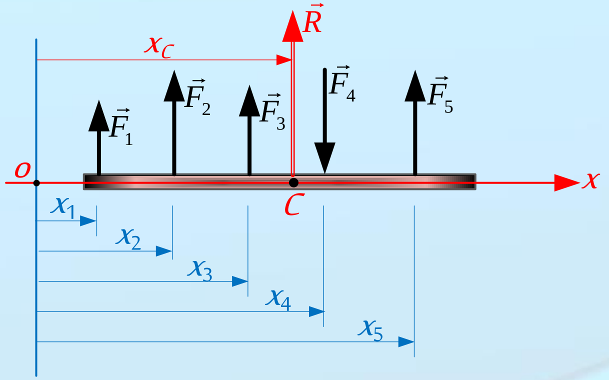
رأينا سابقاً أنه يمكن اختزال جملة من قوى غير متوازنة إلى قوة وحيدة تدعى المحصلة ونرمز لها $\vec{R}$، شريطة أن لا يؤول الاختزال إلى مزدوجة.

For parallel forces, the direction of the resultant R ($\vec{R}$) coincides with the direction of the forces.

وفي حالة القوى المتوازية تكون المحصلة موازية لهذه القوى

1.1 Concentrated Forces: القوى المركزة

a) Parallel forces acting on a line



$$\vec{R} = \sum \vec{F}_i \quad \Rightarrow R = \sum F_i = F_1 + F_2 + F_3 - F_4 + F_5$$

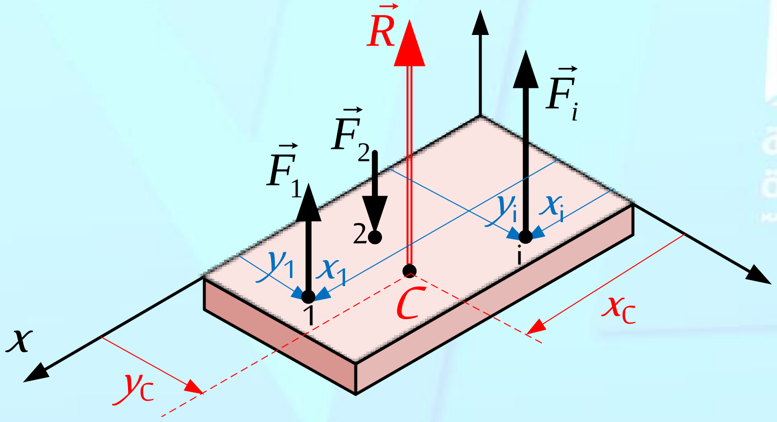
The action line of the resultant R ($\vec{R}$) can be found from

$$x_C R = \sum x_i F_i \qquad \Rightarrow x_C R = x_1 F_1 + x_2 F_2 + x_3 F_3 - x_4 F_4 + x_5 F_5 +$$

$$\Rightarrow x_C = \frac{x_1 F_1 + x_2 F_2 + x_3 F_3 - x_4 F_4 + x_5 F_5}{R} \quad \Rightarrow \quad \boxed{x_C = \frac{\sum x_i F_i}{\sum F_i}}$$

b) Parallel forces acting on a Plane

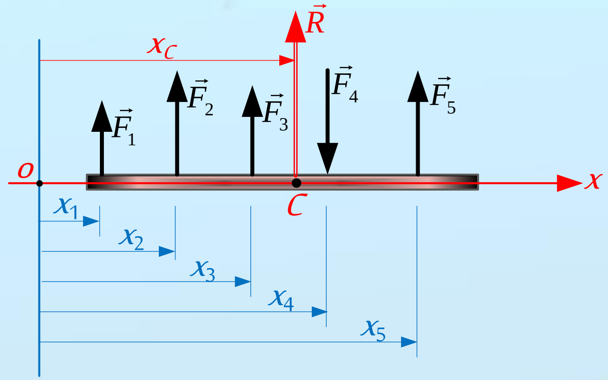
$$R = \sum F_i \qquad x_C = \frac{\sum x_i F_i}{\sum F_i} \qquad y_C = \frac{\sum y_i F_i}{\sum F_i}$$



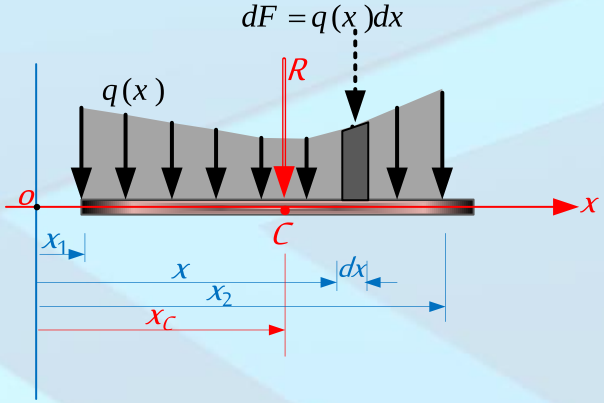
1.2 Distributed Forces:

1.2.1 Line Forces: Dim. :$[q] \equiv [F]/[L]$, S.I. Units: N/m, kN/m, . . .

$$R = \sum dF_i = \int_{x_1}^{x_2} q(x)\,dx$$



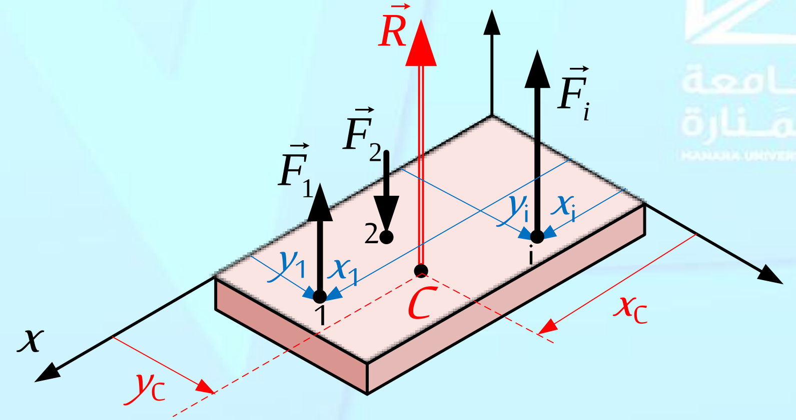
$$x_C = \frac{\sum x_i F_i}{\sum F_i} = \frac{\int_{x_1}^{x_2} x\,q(x)\,dx}{\int_{x_1}^{x_2} q(x)\,dx}$$



1.2.2 Area (Surface) Forces: Dim. :$[p] \equiv [F]/[L^2]$, S.I. Units: N/m^2, kN/m^2, . . .

$$x_C = \frac{\sum x_i F_i}{\sum F_i}$$

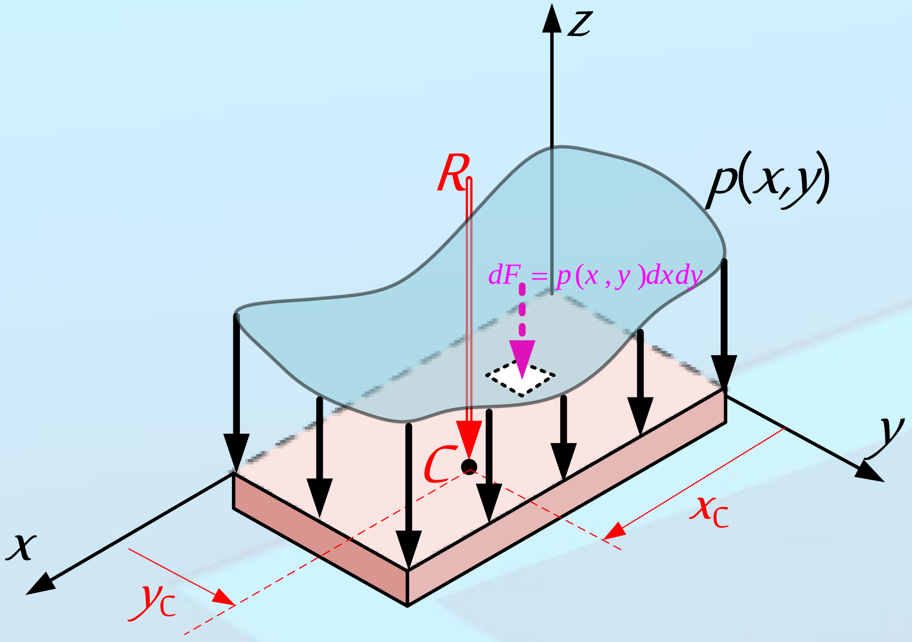
$$y_C = \frac{\sum y_i F_i}{\sum F_i}$$



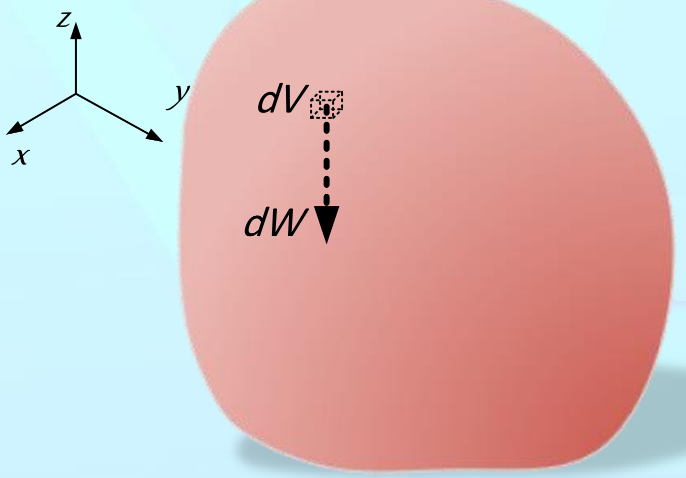
$$R = \sum dF = \int_A p(x,y)dxdy$$

$$x_C = \frac{\int_A xp(x,y)dxdy}{\int_A p(x,y)dxdy}$$

$$y_C = \frac{\int_A yp(x,y)dxdy}{\int_A p(x,y)dxdy}$$



2. Center of Gravity and Center of Mass.



Center of Gravity

$$W = \sum dW = \int_V \gamma(x,y,z)\,dx\,dy\,dz$$

$$x_C = \frac{\int_V x\,\gamma(x,y,z)\,dx\,dy\,dz}{\int_V \gamma(x,y,z)\,dx\,dy\,dz}$$

$$y_C = \frac{\int_V y\,\gamma(x,y,z)\,dx\,dy\,dz}{\int_V \gamma(x,y,z)\,dx\,dy\,dz}$$

$$z_C = \frac{\int_V z\,\gamma(x,y,z)\,dx\,dy\,dz}{\int_V \gamma(x,y,z)\,dx\,dy\,dz}$$

Center of Mass

$$M = \sum dM = \int_V \rho(x,y,z)\,dx\,dy\,dz$$

$$x_C = \frac{\int_V x\,\rho(x,y,z)\,dx\,dy\,dz}{\int_V \rho(x,y,z)\,dx\,dy\,dz}$$

$$y_C = \frac{\int_V y\,\rho(x,y,z)\,dx\,dy\,dz}{\int_V \rho(x,y,z)\,dx\,dy\,dz}$$

$$z_C = \frac{\int_V z\,\rho(x,y,z)\,dx\,dy\,dz}{\int_V \rho(x,y,z)\,dx\,dy\,dz}$$

$$\gamma(x,y,z)$$

Weight per Volume Unit الوزن الحجمي

Dim. :$[\gamma] \equiv [F]/[L^3]$, S.I. N/m^3, kN/m^3, . . .

$$\rho(x,y,z)$$

Mass per Volume Unit الكتلة الحجمية

Dim. :$[\rho] \equiv [kg]/[L^3]$, S.I. kg/m^3, kg/m^3, . . .

On Earth عند سطح الأرض

$$\gamma(x,y,z) = g\,\rho(x,y,z)$$

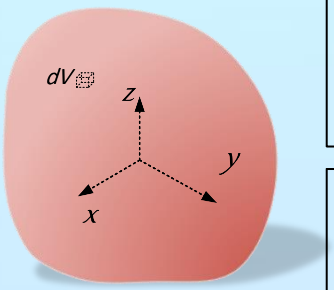
3. Centroids: Center of a Volume, Center of an Area and Center of a Line.

When $\gamma(x,y,z)$ and $\rho(x,y,z)$ are constant (homogeneous materials), the gravity and mass centers become geometric centers "Centroids". The previous equations take the forms

تكون الكتلة الحجمية والوزن الحجمي ثابتان في المواد المتجانسة وعندئذ يصبح مركز الثقل أو مركز الكتلة، مجرد مركز جيومتري. وتكون العلاقات السابقة:

For Volumes

$$V=\sum V_i=\sum\int_{V_i} dx\,dy\,dz$$



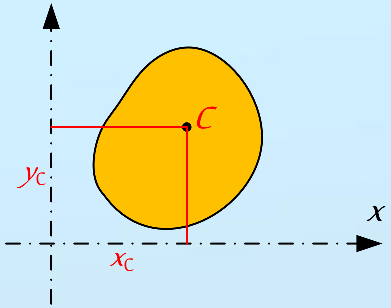
$$x_C=\frac{\int_V x\,dx\,dy\,dz}{V}$$

$$y_C=\frac{\int_V y\,dx\,dy\,dz}{V}$$

$$z_C=\frac{\int_V z\,dx\,dy\,dz}{V}$$

For Areas

$$A=\sum A_i=\sum\int_{A_i} dx\,dy$$



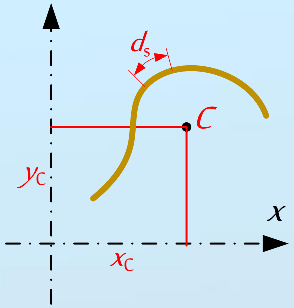
$$x_C=\frac{\int_A x\,dx\,dy}{A}$$

$$y_C=\frac{\int_A y\,dx\,dy}{A}$$

$$z_C=\frac{\int_A z\,dx\,dy}{A}$$

For Lines

$$s=\int_S ds=\int_S\sqrt{dx^2+dy^2}$$

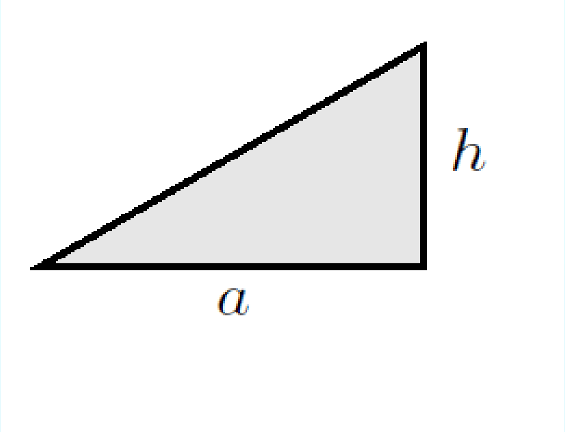


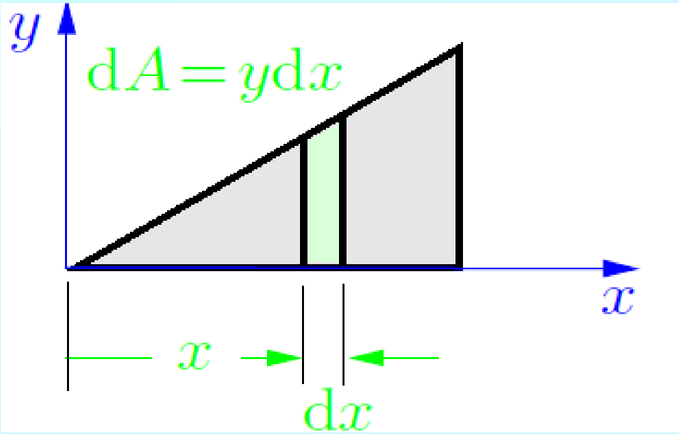
$$x_C=\frac{\int_S x\,ds}{S}$$

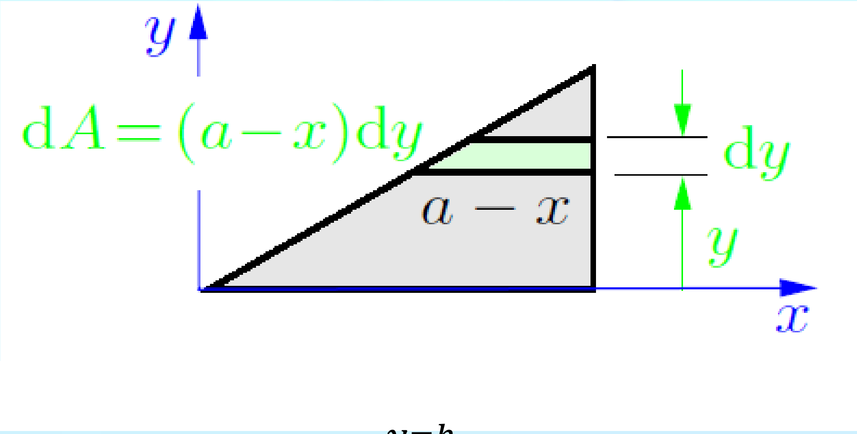
$$y_C=\frac{\int_S y\,ds}{S}$$

$$z_C=\frac{\int_S z\,ds}{S}$$

Example 1. Locate the centroid of a rectangular triangle with baseline *a* and height *h*.







$$A = \iint_A dA \Rightarrow A = \int_{x=0}^{x=a} y dx$$

$$\Rightarrow A = \int_{x=0}^{x=a} \left(\frac{h}{a}x\right) dx$$

$$= \frac{h}{a}\int_{x=0}^{x=a} x dx = \frac{h}{a}\left[\frac{x^2}{2}\right]_0^a = \frac{ha}{2}$$

$$x_C = \frac{\iint_A x dA}{A} = \frac{\int_{x=0}^{x=a} x(ydx)}{A}$$

$$= \frac{\int_{x=0}^{x=a} x(\frac{h}{a}xdx)}{A}$$

$$= \frac{\frac{h}{a}\int_{x=0}^{x=a} x^2 dx}{A} = \frac{\frac{h}{a}\left[\frac{x^3}{3}\right]_0^a}{A} = \frac{2a}{3}$$

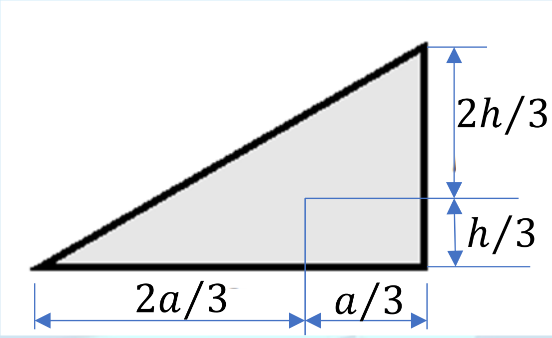
$$y_C = \frac{\iint_A y dA}{A} = \frac{\int_{y=0}^{y=h} y[(a-x)dy]}{A}$$

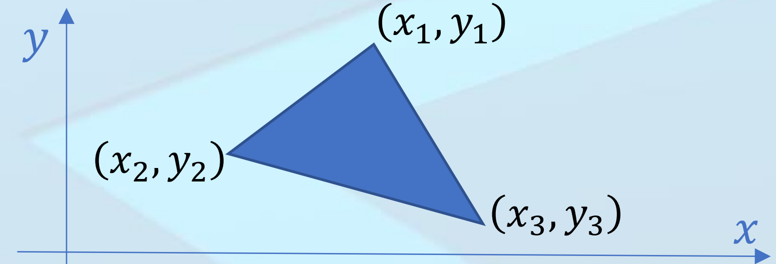
$$= \frac{\int_{y=0}^{y=h} y[\left(a - \frac{a}{h}y\right) dy]}{A}$$

$$= \frac{\frac{a}{h}\int_{y=0}^{y=h} y[(hy - y^2)dy]}{A} = \frac{\frac{a}{h}\left[\frac{hy^2}{2} - \frac{y^3}{3}\right]_0^h}{A} = \frac{h}{3}$$

$$A = \iint_A dA \Rightarrow A = \int_{y=0}^{y=h} (a-x)dy$$

$$= \int_{y=0}^{y=h} \left(a - \frac{a}{h}y\right) dy = \frac{a}{h}\left[hy - \frac{y^2}{2}\right]_0^h = \frac{ah}{2}$$

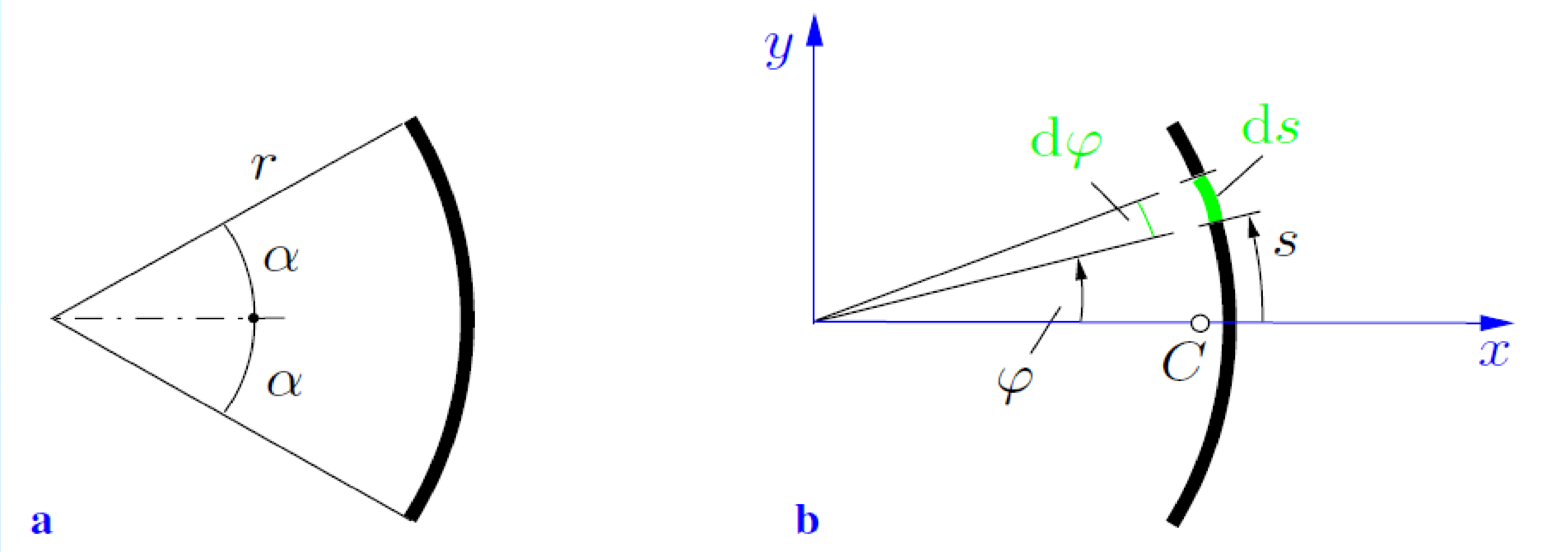




$$x_C = \frac{x_1 + x_2 + x_3}{3} \qquad y_C = \frac{y_1 + y_2 + y_3}{3}$$

Example.

A wire is bent into the shape of a circular arc with an opening angle 2α



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4. Examples and Exercises. Centroids of plane areas and lines

Three Remarks before starting

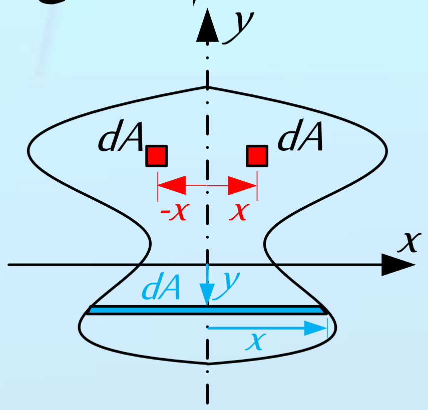
1. The term “centroid” is used when the material factors (γ, ρ) are omitted, i.e., when one is concerned with geometrical considerations only.

يستخدم مصطلح "المركز أو مركز الشكل" عندما تكون المادة متجانسة وينصب اهتمامنا على الجيوموتري فقط

2. If the area has an axis of symmetry, the centroid of the area lies on this axis.

$$x_C = \frac{\int_A x\,dx\,dy}{A} = 0$$

$$y_C = \frac{\int_A y\,dA}{A} = \frac{\int_A 2xy\,dy}{A}$$

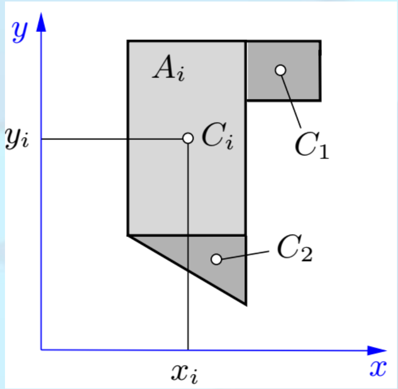


إذا كان للشكل محور تناظر أو أكثر فإن المركز يقع على أي من هذه المحاور

3. For area composed of several parts of simple shape. The coordinates x_i, y_i of the centroids C_i and the areas A_i of the individual parts are assumed to be known.

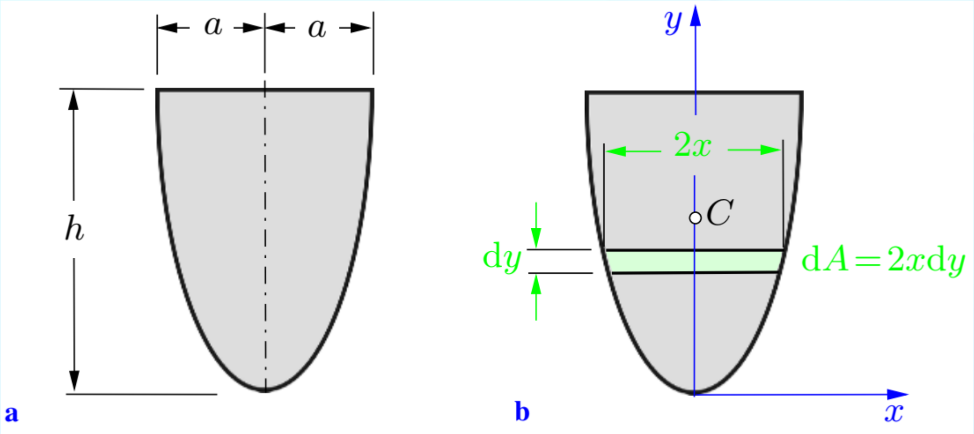
$$x_C = \frac{\sum_i x_i A_i}{\sum_i A_i}$$

$$y_C = \frac{\sum_i y_i A_i}{\sum_i A_i}$$



إذا كان الشكل مجمعا من أشكال بسيطةمعرفة ومراكزها معلومة، يستنتج مركزه من المعادلات البسيطة المبينة جانبا كما سنرى في أمثلة لاحقة.

Example 1. Locate the centroid of the area that is bounded by a parabola, as in figure (a).



Solution

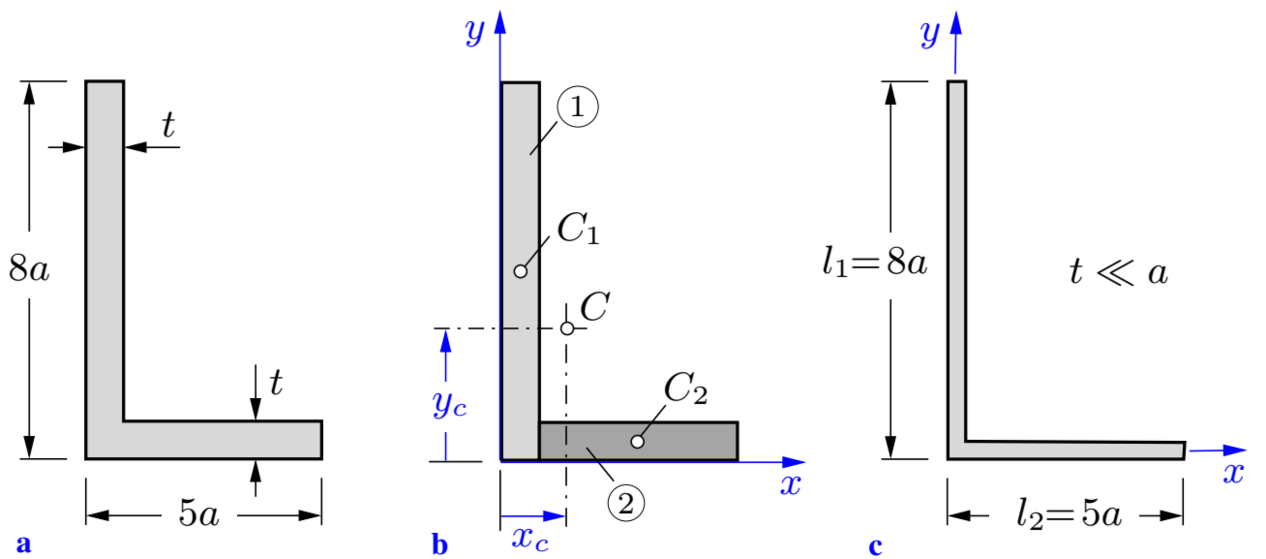
We use the coordinate system as shown in (b). Since the y -axis is an axis of symmetry, the centroid C lies on it: $x_c = 0$. To determine the coordinate y_c from:

$$y_C = \frac{\int_A y dA}{A} = \frac{\int_A 2xydy}{A}$$

but $A = \int_A 2xdy = \int_0^h \ldots$ and $y = cx^2$, with $x = a$ when $y = h$, so $c = ?$ and $y = ?$

Example 2. Find the centroid of the L-shaped area in Fig. (a), then in Fig. (c).

Solution: We choose a coordinate system and consider the area to be composed of two rectangles (b):



$A_1 = 8at,\qquad A_2 = (5a-t)t$

The coordinates of their respective centroids are given by

$$x_1 = t/2,\; y_1 = 4a\;;\qquad x_2 = t + (5a-t)/2 = (5a+t)/2,\; y_2 = t/2$$

$$x_C = \frac{\sum_2 x_i A_i}{\sum_2 A_i} = \frac{(t/2)(8at) + [(5a+t)/2][(5a-t)t]}{(8at) + [(5a-t)t]}$$

$$y_C = \frac{\sum_2 y_i A_i}{\sum_2 A_i} = \frac{(4a)(8at) + (t/2)[(5a-t)t]}{(8at) + [(5a-t)t]}$$

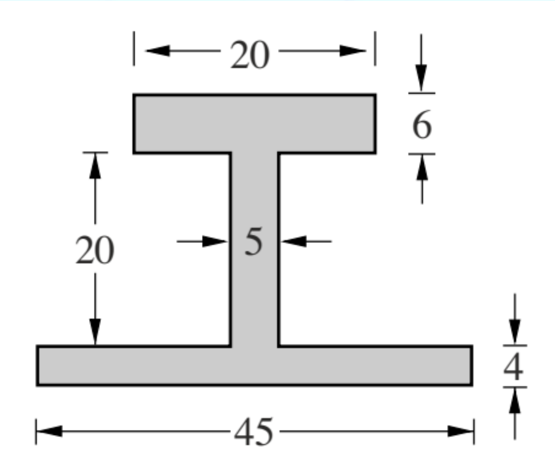
$$x_C = \frac{4at^2 + (25a^2 - t^2)(t/2)}{13at - t^2} = \frac{25a^2 + 8at - t^2}{26a - 2t}$$

$$y_C = \frac{32a^2 t + (5a-t)(t^2/2)}{13at - t^2} = \frac{64a^2 + 5at - t^2}{26a - 2t}$$

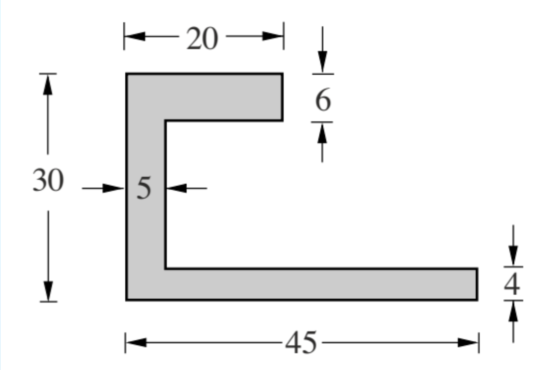
For $t << a$, as in Fig. (c), the areas become lines $x_C = \frac{25a}{26}$ and $y_C = \frac{32a}{13}$

Exercises.

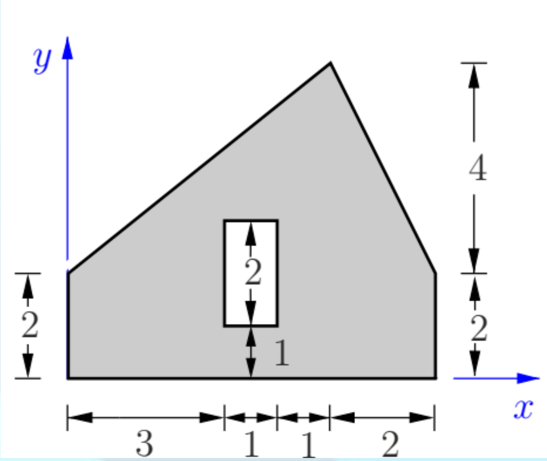
1. Locate the centroids of the depicted profile. The measurements are given in mm.



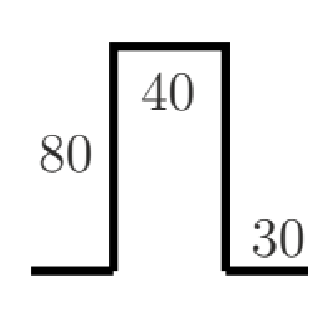
2. Locate the centroids of the depicted profile. The measurements are given in mm.



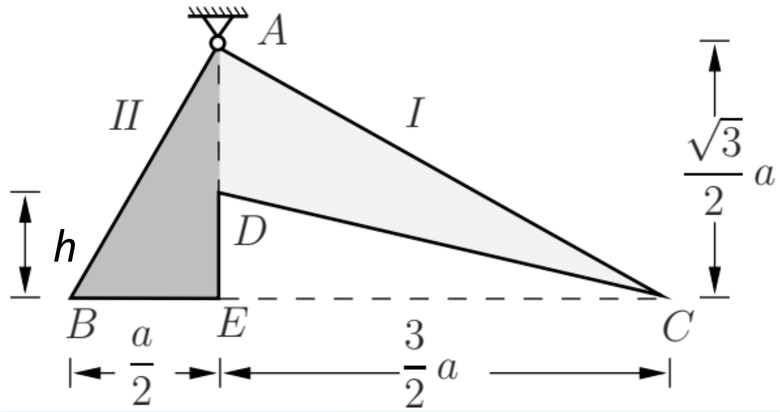
3. Locate the centroid of the depicted area with a rectangular cutout. The measurements are given in cm.



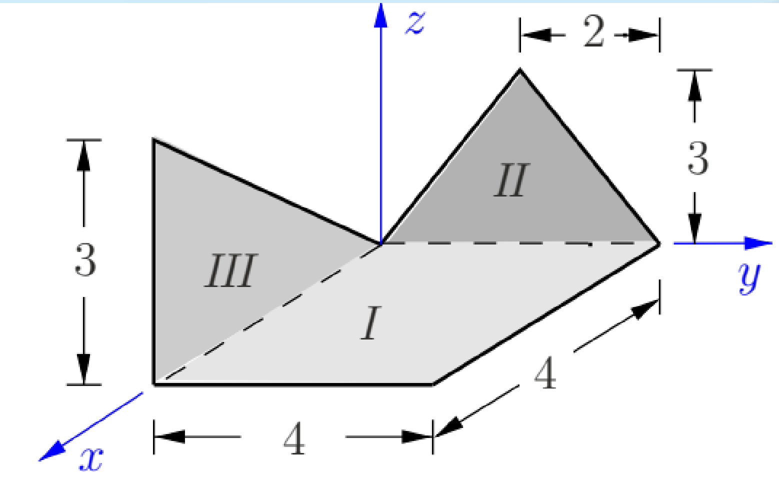
4. A wire with constant thickness is deformed into the depicted figure. The measurements are given in mm. Locate the centroid.



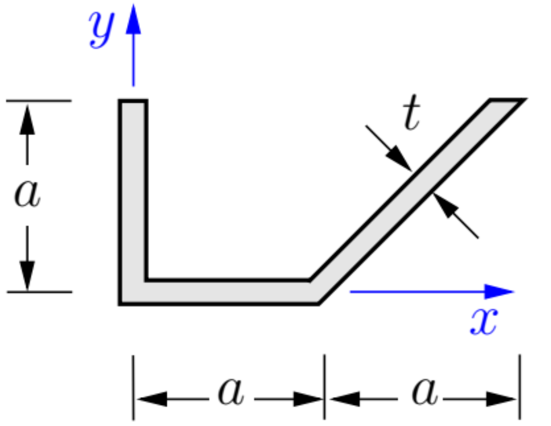
5. From the triangular-shaped metal sheet *ABC*, the triangle *CDE* has been cut out. The system is pin supported in *A*. Determine *h* such that *BC* adjusts horizontal.



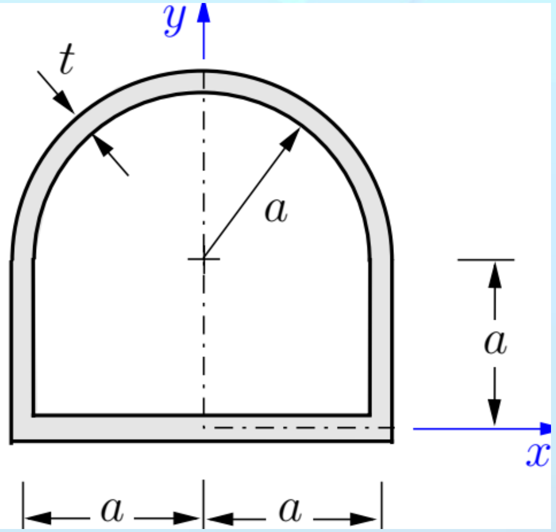
6. A thin sheet with constant thickness and density, consisting of a square and two triangles, is bent to the depicted figure (measurements in cm). Locate the center of gravity



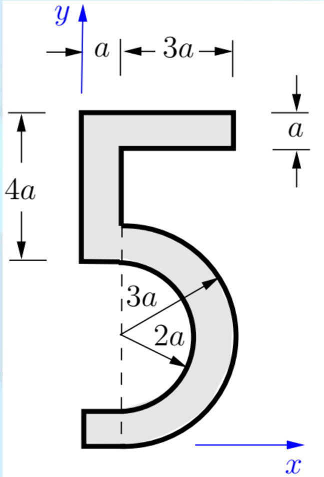
7. Locate the centroids of the thin-walled profiles ($t \ll a$) as shown in Fig.



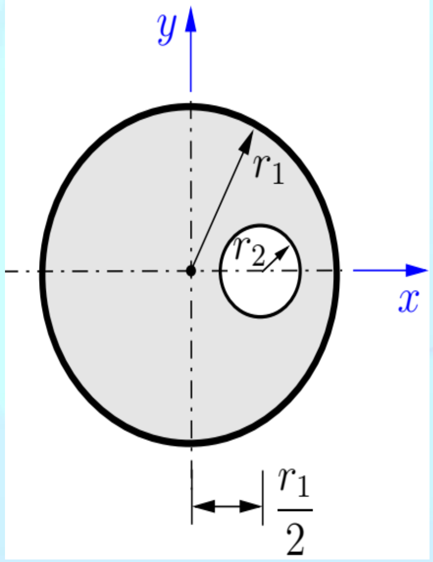
8. Locate the centroids of the thin-walled profiles ($t \ll a$) as shown in Fig.



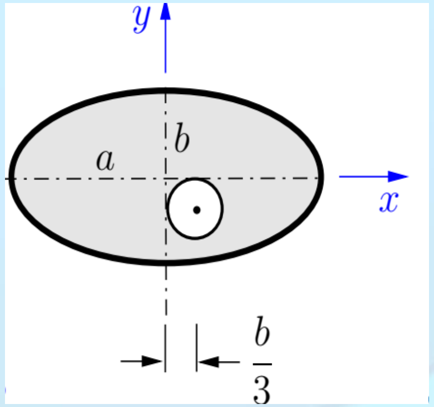
9. Determine the coordinates of the centroid *C* of the number 5.



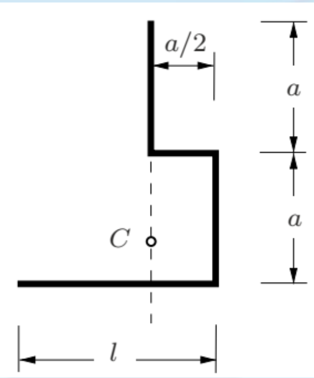
10. A circular area is removed from a circle. Locate the centroids of the remaining areas.

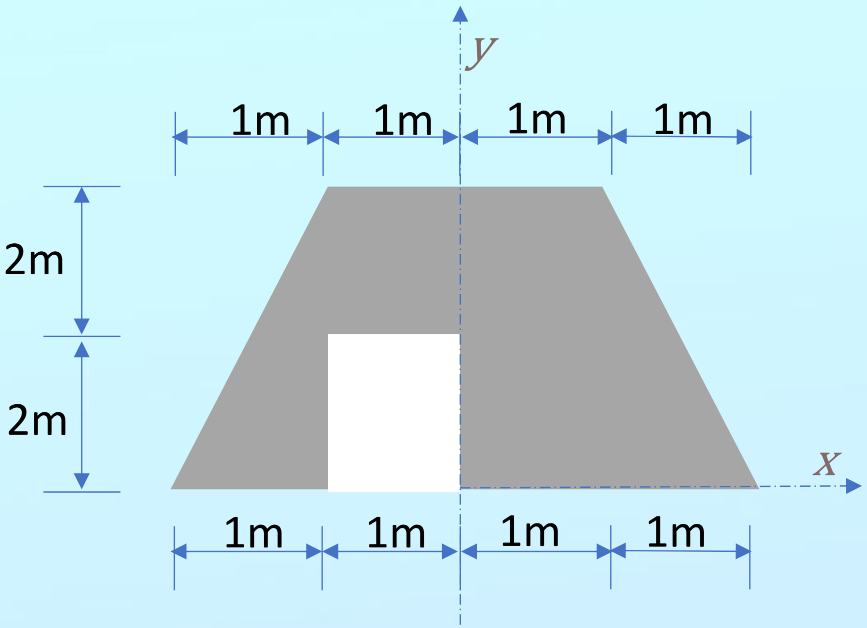


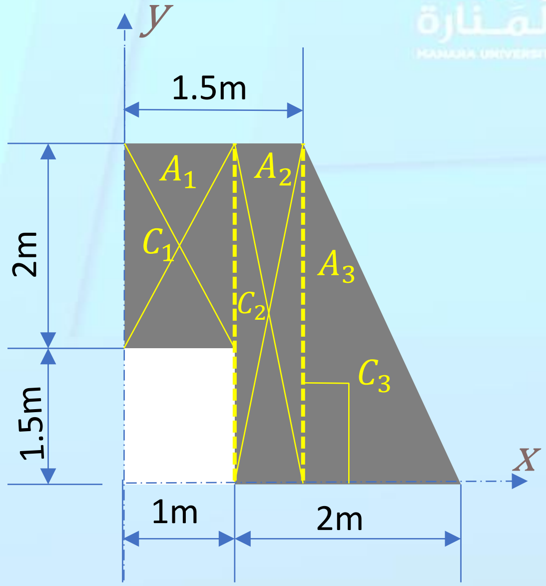
11. A circular area is removed from an ellipse. Locate the centroids of the remaining areas.

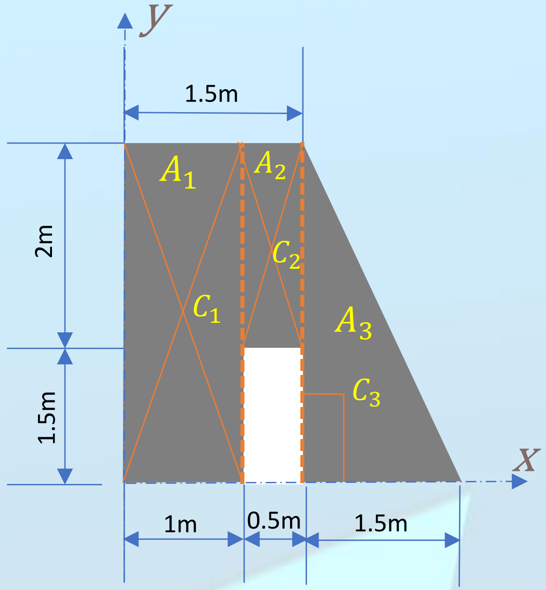


12. The depicted stirrer consists of a homogenous wire that rotates about the sketched vertical axis. Determine the length *l* , such that the center of mass *C* is located on the rotation axis.




y
1m
1m
1m
1m
2m
2m
x
1m
1m
1m
1m


y
1.5m
A_1
A_2
C_1
A_3
C_2
C_3
2m
1.5m
x
1m
2m


y
1.5m
A_1
A_2
C_2
C_1
A_3
C_3
2m
1.5m
x
1m
0.5m
1.5m