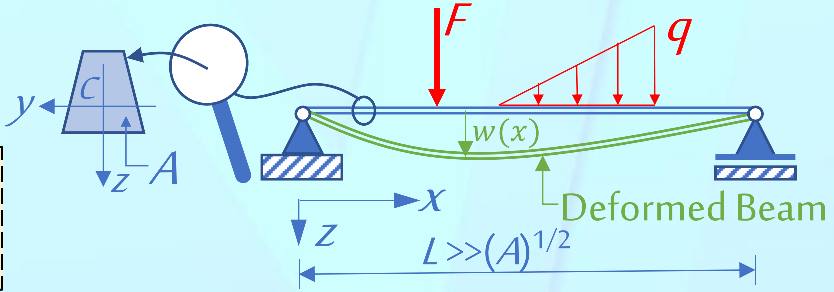
# Deflection Curve

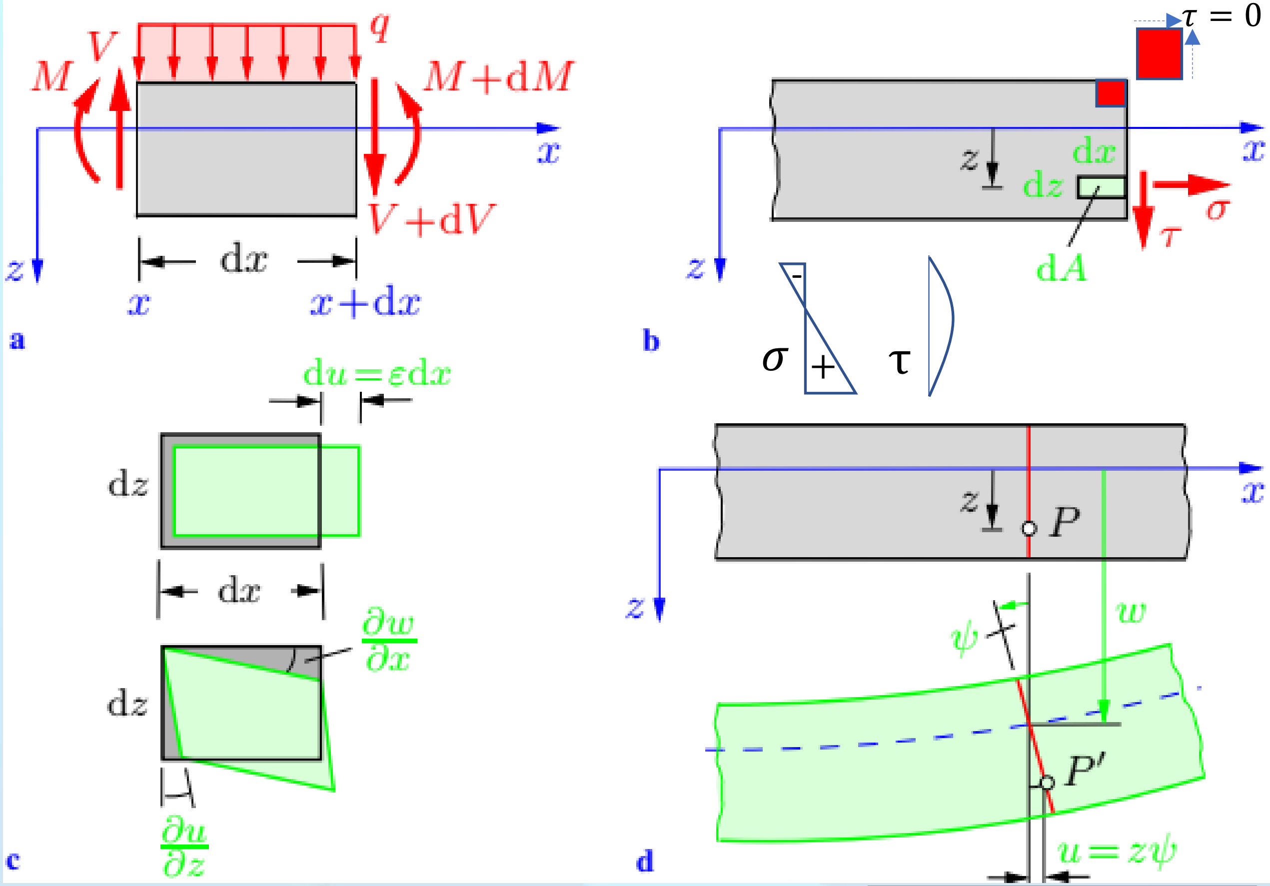
## 1 Differential Equation of the Deflection Curve




Two Eq. Eqs. $\dfrac{dV}{dx} = -q(x)$ $\quad \dfrac{dM}{dx} = V(x)$ $\quad \Rightarrow \dfrac{d^2M}{dx^2} = -q(x)$

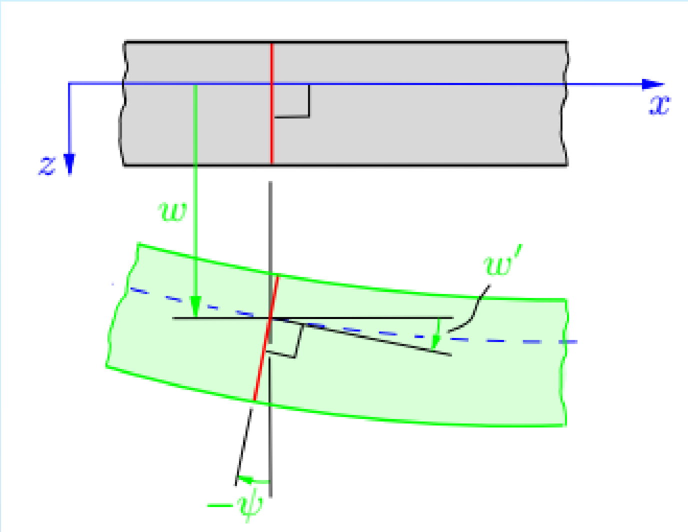
$$\sigma = E\varepsilon = E\frac{\partial u}{\partial x} = E\psi' z \qquad M = EI_y\psi' \qquad \sigma = \frac{M}{I_y}z$$

$$\tau = G\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) = G(\psi(x) + w') \qquad \Rightarrow \tau = \tau(x)??$$




**Euler-Bernoulli assumption**

المقطع المستوي الناظم على المحور الطولي، يبقى مستويا وعموديا على المحور الطولي المنحني



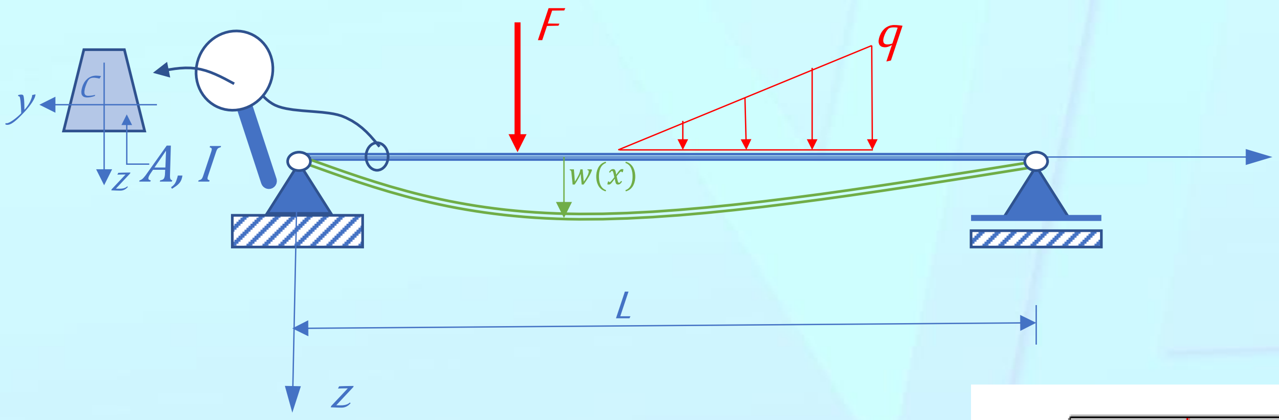

$$\gamma = \psi(x) + w' = 0$$

$$\psi(x) = -w'$$

$$M = EI_y\psi'$$
$$M = -EI_y w''$$

$$\Rightarrow \frac{d^2w}{dx^2} = -\frac{M}{EI_y} \qquad \Rightarrow \frac{d^4w}{dx^4} = -\frac{d^2}{dx^2}\left(\frac{M}{EI_y}\right) \qquad EI_y = Const \qquad \Rightarrow \frac{d^4w}{dx^4} = \frac{q(x)}{EI_y}$$

# Euler-Bernoulli bending theory

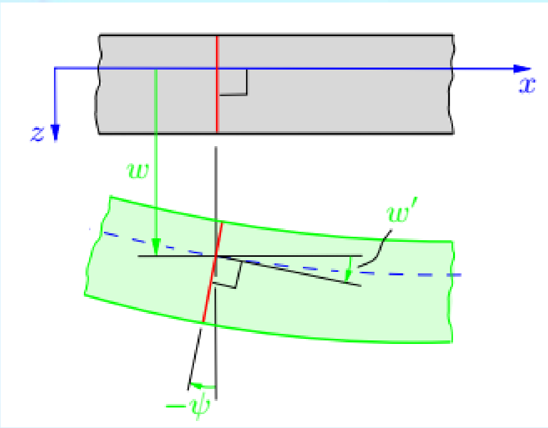



Knowns Functions of $x$:

$q(x)$,

$EI_y(x)$ variable section Or

$EI_y = Const.$ uniform section




Unknowns Functions of $x$, to be calculated:

$V(x)$: Shear force (قوة القص)

$M(x)$: Bending moment (عزم الانعطاف)

$\psi(x)$: Section rotaion or deflection slope (دوران المقطع أو ميل التدلي)

$w(x)$: deflection or Elastic line (التدلي أو الخط المرن)

Using four first order ordinary differential equations written with the prime convention: $d(\,)/dx = (\,)'$

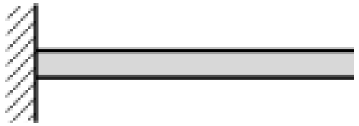
| $V' = -q(x)$ | $M' = V(x)$ | $\psi' = \dfrac{M(x)}{EI_y(x)}$ | $w' = -\psi(x)$ |
|---|---|---|---|
| Two Eq. Eqs. | | Material behaviour Eq. | Kinematic Eq. |

The Solution needs four boundary conditions (Integration constants) determined by the support types

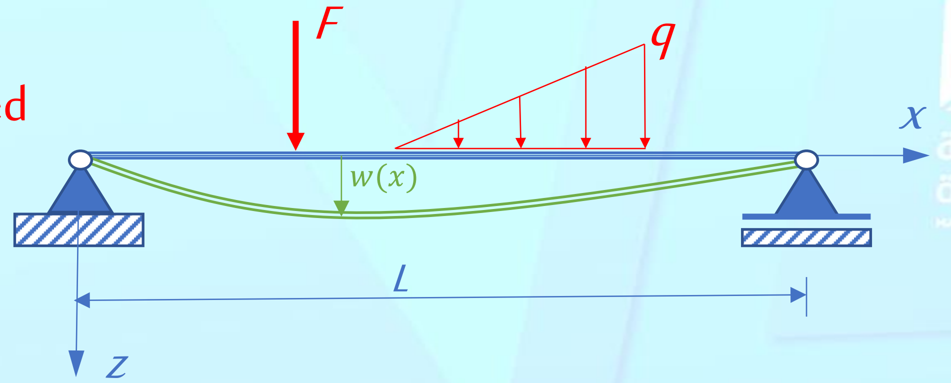
يتطلب الحل معرفة أربعة شروط طرفية (ثوابت تكامل) تحدد من أنماط المساند

## Table Boundary conditions

## جدول شروط الاستناد الطرفية

| Support | $w$ | $w'$ | $M$ | $V$ |
|---|---|---|---|---|
| pin  | $0$ | $\neq 0$ | $0$ | $\neq 0$ |
| parallel motion  | $\neq 0$ | $0$ | $\neq 0$ | $0$ |
| fixed end  | $0$ | $0$ | $\neq 0$ | $\neq 0$ |
| free end  | $\neq 0$ | $\neq 0$ | $0$ | $0$ |

- The four first order ordinary differential equations can be combined into two second order ordinary differential equations as:




$$V' = -q(x) \qquad M' = V(x) \qquad \Rightarrow M'' = -q(x)$$

$$\psi' = \frac{M(x)}{EI_y(x)} \qquad w' = -\psi(x) \qquad \Rightarrow w'' = -\frac{M(x)}{EI_y(x)}$$

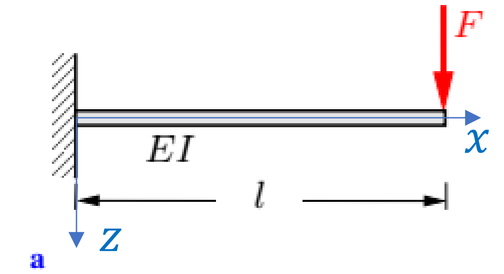
$$\Rightarrow [EI_y(x)w'']'' = q(x)$$

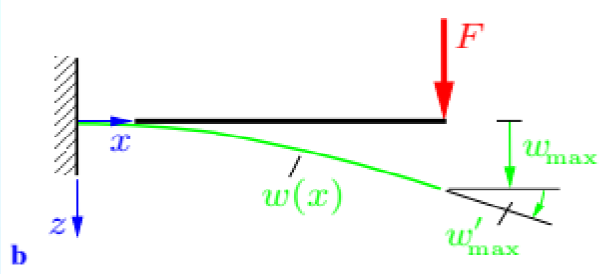
- The two second order ordinary differential equations can be combined into one fourth order ordinary differential equations as:
- This last equation can be simplified when the section is uniform: $EI_y = const.$ as $\Rightarrow EI_y w^{IV} = q(x)$
- In the three forms of the equations, the solution needs the four boundary conditions as it will be shown in the examples.
- In the last form (the fourth order equation is used to determine the deflection $w$, the others unknowns are determined by:

$\psi(x) = -w'$ $\qquad$ $M(x) = -EI_y(x)w''$ $\qquad$ $V(x) = -[EI_y(x)w'']'$ $\qquad$ If $EI_y = Const$ $\qquad$ $V(x) = -EI_y w'''$

## 4.4.2 Beams with one Region of Integration

It will be now shown with the aid of several examples how the differential equations can be used to obtain the deflection curve. In this section we restrict ourselves to beams where the integration can be performed in *one* region, i.e., we assume that each of the quantities $q(x), V(x), M(x), w(x)$ and $w(x)$ is given by *one* function for the entire length of the beam.




Ex.1 A cantilever beam (flexural rigidity $EI$ ) subjected to a concentrated force $F$(Fig.a). Since the system is statically determinate, the bending moment can be calculated from the equilibrium conditions.

With the coordinate system shown in Fig. a, $M = -F(l - x)$. Introducing into $M(x) = -EIw''$ to get

$$EIw'' = F(l - x).$$ integrating twice yields

$$EIw' = F\left(lx - \frac{x^2}{2}\right) + C_1$$

$$EIw = F\left(\frac{lx^2}{2} - \frac{x^3}{6}\right) + C_1 x + C_2$$

The geometrical boundary conditions: $w(0) = 0, w'(0) = 0$

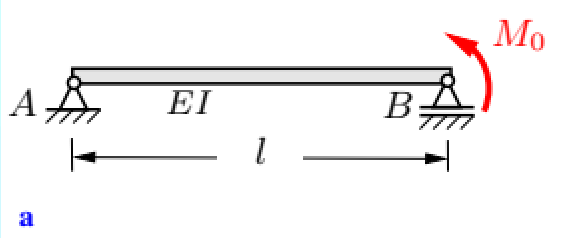
lead to the constants of integration: $C_1 = 0, C_2 = 0$ .

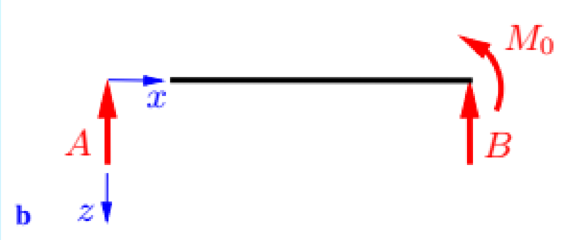
Hence, the slope and the deflection are obtained as

$$w' = \frac{Fl^2}{2EI}\left(\frac{2x}{l} - \frac{x^2}{l^2}\right), \quad w = \frac{Fl^3}{6EI}\left(3\frac{x^2}{l^2} - \frac{x^3}{l^3}\right)$$

maximum slope & maximum deflection (at $x = l$, Fig.b) are $w'_{max} = \dfrac{Fl^2}{2EI}$ $\quad w_{max} = \dfrac{Fl^3}{3EI}$

Ex.2 A simply supported beam (bending stiffness $EI$) $is$ loaded by a moment $M_0$ (Fig. a). Determine the location and magnitude of the maximum deflection.



Ex.3 Consider three beams (bending stiffness $EI$) subjected to a constant line load $q_0$. The supports in the three cases are different; the systems in the (Figs. a & b) are statically determinate, the system in (Fig. c) is statically indeterminate.

Since in (c) the bending moment can not be calculated from Eqm. conditions, the 4th order Diff. Eq. $EI_y w^{IV} = q(x)$ will be used in all three cases. A coordinate system is introduced, integration is done 4 times starting from :

$$EI_y w^{IV} = q_0$$

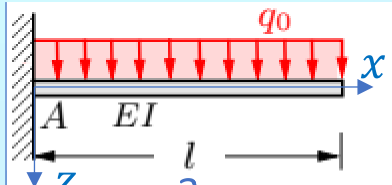
$$EIw''' = -V = q_0 x + C_1,$$

$$EIw'' = -M = \frac{1}{2} q_0 x^2 + C_1 x + C_2,$$

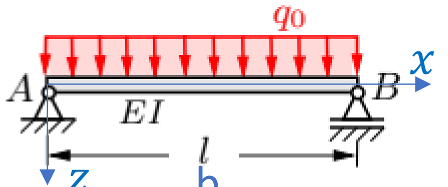
$$EIw' = \frac{1}{6} q_0 x^3 + \frac{1}{2} C_1 x^2 + C_2 x + C_3,$$

$$EIw = \frac{1}{24} q_0 x^4 + \frac{1}{6} C_1 x^3 + \frac{1}{2} C_2 x^2 + C_3 x + C_4,$$

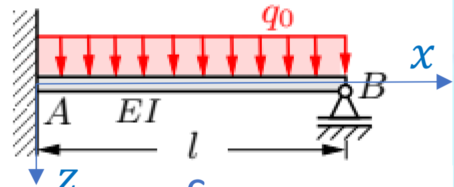
These equations are independent of the supports & therefore are valid for all cases. Different boundary conditions lead to different constants of integration:



a

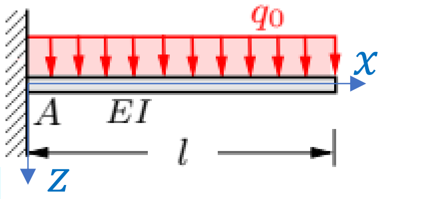
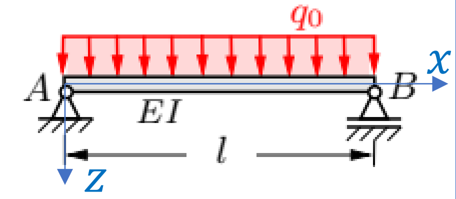
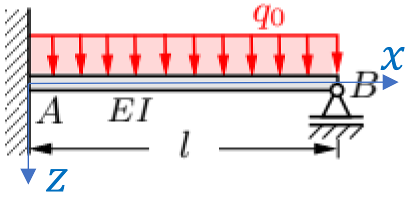
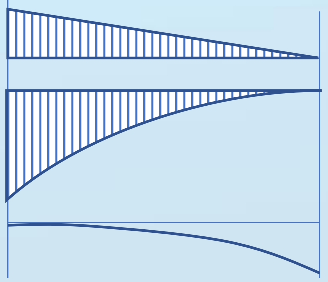
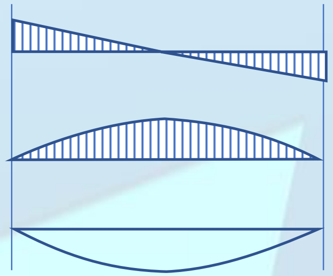
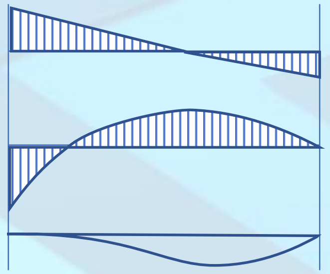


b



c

| | a | b | c |
|---|---|---|---|
| Boundary conditions | $w'(0) = 0 \rightarrow C_3 = 0$<br>$w(0) = 0 \rightarrow C_4 = 0$<br>$V(l) = 0 \rightarrow C_1 = -q_0 l$<br>$M(l) = 0 \rightarrow C_2 = \frac{1}{2} q_0 l^2$ | $M(0) = 0 \rightarrow C_2 = 0$<br>$w(0) = 0 \rightarrow C_4 = 0$<br>$M(l) = 0 \rightarrow C_1 = -\frac{1}{2} q_0 l$<br>$w(l) = 0 \rightarrow C_3 = \frac{1}{24} q_0 l^3$ | $w'(0) = 0 \rightarrow C_3 = 0$<br>$w(0) = 0 \rightarrow C_4 = 0$<br>$M(l) = 0 \rightarrow$<br>$\frac{1}{2} q_0 l^2 + C_1 l + C_2 = 0$<br>$w(l) = 0 \rightarrow$<br>$\frac{1}{24} q_0 l^4 + \frac{1}{6} C_1 l^3 + \frac{1}{2} C_2 l^2 = 0$<br>$C_1 = -\frac{5}{8} q_0 l$ & $C_2 = \frac{1}{8} q_0 l^2$<br>Indeterminate! no more |
| The deflection function is given by: $w(x) = \frac{q_0 l^4}{24EI} \times$ | $\left[\left(\frac{x}{l}\right)^4 - 4\left(\frac{x}{l}\right)^3 + 6\left(\frac{x}{l}\right)^2\right]$ | $\left[\left(\frac{x}{l}\right)^4 - 2\left(\frac{x}{l}\right)^3 + \left(\frac{x}{l}\right)\right]$ | $\left[\left(\frac{x}{l}\right)^4 - \frac{5}{2}\left(\frac{x}{l}\right)^3 + \frac{3}{2}\left(\frac{x}{l}\right)^2\right]$ |
| Maximum deflection is given by: $w_{max} =$ | $\frac{q_0 l^4}{8EI} = 0.125 \frac{q_0 l^4}{EI}$ | $\frac{5 q_0 l^4}{348EI} = 0.0130 \frac{q_0 l^4}{EI}$ | $0.00542 \frac{q_0 l^4}{EI}$ at $x = 0.578l$ |

| | | | |
|---|---|---|---|
| |  |  |  |
| The deflection function is : $w(x) = \frac{q_0 l^4}{24EI} \times$ | $\left[\left(\frac{x}{l}\right)^4 - 4\left(\frac{x}{l}\right)^3 + 6\left(\frac{x}{l}\right)^2\right]$ | $\left[\left(\frac{x}{l}\right)^4 - 2\left(\frac{x}{l}\right)^3 + \left(\frac{x}{l}\right)\right]$ | $\left[\left(\frac{x}{l}\right)^4 - \frac{5}{2}\left(\frac{x}{l}\right)^3 + \frac{3}{2}\left(\frac{x}{l}\right)^2\right]$ |
| Maximum deflection is : $w_{max} =$ | $\frac{q_0 l^4}{8EI} = 0.125\frac{q_0 l^4}{EI}$ | $\frac{5q_0 l^4}{348EI} = 0.0130\frac{q_0 l^4}{EI}$ | $0.00542\frac{q_0 l^4}{EI}$ at $x = 0.578l$ |
| $w'(x) = \frac{q_0 l^3}{6EI} \times$ | $\left[\left(\frac{x}{l}\right)^3 - 3\left(\frac{x}{l}\right)^2 + 3\left(\frac{x}{l}\right)\right]$ | $\left[\left(\frac{x}{l}\right)^3 - \frac{3}{2}\left(\frac{x}{l}\right)^2 + \frac{1}{4}\right]$ | $\left[\left(\frac{x}{l}\right)^3 - \frac{15}{8}\left(\frac{x}{l}\right)^2 + \frac{3}{4}\left(\frac{x}{l}\right)\right]$ |
| $M = -EIw'' = -\frac{q_0 l^2}{2} \times$ | $\left[\left(\frac{x}{l}\right)^2 - 2\left(\frac{x}{l}\right) + 1\right]$ | $\left[\left(\frac{x}{l}\right)^2 - \left(\frac{x}{l}\right)\right]$ | $\left[\left(\frac{x}{l}\right)^2 - \frac{5}{4}\left(\frac{x}{l}\right) + \frac{1}{4}\right]$ |
| $V = -EIw''' = -q_0 l \times$ | $\left[\left(\frac{x}{l}\right) - 1\right]$ | $\left[\left(\frac{x}{l}\right) - \frac{1}{2}\right]$ | $\left[\left(\frac{x}{l}\right) - \frac{5}{8}\right]$ |
| Support reactions are: | $A_z = V(0) = q_0 l\ (\uparrow)$<br>$A_M = -M(0) = \frac{q_0 l^2}{2}\ (\circlearrowleft)$ | $A_z = B_z = \frac{1}{2} q_0 l\ (\uparrow)$ | $A_z = V(0) = \frac{5}{8} q_0 l\ (\uparrow), B_z = -V(l) = \frac{3}{8} q_0 l\ (\uparrow)$<br>$A_M = -M(0) = \frac{q_0 l^2}{8}\ (\circlearrowleft)$ |
| |  |  |  |

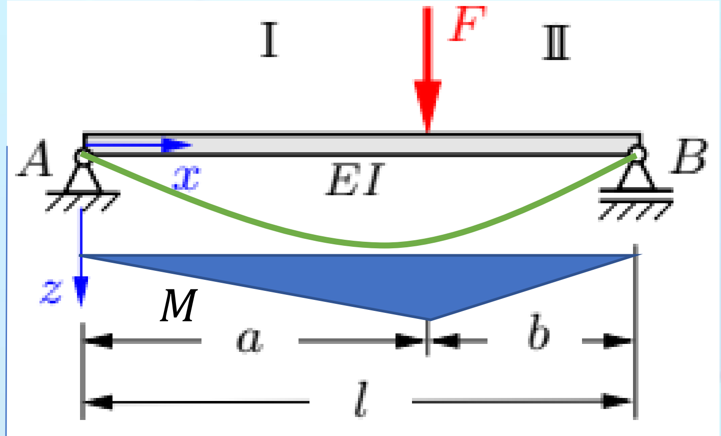
### 4.4.3 Beams with several Regions of Integration

Frequently, one or several of the quantities $q, V, M, w', w$ or the flexural rigidity $EI$ are given through different functions of $x$ in different portions of the beam. In this case the beam must be divided into several regions and the integration has to be performed separately in each of these regions.

The constants of integration can be calculated from both, boundary conditions and *matching conditions*, also called *continuity conditions.* The treatment of such problems will be illustrated by means of the following example.

**Ex. 4** A simply supported beam is subjected to a concentrated force $F$ at $x = a$ (Fig.).

Determine the deflection $w$ at the location $x = a$.




$$M(x) = \begin{cases} F\frac{b}{l}x: & 0 \le x \le a \\ F\frac{a}{l}(l-x): & a \le x \le l \end{cases}$$

In Region I: $0 \le x \le a$

$$EIw'' = -M = -F\frac{b}{l}x$$

$$EIw' = -F\frac{b}{l}\frac{x^2}{2} + C_1$$

$$EIw = -F\frac{b}{l}\frac{x^3}{6} + C_1 x + C_2$$

$$w_I(0) = 0 \Rightarrow C_2 = 0$$

In Region II: $a \le x \le l$

$$EIw'' = -M = -F\frac{a}{l}(l-x)$$

$$EIw' = F\frac{a}{l}\frac{(l-x)^2}{2} + C_3$$

$$EIw = -F\frac{a}{l}\frac{(l-x)^3}{6} - C_3(l-x) + C_4$$

$$w_{II}(0) = 0 \Rightarrow C_4 = 0$$

$$w'_I(a) = w'_{II}(a)$$

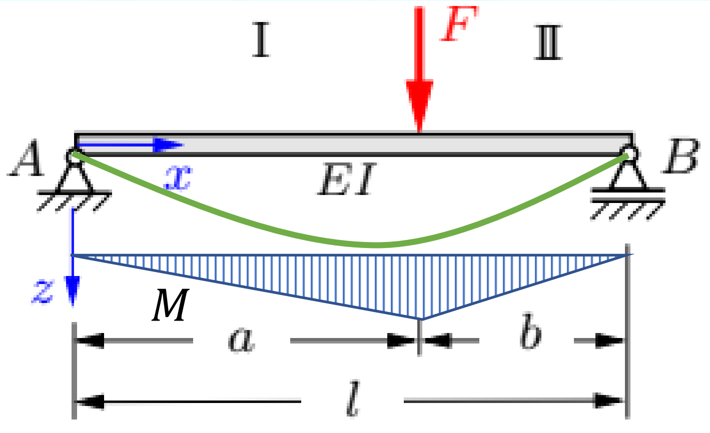
$$w_I(a) = w_{II}(a)$$

$$w'_I(a) = w'_{II}(a) \quad \Rightarrow -F\frac{b}{l}\frac{a^2}{2} + C_1 = F\frac{a}{l}\frac{b^2}{2} + C_3 \quad \Rightarrow C_1 - C_3 = F\frac{ab}{2}$$

$$w_I(a) = w_{II}(a) \quad \Rightarrow -F\frac{b}{l}\frac{a^3}{6} + C_1 a = -F\frac{a}{l}\frac{b^3}{6} - C_3 \mathrm{b} \Rightarrow C_1 a + C_3 \mathrm{b} = F\frac{ab}{2}\frac{(a-b)}{3}$$

$$\Rightarrow \begin{cases} C_1 = \frac{Fab(a+2b)}{6l} \\ C_3 = -\frac{Fab(2a+b)}{6l} \end{cases}$$

**Ex. 4** A simply supported beam is subjected to a concentrated force $F$ at $x = a$ (Fig.). Determine the deflection $w$ at $x = a$.




$$M(x) = \begin{cases} F\frac{b}{l}x: & \text{In Region I} \\ F\frac{a}{l}(l-x) & \text{In Region II} \end{cases}$$

Boundary Conditions: $w(0) = 0, w(l) = 0$

Continuity Conditions of Elastic line :

$w'(a_l) = w'(a_r)$ & $w(a_l) = w(a_r)$

In Region I: $0 \le x \le a$ | $EIw'' = -M = -F\frac{b}{l}x$ | $EIw' = -F\frac{b}{l}\frac{x^2}{2} + C_1$ | $EIw = -F\frac{b}{l}\frac{x^3}{6} + C_1x + C_2$

In Region II: $a \le x \le l$ | $EIw'' = -M = -F\frac{a}{l}(l-x)$ | $EIw' = F\frac{a}{l}\frac{(l-x)^2}{2} + C_3$ | $EIw = -F\frac{a}{l}\frac{(l-x)^3}{6} - C_3(l-x) + C_4$

Boundary Conditions give: $w(0) = 0 \Rightarrow C_2 = 0$ & $w(l) = 0 \Rightarrow C_4 = 0$

Continuity Conditions give:

$$\begin{cases} w'(a_l) = w'(a_r) \Rightarrow -F\frac{b}{l}\frac{a^2}{2} + C_1 = F\frac{a}{l}\frac{b^2}{2} + C_3 \Rightarrow C_1 - C_3 = \frac{Fab}{2} \\ w(a_l) = w(a_r) \Rightarrow -F\frac{b}{l}\frac{a^3}{6} + C_1a = -F\frac{a}{l}\frac{b^3}{6} - C_3b \Rightarrow aC_1 + bC_3 = \frac{Fab}{2}\left(\frac{a-b}{3}\right) \end{cases}$$

$$C_1 = \frac{Fab}{6l}(a+2b) \qquad C_3 = -\frac{Fab}{6l}(2a+b)$$

In Region I: $w = \frac{Fbl^2}{6EI}\left\{\left[1-\left(\frac{b}{l}\right)^2\right]\left(\frac{x}{l}\right) - \left(\frac{x}{l}\right)^3\right\}$ & $w' = \frac{Fbl}{6EI}\left\{\left[1-\left(\frac{b}{l}\right)^2\right] - 3\left(\frac{x}{l}\right)^2\right\}$

In Region II: $w = \frac{Fal^2}{6EI}\left\{\left[1-\left(\frac{a}{l}\right)^2\right]\left(\frac{l-x}{l}\right) - \left(\frac{l-x}{l}\right)^3\right\}$ & $w' = \frac{Fal}{6EI}\left\{\left[1-\left(\frac{a}{l}\right)^2\right] - 3\left(\frac{l-x}{l}\right)^2\right\}$

$$w(a) = \frac{Fa^2b^2}{3EIl}$$

$$w'(a) = \frac{Fab(b-a)}{3EIl}$$

a) Determine the deflection function of the cantilever beam shown in the figure

b) Find the deflection at the free end

c) Find the slope at the free end