CECC122: Linear Algebra and Matrix Theory

Exercises 3: Matrices: Part B

Ramez Koudsieh, Ph.D.

Faculty of Engineering
Department of Informatics
Manara University

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Find a sequence of elementary matrices that can be used to write the matrix in row-echelon form

1 $$A=\begin{bmatrix} 0 & 1 & 7 \\ 5 & 10 & -5 \end{bmatrix}$$

$$A_1 = r_{12}(A) = \begin{bmatrix} 5 & 10 & -5 \\ 0 & 1 & 7 \end{bmatrix} \Rightarrow E_1 = r_{12}(I_2) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$A_2 = r_1^{(1/5)}(A_1) = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 7 \end{bmatrix} \Rightarrow E_2 = r_1^{(1/5)}(I_2) = \begin{bmatrix} 0 & 1/5 \\ 1 & 0 \end{bmatrix}$$

$$\text{So, } B = E_2 E_1 A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 7 \end{bmatrix}$$

2 $$A=\begin{bmatrix}1 & -2 & -1 & 0\\ 0 & 4 & 8 & -4\\ -6 & 12 & 8 & 1\end{bmatrix}$$

$$A_1=r_{13}^{(6)}(A)=\begin{bmatrix}1 & -2 & -1 & 0\\ 0 & 4 & 8 & -4\\ 0 & 0 & 2 & 1\end{bmatrix}\Rightarrow E_1=r_{13}^{(6)}(I_3)=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 6 & 0 & 1\end{bmatrix}$$

$$A_2=r_{2}^{(1/4)}(A_1)=\begin{bmatrix}1 & -2 & -1 & 0\\ 0 & 1 & 2 & -1\\ 0 & 0 & 2 & 1\end{bmatrix}\Rightarrow E_2=r_{2}^{(1/4)}(I_3)=\begin{bmatrix}1 & 0 & 0\\ 0 & 1/4 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$A_3=r_{3}^{(1/2)}(A_2)=\begin{bmatrix}1 & -2 & -1 & 0\\ 0 & 1 & 2 & -1\\ 0 & 0 & 1 & 1/2\end{bmatrix}\Rightarrow E_3=r_{3}^{(1/2)}(I_3)=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1/2\end{bmatrix}$$

So, $B = E_3E_2E_1A = \begin{bmatrix} 1 & -2 & -1 & 0 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & 1/2 \end{bmatrix}$

Find the inverse of the matrix using elementary matrices

1 $A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$

Find a sequence of elementary row operations that can be used to rewrite A in reduced row-echelon form

$$A_1 = r_{12}(A) = \begin{bmatrix} 1 & 0 \\ 3 & -2 \end{bmatrix} \Rightarrow E_1 = r_{12}(I_2) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$A_2 = r_{12}^{(-3)}(A_1) = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \Rightarrow E_2 = r_{12}^{(-3)}(I_2) = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$$

$$A_3 = r_2^{(-1/2)}(A_2) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \Rightarrow E_3 = r_{12}^{(-1/2)}(I_2) = \begin{bmatrix} 1 & 0 \\ 0 & -1/2 \end{bmatrix}$$

So, $B = I_2 = E_3 E_2 E_1 A \quad \Rightarrow A^{-1} = E_3 E_2 E_1$

$$A^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -1/2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3/2 & -1/2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1/2 & 3/2 \end{bmatrix}$$

2 $$A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 6 & -1 \\ 0 & 0 & 4 \end{bmatrix}$$

$$A_1 = r_2^{(1/6)}(A) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1/6 \\ 0 & 0 & 4 \end{bmatrix} \quad \Rightarrow E_1 = r_2^{(1/6)}(I_3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/6 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A_2 = r_3^{(1/4)}(A) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1/6 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow E_2 = r_3^{(1/4)}(I_3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/4 \end{bmatrix}$$

$$A_3 = r_{31}^{(1)}(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1/6 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow E_3 = r_{31}^{(1)}(I_3) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A_4 = r_{32}^{(1/6)}(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow E_4 = r_{32}^{(1/6)}(I_3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1/6 \\ 0 & 0 & 1 \end{bmatrix}$$

So, $B = I_3 = E_4 E_3 E_2 E_1 A \quad \Rightarrow A^{-1} = E_4 E_3 E_2 E_1$

$$A^{-1} = E_4 E_3 E_2 E_1$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1/6 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/6 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1/6 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/6 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 1/4 \\ 0 & 1 & 1/24 \\ 0 & 0 & 1/4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/6 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1/4 \\ 0 & 1/6 & 1/24 \\ 0 & 0 & 1/4 \end{bmatrix}$$

Find a sequence of elementary matrices whose product is the given nonsingular matrix

1 $A=\begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$

Find a sequence of elementary row operations that can be used to rewrite A in reduced row-echelon form

$$A_1 = r_{12}(A) = \begin{bmatrix} 1 & 0 \\ 3 & -2 \end{bmatrix} \Rightarrow E_1 = r_{12}(I_2) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$A_2 = r_{12}^{(-3)}(A_1) = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \Rightarrow E_2 = r_{12}^{(-3)}(I_2) = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$$

$$A_3 = r_2^{(-1/2)}(A_2) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow E_3 = r_{12}^{(-1/2)}(I_2) = \begin{bmatrix} 1 & 0 \\ 0 & -1/2 \end{bmatrix}$$

So, $B = I_2 = E_3E_2E_1A \quad \Rightarrow A = E_1^{-1}E_2^{-1}E_3^{-1}$

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$$

2 $$A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 6 & -1 \\ 0 & 0 & 4 \end{bmatrix}$$

$$A_1 = r_2^{(1/6)}(A) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1/6 \\ 0 & 0 & 4 \end{bmatrix} \Rightarrow E_1 = r_2^{(1/6)}(I_3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/6 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A_2 = r_3^{(1/4)}(A_1) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1/6 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow E_2 = r_3^{(1/4)}(I_3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/4 \end{bmatrix}$$

$$A_3 = r_{31}^{(1)}(A_2) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1/6 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow E_3 = r_{31}^{(1)}(I_3) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A_4 = r_{32}^{(1/6)}(A_3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow E_4 = r_{32}^{(1/6)}(I_3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1/6 \\ 0 & 0 & 1 \end{bmatrix}$$

So, $B = I_3 = E_4 E_3 E_2 E_1 A \quad \Rightarrow A = E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1}$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1/6 \\ 0 & 0 & 1 \end{bmatrix}$$

Find an LU-factorization of the matrix

1 $$A=\begin{bmatrix} 3 & 0 & 1 \\ 6 & 1 & 1 \\ -3 & 1 & 0 \end{bmatrix}$$

$$A_1 = r_{12}^{(-2)}(A) = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 1 & -1 \\ -3 & 1 & 0 \end{bmatrix} \Rightarrow E_1 = r_{12}^{(-2)}(I_3) = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A_2 = r_{13}^{(1)}(A_1) = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \Rightarrow E_2 = r_{13}^{(1)}(I_3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$A_3 = r_{23}^{(-1)}(A_2) = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix} \Rightarrow E_3 = r_{23}^{(-1)}(I_3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

So, $U = E_3 E_2 E_1 A \quad \Rightarrow A = E_1^{-1} E_2^{-1} E_3^{-1} U = LU$

$$L = E_1^{-1} E_2^{-1} E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix} = LU$$

Use an LU-factorization of the coefficient matrix to solve the linear system

1

$$\begin{aligned} 2x + y \phantom{{}+z} &= 1 \\ y - z &= 2 \\ -2x + y + z &= -2 \end{aligned}$$

$$A=\begin{bmatrix}2&1&0\\0&1&-1\\-2&1&1\end{bmatrix}$$

$$A_1=r_{13}^{(1)}(A)=\begin{bmatrix}2&1&0\\0&1&-1\\0&2&1\end{bmatrix}\quad\Rightarrow E_1=r_{13}^{(1)}(I_3)=\begin{bmatrix}1&0&0\\0&1&0\\1&0&1\end{bmatrix}$$

$$A_2=r_{23}^{(-2)}(A_1)=\begin{bmatrix}2&0&1\\0&1&-1\\0&0&3\end{bmatrix}\quad\Rightarrow E_2=r_{23}^{(-2)}(I_3)=\begin{bmatrix}1&0&0\\0&1&0\\0&-2&1\end{bmatrix}$$

So, $U=E_2E_1A \quad\Rightarrow A=E_1^{-1}E_2^{-1}U=LU$

$$L=E_1^{-1}E_2^{-1}=\begin{bmatrix}1&0&0\\0&1&0\\-1&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&2&1\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\-1&2&1\end{bmatrix}$$

$$A=\begin{bmatrix}1&0&0\\0&1&0\\-1&2&1\end{bmatrix}\begin{bmatrix}2&0&1\\0&1&-1\\0&0&3\end{bmatrix}=LU$$

$$L\boldsymbol{y}=\boldsymbol{b}:$$

$$\begin{bmatrix}1&0&0\\0&1&0\\-1&2&1\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}=\begin{bmatrix}1\\2\\-2\end{bmatrix}\quad\Rightarrow y_1=1,\ y_2=2,\ y_3=-5$$

$$U\boldsymbol{x}=\boldsymbol{y}:$$

$$\begin{bmatrix}2&0&1\\0&1&-1\\0&0&3\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}1\\2\\-5\end{bmatrix}\quad\Rightarrow x_1=\frac{1}{3},\ x_2=\frac{1}{3},\ x_3=-\frac{5}{3}$$

Find a sequence of elementary matrices that can be used to write the matrix in row-echelon form

1. $A = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 5 & -1 \\ 3 & -2 & -4 \end{bmatrix}$

2. $A = \begin{bmatrix} 0 & 3 & -3 & 6 \\ 1 & -1 & 2 & -2 \\ 0 & 0 & 2 & 2 \end{bmatrix}$

Find the inverse of the matrix using elementary matrices

1. $A = \begin{bmatrix} 3 & -1 \\ 1 & -1 \end{bmatrix}$

2. $A = \begin{bmatrix} 2 & 3 & 1 \\ 2 & -3 & -3 \\ 4 & 0 & 3 \end{bmatrix}$

Find a sequence of elementary matrices whose product is the given nonsingular matrix

1 $A = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$

2 $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 4 \end{bmatrix}$

Find an LU-factorization of the matrix

1 $A = \begin{bmatrix} 3 & 0 & 1 \\ 6 & 1 & 1 \\ -3 & 1 & 0 \end{bmatrix}$

Use an LU-factorization of the coefficient matrix to solve the linear system

1

$$\begin{aligned} 2x + y \quad &= 1 \\ y - z &= 2 \\ -2x + y + z &= -2 \end{aligned}$$