

CECC122: Linear Algebra and Matrix Theory

Exercises 9: Linear Transformations: Part A

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Find (a) the image of v and (b) the preimage of w

① $T(v_1, v_2) = (v_1 + v_2, v_1 - v_2)$, $v = (3, -4)$, $w = (3, 19)$

(a) The image of v is $T(3, -4) = (3 + (-4), 3 - (-4)) = (-1, 7)$

(b) If $T(v_1, v_2) = (v_1 + v_2, v_1 - v_2) = (3, 19)$, then

$$v_1 + v_2 = 3$$

$$v_1 - v_2 = 19$$

Which implies that $v_1 = 11$ and $v_2 = -8$. So, the preimage of w is $(11, -8)$

② $T(v_1, v_2) = (2v_2 - v_1, v_1, v_2)$, $v = (0, 6)$, $w = (3, 1, 2)$

(a) The image of v is $T(0, 6) = (2(6) - 0, 0, 6) = (12, 0, 6)$

(b) If $T(v_1, v_2) = (2v_2 - v_1, v_1, v_2) = (3, 1, 2)$, then

$$2v_2 - v_1 = 3$$

$$v_1 = 1$$

$$v_2 = 2$$

Which implies that the preimage of w is $(1, 2)$

③ $T(v_1, v_2, v_3) = (2v_1 + v_2, 2v_2 - 3v_1, v_1 - v_3), v = (-4, 5, 1), w = (4, 1, -1)$

(a) The image of v is

$$T(-4, 5, 1) = (2(-4) + 5, 2(5) - 3(-4), -4 - 1) = (-3, 22, -5)$$

(b) If $T(v_1, v_2, v_3) = (2v_1 + v_2, 2v_2 - 3v_1, v_1 - v_3) = (4, 1, -1)$, then

$$2v_1 + v_2 = 4$$

$$-3v_1 + 2v_2 = 1$$

$$v_1 - v_3 = -1$$

Which implies that $v_1 = 1, v_2 = 2$ and $v_3 = 2$. So, the preimage of w is $(1, 2, 2)$

④ $T(v_1, v_2, v_3) = (4v_2 - v_1, 4v_1 + 5v_2), v = (2, -3, -1), w = (3, 9)$

(a) The image of v is

$$T(2, -3, -1) = (4(-3) - 2, 4(2) + 5(-3)) = (-14, -7)$$

(b) If $T(v_1, v_2, v_3) = (4v_2 - v_1, 4v_1 + 5v_2) = (3, 9)$, then

$$-v_1 + 4v_2 = 3$$

$$4v_1 + 5v_2 = 9$$

Which implies that $v_1 = 1$, $v_2 = 1$ and $v_3 = t$, where t is any real number. So, the preimage of w is $\{(1, 1, t): t \text{ is any real number}\}$

Let $T: R^3 \rightarrow R^3$ be a linear transformation such that $T(1, 0, 0) = (2, 4, -1)$, $T(0, 1, 0) = (1, 3, -2)$, and $T(0, 0, 1) = (0, -2, 2)$. Find $T(2, -4, 1)$

$$(2, -4, 1) = 2(1, 0, 0) - 4(0, 1, 0) + (0, 0, 1)$$

$$\begin{aligned} T(2, -4, 1) &= 2T(1, 0, 0) - 4T(0, 1, 0) + T(0, 0, 1) \\ &= 2(2, 4, -1) - 4(1, 3, -2) + (0, -2, 2) = (0, -6, 8) \end{aligned}$$

Let $T: R^3 \rightarrow R^3$ be a linear transformation such that $T(1, 1, 1) = (2, 0, -1)$, $T(0, -1, 2) = (-3, 2, -1)$, and $T(1, 0, 1) = (1, 1, 0)$. Find $T(2, 1, 0)$

$$(2, 1, 0) = 0(1, 1, 1) - (0, -1, 2) + 2(0, 0, 1)$$

$$\begin{aligned} (2, 1, 0) &= 0 \cdot T(1, 1, 1) - T(0, -1, 2) + 2T(0, 0, 1) \\ &= (0, 0, 0) - (-3, 2, -1) + 2(1, 1, 0) = (5, 0, 1) \end{aligned}$$

Let $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that $T(1, 0) = (1, 0)$ and $T(0, 1) = (0, 0)$. (a) Determine $T(x, y)$ for (x, y) in $\mathbb{R}^2$. (b) Give a geometric description of T

$$(a) \quad T(x, y) = T[x(1, 0) + y(0, 1)] = xT(1, 0) + yT(0, 1) = x(1, 0) + y(0, 0) = (x, 0)$$

(b) T is the projection onto the x -axis

Find the kernel of the linear transformation

① $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2, T(x, y) = (x + 2y, y - x)$

$T(x, y) = (x + 2y, y - x) = (0, 0)$ yields the trivial solution $x = y = 0$. So, $\ker(T) = \{(0, 0)\}$

② $T: R^2 \rightarrow R^2, T(x, y) = (x + y, y - x)$

$T(x, y) = (x + y, y - x) = (0, 0)$ yields solutions of the form $x = y$. So, $\ker(T) = \{(x, x) : x \in R\}$

③ $T: R^3 \rightarrow R^3, T(x, y, z) = (x, 0, z)$

$T(x, y, z) = (x, 0, z)$. The kernel consists of all vectors lying on the y -axis. That is, $\ker(T) = \{(0, y, 0) : y \text{ is a real number}\}$

The linear transformation T is defined by $T(x) = Ax$. Find the (a) kernel of T , (b) nullity(T), and (c) rank(T)

① $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

② $A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$

③ $A = \begin{bmatrix} 1 & 2 \\ -1 & -2 \\ 1 & 1 \end{bmatrix}$

$$T(\mathbf{v}) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

(a) has only the trivial solution $v_1 = v_2 = 0$, the kernel is $\{(0, 0)\}$

(b) $\text{nullity}(T) = \dim(\ker(T)) = 0$

(c) $\text{rank}(T) = \dim(\text{domain of } T) - \dim(\ker(T)) = 2 - 0 = 2$

$$T(\mathbf{v}) = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

(a) has solutions of the form $(-4t, -2t, t)$, where t is any real number

(b) $\text{nullity}(T) = \dim(\ker(T)) = 1$

(c) $\text{rank}(T) = \dim(\text{domain of } T) - \dim(\ker(T)) = 3 - 1 = 2$

$$T(\mathbf{v}) = \begin{bmatrix} 1 & 2 \\ -1 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

(a) has only the trivial solution $v_1 = v_2 = 0$, the kernel is $\{(0, 0)\}$

(b) $\text{nullity}(T) = \dim(\ker(T)) = 0$

(c) $\text{rank}(T) = \dim(\text{domain of } T) - \dim(\ker(T)) = 2 - 0 = 2$

Let $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a L.T. given by $T(\mathbf{x}) = A\mathbf{x}$. Find the nullity and rank of T to determine whether T is one-to-one, onto, or neither

$$(a) A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}, \quad (b) B = \begin{bmatrix} 5 & -3 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad (c) C = \begin{bmatrix} 1 & -2 & 2 \\ 4 & 1 & 8 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \xrightarrow{\text{G.-J. Elimination}} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 5 & -3 \\ 1 & 1 \\ 1 & -1 \end{bmatrix} \xrightarrow{\text{G.-J. Elimination}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 2 \\ 4 & 1 & 8 \end{bmatrix} \xrightarrow{\text{G.-J. Elimination}} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix}$$

	$\dim(\text{domain of } T)$	$\text{rank}(T)$	$\text{nullity}(T)$	one-to-one	onto
(a)	2	1	1	No	No
(b)	2	2	0	Yes	No
(c)	3	2	1	No	Yes

Let $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a reflection in the line $y = -x$. Find the image of each vector (a) $(-1, 2)$, (b) $(2, 3)$, (c) $(a, 0)$

The standard matrix for T is $A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

$$(a) \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \Rightarrow T(-1, 2) = (-2, 1)$$

$$(b) \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -3 \\ -2 \end{bmatrix} \Rightarrow T(2, 3) = (-3, -2)$$

$$(c) \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} a \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -a \end{bmatrix} \Rightarrow T(a, 0) = (0, -a)$$

Let $T(1, 0) = (2, 0)$ and $T(0, 1) = (0, 1)$. (a) Determine $T(x, y)$ for any (x, y) , (b) Give a geometric description of T

(a) $T(x, y) = xT(1, 0) + yT(0, 1) = (2x, y)$

(b) T is a horizontal expansion

Let $T(1, 0) = (1, 1)$ and $T(0, 1) = (0, 1)$. (a) Determine $T(x, y)$ for any (x, y) , (b) Give a geometric description of T

(a) $T(x, y) = xT(1, 0) + yT(0, 1) = (x, x + y)$

(b) T is a vertical shear ($k = 1$)

Find (a) the image of v and (b) the preimage of w

1. $T(v_1, v_2) = (2v_1 + v_2, v_1 - 2v_2)$, $v = (1, -1)$, $w = (3, -1)$
2. $T(v_1, v_2) = (v_1, 2v_2 - v_1, v_2)$, $v = (0, 4)$, $w = (2, 4, 3)$
3. $T(v_1, v_2, v_3) = (v_2 - v_1, v_1 + v_2, 2v_1)$, $v = (2, 3, 0)$, $w = (-11, -1, 10)$
4. $T(v_1, v_2, v_3) = (2v_1 + v_2, v_1 - v_2)$, $v = (2, 1, 4)$, $w = (-1, 2)$

Let $T: R^3 \rightarrow R^3$ be a linear transformation such that $T(1, 0, 0) = (2, 0, -1)$, $T(0, 1, 0) = (-1, 3, 0)$, and $T(0, 0, 1) = (0, -2, 0)$. Find $T(2, -4, 1)$

Let $T: R^3 \rightarrow R^3$ be a linear transformation such that $T(1, 1, 1) = (2, 0, -1)$, $T(1, 1, 0) = (-3, 2, -1)$, and $T(1, 0, 0) = (1, 1, 0)$. Find $T(3, 2, 1)$

Let $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that $T(1, 0) = (0, 0)$ and $T(0, 1) = (0, 1)$. (a) Determine $T(x, y)$ for (x, y) in $\mathbb{R}^2$. (b) Give a geometric description of T

Find the kernel of the linear transformation

1. $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$, $T(x, y) = (x + y, x - y)$
2. $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$, $T(x, y) = (x + y, x - 2y)$
3. $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$, $T(x, y, z) = (x, y, x + y)$

The linear transformation T is defined by $T(x) = Ax$. Find the (a) kernel of T , (b) nullity(T), and (c) rank(T)

① $A = \begin{bmatrix} 1 & 2 \\ -3 & 6 \end{bmatrix}$

② $A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & 1 \end{bmatrix}$

③ $A = \begin{bmatrix} 1 & 1 \\ -1 & 2 \\ 0 & 1 \end{bmatrix}$

Let $T: R^n \rightarrow R^m$ be a L.T. given by $T(x) = Ax$. Find the nullity and rank of T to determine whether T is one-to-one, onto, or neither

① $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

② $A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 0 \\ -1 & -2 & 2 \end{bmatrix}$

③ $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 2 & 4 \\ 0 & 4 & 1 \end{bmatrix}$

Let $T: R^2 \rightarrow R^2$ be a reflection in the line $y = x$. Find the image of each vector (a) $(-1, 3)$, (b) $(1, 3)$, (c) (a, b)

Let $T(1, 0) = (0, 2)$ and $T(0, 1) = (1, 0)$. (a) Determine $T(x, y)$ for any (x, y) , (b) Give a geometric description of T

Let $T(1, 0) = (1, 1)$ and $T(0, 1) = (1, 0)$. (a) Determine $T(x, y)$ for any (x, y) , (b) Give a geometric description of T