



Calculus 1

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Calculus 1

Lecture 9

Integration Techniques

Chapter 5

Integrals

5.4 Integration by substitution

5.5 Trigonometric Integrals

5.6 Partial Fractions

$$\int f(x) dx = F(x) \quad \Leftrightarrow \quad F'(x) = f(x)$$

$$\int f(x) dx = F(x) + C$$

$$1) \int c f(x) dx = c \int f(x) dx \quad ; \quad c = \text{constant}$$

$$2) \int [f_1(x) + f_2(x) - f_3(x)] dx = \int f_1(x) dx + \int f_2(x) dx - \int f_3(x) dx$$

$$3) \left[\int f(x) dx \right]' = f(x)$$

THEOREM 6—The Substitution Rule

If $u = g(x)$ is a differentiable function whose range is an interval I , and f is continuous on I , then

$$\int f(g(x)) \cdot g'(x) dx = \int f(u) du.$$

EXAMPLE

Find $\int \sec^2(5x + 1) \cdot 5 dx$

Solution We substitute $u = 5x + 1$ and $du = 5 dx$. Then,

$$\int \sec^2(5x + 1) \cdot 5 dx = \int \sec^2 u du$$

$$= \tan u + C$$

$$= \tan(5x + 1) + C.$$

Let $u = 5x + 1$, $du = 5 dx$.

$$\frac{d}{du} \tan u = \sec^2 u$$

Substitute $5x + 1$ for u . ■

EXAMPLE

$$\int x^2 \cos x^3 dx$$

$$\int x^2 \cos x^3 dx = \int \cos x^3 \cdot x^2 dx$$

$$= \int \cos u \cdot \frac{1}{3} du$$

Let $u = x^3$, $du = 3x^2 dx$,
 $(1/3) du = x^2 dx$.

$$= \frac{1}{3} \int \cos u du$$

$$= \frac{1}{3} \sin u + C$$

Integrate with respect to u .

$$= \frac{1}{3} \sin x^3 + C$$

Replace u by x^3 .



indefinite integrals and the Substitution method

EXAMPLE

Evaluate $\int x\sqrt{2x+1} \, dx$.

Solution Our previous experience with the integral in Example 2 suggests the substitution $u = 2x + 1$ with $du = 2 \, dx$. Then

$$\sqrt{2x+1} \, dx = \frac{1}{2} \sqrt{u} \, du.$$

However, in this example the integrand contains an extra factor of x that multiplies the term $\sqrt{2x+1}$. To adjust for this, we solve the substitution equation $u = 2x + 1$ for x to obtain $x = (u - 1)/2$, and find that

$$x\sqrt{2x+1} \, dx = \frac{1}{2}(u - 1) \cdot \frac{1}{2} \sqrt{u} \, du.$$

The integration now becomes

$$\int x\sqrt{2x+1} \, dx = \frac{1}{4} \int (u - 1)\sqrt{u} \, du = \frac{1}{4} \int (u - 1)u^{1/2} \, du \quad \text{Substitute.}$$

$$= \frac{1}{4} \int (u^{3/2} - u^{1/2}) \, du \quad \text{Multiply terms.}$$

$$= \frac{1}{4} \left(\frac{2}{5} u^{5/2} - \frac{2}{3} u^{3/2} \right) + C \quad \text{Integrate.}$$

$$= \frac{1}{10} (2x + 1)^{5/2} - \frac{1}{6} (2x + 1)^{3/2} + C. \quad \text{Replace } u \text{ by } 2x + 1. \quad \blacksquare$$

indefinite integrals and the Substitution method

EXAMPLE

Evaluate $\int \frac{2z dz}{\sqrt[3]{z^2 + 1}}$.

Substitute $u = z^2 + 1$.

$$\begin{aligned}\int \frac{2z dz}{\sqrt[3]{z^2 + 1}} &= \int \frac{du}{u^{1/3}} \\ &= \int u^{-1/3} du \\ &= \frac{u^{2/3}}{2/3} + C \\ &= \frac{3}{2} u^{2/3} + C \\ &= \frac{3}{2} (z^2 + 1)^{2/3} + C\end{aligned}$$

Let $u = z^2 + 1$,
 $du = 2z dz$.

In the form $\int u^a du$

Integrate.

Replace u by $z^2 + 1$.

EXAMPLE

Evaluate

$$\int \frac{dx}{(1 + \sqrt{x})^3}.$$

$$\begin{aligned} \int \frac{dx}{(1 + \sqrt{x})^3} &= \int \frac{2(u - 1) du}{u^3} \\ &= \int \left(\frac{2}{u^2} - \frac{2}{u^3} \right) du \end{aligned}$$

$$\begin{aligned} u &= 1 + \sqrt{x}, \quad du = \frac{1}{2\sqrt{x}} dx; \\ dx &= 2\sqrt{x} du = 2(u - 1) du \end{aligned}$$

$$\begin{aligned} &= -\frac{2}{u} + \frac{1}{u^2} + C \\ &= \frac{1 - 2u}{u^2} + C \\ &= \frac{1 - 2(1 + \sqrt{x})}{(1 + \sqrt{x})^2} + C \\ &= C - \frac{1 + 2\sqrt{x}}{(1 + \sqrt{x})^2}. \end{aligned}$$



Trigonometric Integrals

$$\begin{aligned}\cos(A + B) &= \cos A \cos B - \sin A \sin B \\ \sin(A + B) &= \sin A \cos B + \cos A \sin B\end{aligned}$$

EXAMPLE

Evaluate the integral

$$\int (\cos x \sin 2x + \sin x \cos 2x) dx.$$

$$\int (\cos x \sin 2x + \sin x \cos 2x) dx = \int (\sin (x + 2x)) dx$$

$$= \int \sin 3x dx$$

$$= \int \frac{1}{3} \sin u du$$

$$u = 3x, du = 3 dx$$

$$= -\frac{1}{3} \cos 3x + C.$$

Table 8.1, Formula 6



Trigonometric Integrals

$$\int \overbrace{\sin^{2k+1} x}^{\text{Odd}} \cos^n x \, dx = \int \overbrace{(\sin^2 x)^k}^{\text{Convert to cosines}} \overbrace{\cos^n x \sin x}^{\text{Save for } du} \, dx = \int (1 - \cos^2 x)^k \cos^n x \sin x \, dx$$

$$\int \sin^m x \overbrace{\cos^{2k+1} x}^{\text{Odd}} \, dx = \int \overbrace{(\sin^m x)(\cos^2 x)^k}^{\text{Convert to sines}} \overbrace{\cos x}^{\text{Save for } du} \, dx = \int (\sin^m x)(1 - \sin^2 x)^k \cos x \, dx$$

EXAMPLE

Evaluate

$$\int \sin^3 x \cos^2 x \, dx.$$

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Solution This is an example of Case 1.

$$\int \sin^3 x \cos^2 x \, dx = \int \sin^2 x \cos^2 x \sin x \, dx$$

m is odd.

$$= \int (1 - \cos^2 x)(\cos^2 x)(-d(\cos x))$$

$$\sin x \, dx = -d(\cos x)$$

$$= \int (1 - u^2)(u^2)(-du)$$

$$u = \cos x$$

$$= \int (u^4 - u^2) \, du$$

Multiply terms.

$$= \frac{u^5}{5} - \frac{u^3}{3} + C = \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + C$$



EXAMPLE : Evaluate

$$\int \cos^5 x \, dx.$$

$$\int \cos^5 x \, dx = \int \cos^4 x \cos x \, dx = \int (1 - \sin^2 x)^2 d(\sin x) \quad \cos x \, dx = d(\sin x)$$

$$= \int (1 - u^2)^2 du \quad u = \sin x$$

$$= \int (1 - 2u^2 + u^4) du \quad \text{Square } 1 - u^2.$$

$$= u - \frac{2}{3}u^3 + \frac{1}{5}u^5 + C = \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C \quad \blacksquare$$

Trigonometric Integrals

EXAMPLE Evaluate

$$\int \tan^4 x \, dx.$$

Solution

$$\begin{aligned} \int \tan^4 x \, dx &= \int \tan^2 x \cdot \tan^2 x \, dx = \int \tan^2 x \cdot (\sec^2 x - 1) \, dx \\ &= \int \tan^2 x \sec^2 x \, dx - \int \tan^2 x \, dx \\ &= \int \tan^2 x \sec^2 x \, dx - \int (\sec^2 x - 1) \, dx \\ &= \int \tan^2 x \sec^2 x \, dx - \int \sec^2 x \, dx + \int dx \end{aligned}$$

In the first integral, we let

$$u = \tan x, \quad du = \sec^2 x \, dx$$

and have

$$\int u^2 \, du = \frac{1}{3} u^3 + C_1.$$

The remaining integrals are standard forms, so

$$\int \tan^4 x \, dx = \frac{1}{3} \tan^3 x - \tan x + x + C.$$

EXAMPLE

Evaluate

$$\int \tan^4 x \sec^4 x \, dx.$$

Solution

$$\int (\tan^4 x)(\sec^4 x) \, dx = \int (\tan^4 x)(1 + \tan^2 x)(\sec^2 x) \, dx$$

$$\sec^2 x = 1 + \tan^2 x$$

$$= \int (\tan^4 x + \tan^6 x)(\sec^2 x) \, dx$$

$$= \int (\tan^4 x)(\sec^2 x) \, dx + \int (\tan^6 x)(\sec^2 x) \, dx$$

$$= \int u^4 \, du + \int u^6 \, du = \frac{u^5}{5} + \frac{u^7}{7} + C$$

$$u = \tan x, \\ du = \sec^2 x \, dx$$

$$= \frac{\tan^5 x}{5} + \frac{\tan^7 x}{7} + C$$



Products of Sines and Cosines

The integrals

$$\int \sin mx \sin nx \, dx, \quad \int \sin mx \cos nx \, dx, \quad \text{and} \quad \int \cos mx \cos nx \, dx$$

$$\sin mx \sin nx = \frac{1}{2}(\cos[(m - n)x] - \cos[(m + n)x])$$

$$\sin mx \cos nx = \frac{1}{2}(\sin[(m - n)x] + \sin[(m + n)x])$$

$$\cos mx \cos nx = \frac{1}{2}(\cos[(m - n)x] + \cos[(m + n)x])$$

EXAMPLE Evaluate

$$\int \sin 3x \cos 5x \, dx.$$

Solution From Equation (4) with $m = 3$ and $n = 5$, we get

$$\begin{aligned} \int \sin 3x \cos 5x \, dx &= \frac{1}{2} \int [\sin(-2x) + \sin 8x] \, dx \\ &= \frac{1}{2} \int (\sin 8x - \sin 2x) \, dx \\ &= -\frac{\cos 8x}{16} + \frac{\cos 2x}{4} + C. \end{aligned}$$

Integration of Rational Functions by Partial Fractions

$$\int \frac{dx}{1+x} = \ln|1+x| + c$$

$$\int \frac{2x + 3x^2}{1+x^2+x^3} dx = \ln|1+x^2+x^3| + c$$

Find $\int \frac{1}{x(x-1)} dx$

$$\frac{1}{x(x-1)} = \frac{-1}{x} + \frac{1}{x-1}$$

Integration of Rational Functions by Partial Fractions

Solution

$$\frac{1}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1} = \frac{A(x-1) + Bx}{x(x-1)}$$

$$A(x-1) + Bx \equiv 1$$

$$(A+B)x - A \equiv 1 \Rightarrow \left. \begin{array}{l} x^0 : -A = 1 \\ x^1 : A + B = 0 \end{array} \right\} \Rightarrow \begin{array}{l} A = -1 \\ B = 1 \end{array}$$

$$\frac{1}{x(x-1)} = \frac{-1}{x} + \frac{1}{x-1}$$

$$\int \frac{1}{x(x-1)} dx = \int \left(\frac{-1}{x} + \frac{1}{x-1} \right) dx = \ln \left| \frac{x-1}{x} \right| + C$$

Integration of Rational Functions by Partial Fractions

- 1. Divide when improper:** When $N(x)/D(x)$ is an improper fraction (that is, when the degree of the numerator is greater than or equal to the degree of the denominator), divide the denominator into the numerator to obtain

$$\frac{N(x)}{D(x)} = (\text{a polynomial}) + \frac{N_1(x)}{D(x)}$$

Linear factors: For each factor of the form $(px + q)^m$, the partial fraction decomposition must include the following sum of m fractions.

$$\frac{A_1}{(px + q)} + \frac{A_2}{(px + q)^2} + \dots + \frac{A_m}{(px + q)^m}$$

Quadratic factors: For each factor of the form $(ax^2 + bx + c)^n$, the partial fraction decomposition must include the following sum of n fractions.

$$\frac{B_1x + C_1}{ax^2 + bx + c} + \frac{B_2x + C_2}{(ax^2 + bx + c)^2} + \dots + \frac{B_nx + C_n}{(ax^2 + bx + c)^n}$$

Integration of Rational Functions by Partial Fractions

Find A , B , and C in the equation

$$\frac{x - 1}{(x + 1)^3} = \frac{A}{x + 1} + \frac{B}{(x + 1)^2} + \frac{C}{(x + 1)^3}$$

Solution We first clear fractions:

$$x - 1 = A(x + 1)^2 + B(x + 1) + C.$$

Substituting $x = -1$ shows $C = -2$. We then differentiate both sides with respect to x , obtaining

$$1 = 2A(x + 1) + B.$$

Substituting $x = -1$ shows $B = 1$. We differentiate again to get $0 = 2A$, which shows $A = 0$. Hence,

$$\frac{x - 1}{(x + 1)^3} = \frac{1}{(x + 1)^2} - \frac{2}{(x + 1)^3}.$$



Integration of Rational Functions by Partial Fractions

Find $\int \frac{1}{x^2(x+1)} dx$

Solution

$$\frac{1}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1} \Rightarrow \frac{1}{x^2(x+1)} = \frac{Ax(x+1) + B(x+1) + Cx^2}{x^2(x+1)}$$

$$\begin{aligned} Ax(x+1) + B(x+1) + Cx^2 &\equiv 1 \\ (A+C)x^2 + (A+B)x + B &\equiv 1 \Rightarrow \left. \begin{array}{l} x^0 : B = 1 \\ x^1 : A + B = 0 \\ x^2 : A + C = 0 \end{array} \right\} \Rightarrow \begin{array}{l} A = -1 \\ B = 1 \\ C = 1 \end{array} \end{aligned}$$

$$\frac{1}{x^2(x+1)} = \frac{-1}{x} + \frac{1}{x^2} + \frac{1}{x+1}$$

$$\int \frac{1}{x^2(x+1)} dx = \int \left(\frac{-1}{x} + \frac{1}{x^2} + \frac{1}{x+1} \right) dx = \ln \left| \frac{x+1}{x} \right| - \frac{1}{x} + C$$

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Integration of Rational Functions by Partial Fractions

$$\int \frac{x^2 + 4x + 1}{(x - 1)(x + 1)(x + 3)} dx.$$

$$\frac{x^2 + 4x + 1}{(x - 1)(x + 1)(x + 3)} = \frac{A}{x - 1} + \frac{B}{x + 1} + \frac{C}{x + 3}.$$

$$\begin{aligned} x^2 + 4x + 1 &= A(x + 1)(x + 3) + B(x - 1)(x + 3) + C(x - 1)(x + 1) \\ &= A(x^2 + 4x + 3) + B(x^2 + 2x - 3) + C(x^2 - 1) \\ &= (A + B + C)x^2 + (4A + 2B)x + (3A - 3B - C). \end{aligned}$$

$$\text{Coefficient of } x^2: \quad A + B + C = 1$$

$$\text{Coefficient of } x^1: \quad 4A + 2B = 4$$

$$\text{Coefficient of } x^0: \quad 3A - 3B - C = 1$$

Integration of Rational Functions by Partial Fractions

$A = 3/4$, $B = 1/2$, and $C = -1/4$. Hence we have

$$\begin{aligned}\int \frac{x^2 + 4x + 1}{(x - 1)(x + 1)(x + 3)} dx &= \int \left[\frac{3}{4} \frac{1}{x - 1} + \frac{1}{2} \frac{1}{x + 1} - \frac{1}{4} \frac{1}{x + 3} \right] dx \\ &= \frac{3}{4} \ln |x - 1| + \frac{1}{2} \ln |x + 1| - \frac{1}{4} \ln |x + 3| + K,\end{aligned}$$

Integration of Rational Functions by Partial Fractions

$$\frac{6x + 7}{(x + 2)^2} = \frac{A}{x + 2} + \frac{B}{(x + 2)^2}$$

Two terms because $(x + 2)$ is squared

$$\begin{aligned} 6x + 7 &= A(x + 2) + B \\ &= Ax + (2A + B) \end{aligned}$$

Multiply both sides by $(x + 2)^2$.

Equating coefficients of corresponding powers of x gives

$$A = 6 \quad \text{and} \quad 2A + B = 12 + B = 7, \quad \text{or} \quad A = 6 \quad \text{and} \quad B = -5.$$

Therefore,

$$\begin{aligned} \int \frac{6x + 7}{(x + 2)^2} dx &= \int \left(\frac{6}{x + 2} - \frac{5}{(x + 2)^2} \right) dx \\ &= 6 \int \frac{dx}{x + 2} - 5 \int (x + 2)^{-2} dx \\ &= 6 \ln |x + 2| + 5(x + 2)^{-1} + C. \end{aligned}$$

$$\int \frac{6x + 7}{(x + 2)^2} dx.$$

Integration of Rational Functions by Partial Fractions

$$\int \frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3} dx.$$

$$\frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3} = 2x + \frac{5x - 3}{x^2 - 2x - 3}$$

$$\begin{array}{r} 2x \\ x^2 - 2x - 3 \overline{) 2x^3 - 4x^2 - x - 3} \\ \underline{2x^3 - 4x^2 - 6x - 3} \\ 5x - 3 \end{array}$$

$$\begin{aligned} \int \frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3} dx &= \int 2x dx + \int \frac{5x - 3}{x^2 - 2x - 3} dx \\ &= \int 2x dx + \int \frac{2}{x + 1} dx + \int \frac{3}{x - 3} dx \\ &= x^2 + 2 \ln |x + 1| + 3 \ln |x - 3| + C. \end{aligned}$$

Integration of Rational Functions by Partial Fractions

$$\int \frac{-2x + 4}{(x^2 + 1)(x - 1)^2} dx.$$

$$\frac{-2x + 4}{(x^2 + 1)(x - 1)^2} = \frac{Ax + B}{x^2 + 1} + \frac{C}{x - 1} + \frac{D}{(x - 1)^2}.$$

Clearing the equation of fractions gives

$$\begin{aligned} -2x + 4 &= (Ax + B)(x - 1)^2 + C(x - 1)(x^2 + 1) + D(x^2 + 1) \\ &= (A + C)x^3 + (-2A + B - C + D)x^2 \\ &\quad + (A - 2B + C)x + (B - C + D). \end{aligned}$$

Integration of Rational Functions by Partial Fractions

Equating coefficients of like terms gives

$$\text{Coefficients of } x^3: \quad 0 = A + C$$

$$\text{Coefficients of } x^2: \quad 0 = -2A + B - C + D$$

$$\text{Coefficients of } x^1: \quad -2 = A - 2B + C$$

$$\text{Coefficients of } x^0: \quad 4 = B - C + D$$

We solve these equations simultaneously to find the values of A , B , C , and D :

$$-4 = -2A, \quad A = 2$$

Subtract fourth equation from second.

$$C = -A = -2$$

From the first equation

$$B = (A + C + 2)/2 = 1$$

From the third equation and $C = -A$

$$D = 4 - B + C = 1.$$

From the fourth equation

Integration of Rational Functions by Partial Fractions

We substitute these values into Equation (2), obtaining

$$\frac{-2x + 4}{(x^2 + 1)(x - 1)^2} = \frac{2x + 1}{x^2 + 1} - \frac{2}{x - 1} + \frac{1}{(x - 1)^2}.$$

Finally, using the expansion above we can integrate:

$$\begin{aligned}\int \frac{-2x + 4}{(x^2 + 1)(x - 1)^2} dx &= \int \left(\frac{2x + 1}{x^2 + 1} - \frac{2}{x - 1} + \frac{1}{(x - 1)^2} \right) dx \\ &= \int \left(\frac{2x}{x^2 + 1} + \frac{1}{x^2 + 1} - \frac{2}{x - 1} + \frac{1}{(x - 1)^2} \right) dx \\ &= \ln(x^2 + 1) + \tan^{-1} x - 2 \ln |x - 1| - \frac{1}{x - 1} + C. \quad \blacksquare\end{aligned}$$

Thank you for your attention