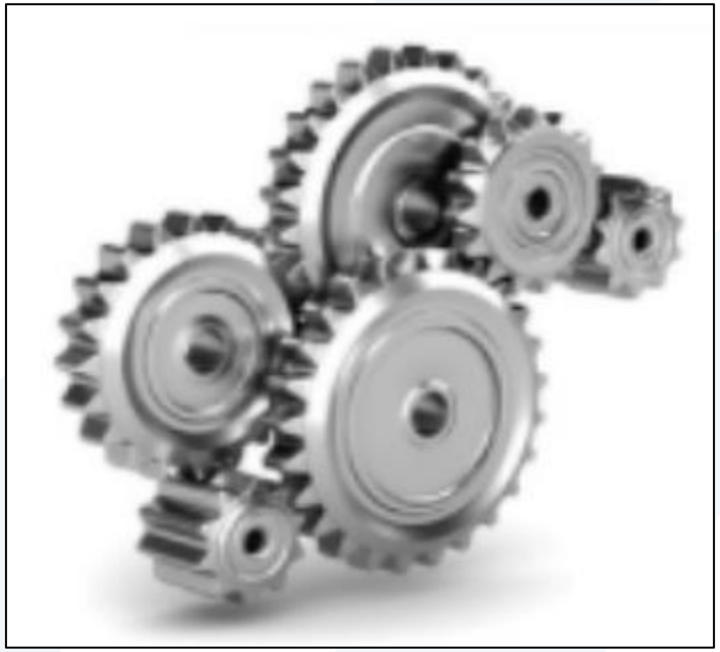
جامعة المنارة

كلية الهندسة

قسم الروبوتيك و الأنظمة الذكية

مقرر نمذجة الميكانيزمات

تقنيات رياضية و برمجية في نمذجة النظم الديناميكية باستخدام نموذج فضاء الحالة و نموذج تابع النقل



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State Space to Transfer Function

Consider the standard state variable description of a control system

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t)$$

Taking Laplace transforms of this equation gives

$$s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s)$$
$$\mathbf{Y}(s) = \mathbf{C}\mathbf{X}(s)$$

Rearranging the expression for $\mathbf{X}(s)$ gives

$$(s\mathbf{I}-\mathbf{A})\mathbf{X}(s) = \mathbf{x}(0) + \mathbf{B}\mathbf{U}(s)$$
$$\mathbf{X}(s) = (s\mathbf{I}-\mathbf{A})^{-1}\mathbf{x}(0) + (s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s)$$

The output equation is given by

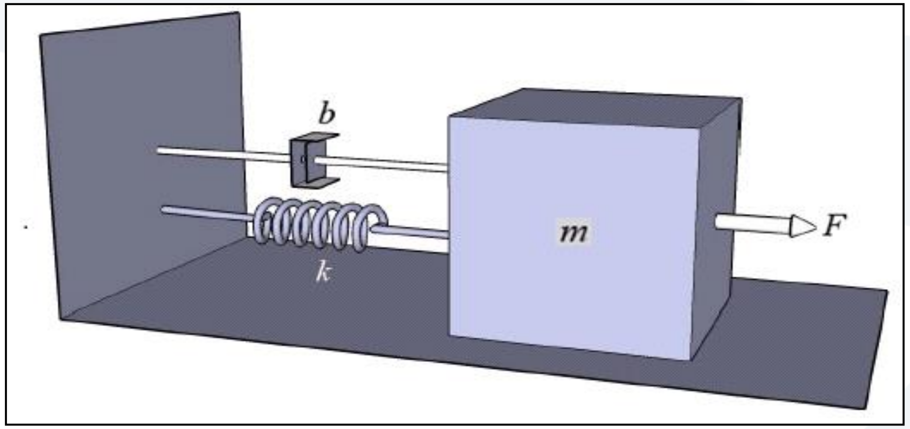
$$\mathbf{Y}(s) = \mathbf{CX}(s) = \mathbf{C}(s\mathbf{I}\text{-}\mathbf{A})^{-1}\mathbf{BU}(s) + \mathbf{C}(s\mathbf{I}\text{-}\mathbf{A})^{-1}\mathbf{x}(0)$$

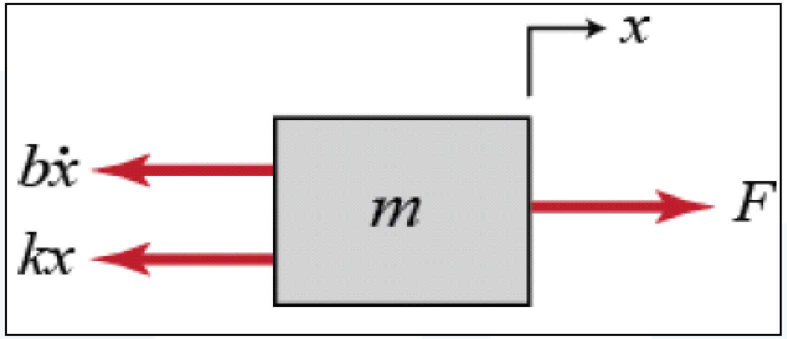
If we set the input conditions to zero, $\mathbf{x}(0) = 0$, we note that the output $\mathbf{Y}(s)$ is related to the input $\mathbf{U}(s)$ as follows

where

$$\mathbf{G}(s) = \mathbf{C}(s\mathbf{I}\text{-}\mathbf{A})^{-1}\mathbf{B}$$

Example





$$\Sigma F_x = F(t) - b\dot{x} - kx = m\ddot{x}$$

m=1;
k=0.1;
b=0.1;
F=1;

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} F(t)$$

$$y = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$$

Example

$$Y(S) = C(SI - A)^{-1}.B.U(s) + C(SI - A)^{-1}.X(0)$$

$$A = \begin{bmatrix} 0 & 1 \\ -0.1 & -0.1 \end{bmatrix}$$

$$B = [0; 1] \quad , \quad C = [1\ 0; 0\ 1]$$

$$SI - A = S\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -0.1 & -0.1 \end{bmatrix}$$

$$SI - A = \begin{bmatrix} S & -1 \\ 0.1 & S + 0.1 \end{bmatrix}$$

$$\det(SI - A) = (S + 0.1)(S) + 0.1$$

$$\det(\mathrm{SI - A}) = S^2 + 0.1S + 0.1$$

$$(SI - A)^{-1} = \frac{1}{S^2 + 0.1S + 0.1}\mathrm{adj}(\mathrm{SI - A})$$

$$(SI - A)^{-1} = \frac{1}{S^2 + 0.1S + 0.1}\begin{bmatrix} S + 0.1 & 1 \\ -0.1 & S \end{bmatrix}$$

$$(SI-A)^{-1} = \begin{bmatrix} \frac{S+0.1}{S^2+0.1S+0.1} & \frac{1}{S^2+0.1S+0.1} \\ \frac{-0.1}{S^2+0.1S+0.1} & \frac{S}{S^2+0.1S+0.1} \end{bmatrix}$$

$$C(SI-A)^{-1}(B) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{S+0.1}{S^2+0.1S+0.1} & \frac{1}{S^2+0.1S+0.1} \\ \frac{-0.1}{S^2+0.1S+0.1} & \frac{S}{S^2+0.1S+0.1} \end{bmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \begin{bmatrix} \frac{S+0.1}{S^2+0.1S+0.1} & \frac{1}{S^2+0.1S+0.1} \\ \frac{-0.1}{S^2+0.1S+0.1} & \frac{S}{S^2+0.1S+0.1} \end{bmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \begin{bmatrix} \frac{1}{S^2+0.1S+0.1} \\ \frac{S}{S^2+0.1S+0.1} \end{bmatrix}$$

Matlab Function ss2tf: State Space to Transfer Function

We note that the function ss2tf (and its counterpart tf2ss) are functions in the control toolbox in MATLAB. The form of the expression for ss2tf is given by

[num,den] = ss2tf(A,B,C,D,iu)

The inputs are the state variable matrices A,B,C and D. If there is no 'D' matrix in the model, then a D matrix must be created with zeros.

The input 'iu' is the input we are interested in, that is input number 1 or 2 , etc. If we need to find the transfer function matrix for all inputs we would have to enter the command several times changing the value of iu.

The output is given in the matrices 'num' and 'den'. The denominator of each transfer function with a particular input will be the same, therefore den is a vector which contains the coefficients of the denominator polynomial.

Using Matlab:

```
A = [0 1;-0.1 -0.1];
B = [0; 1];
C = [1 0;0 1];
D = [0;0];
[num,den]=ss2tf(A,B,C,D,1)
```

```
num =
      0        0    1.0000
      0   1.0000   -0.0000
den =
   1.0000   0.1000   0.1000
```

$$\frac{1}{S^2 + 0.1S + 0.1}$$

$$\frac{S}{S^2 + 0.1S + 0.1}$$

Transfer Function to State Space

The following transfer function describes the dynamics of an actuator.

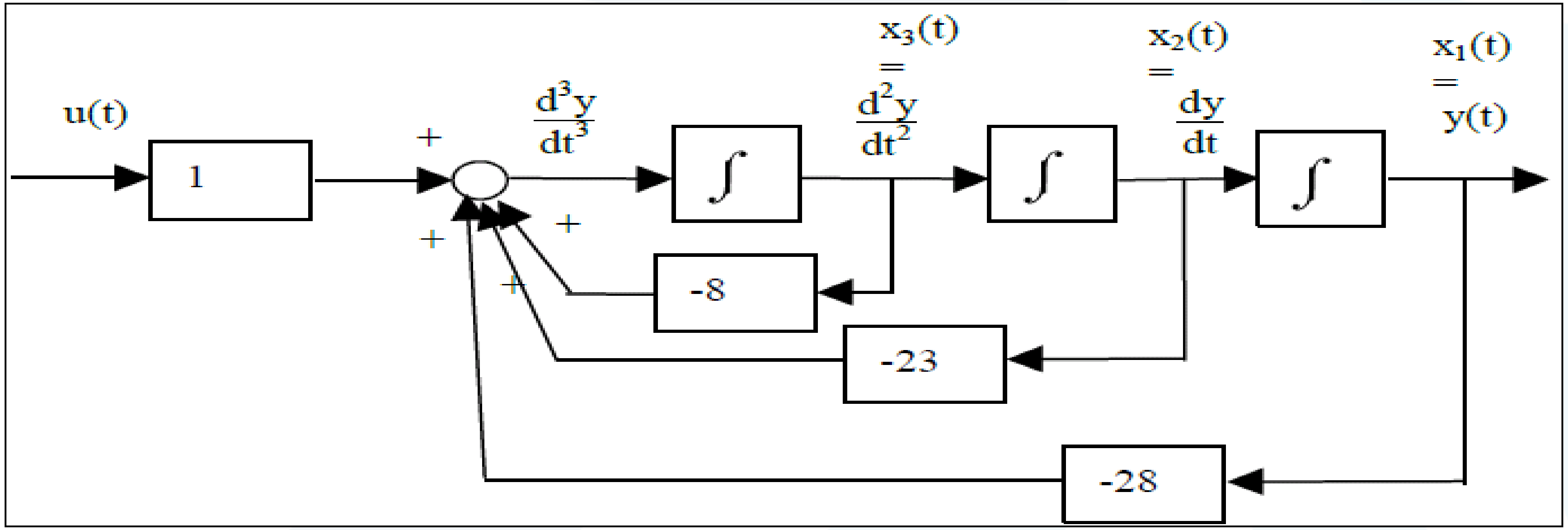
$$Y(s) = \frac{1}{(s+4)(s^2+4s+7)} U(s) = \frac{1}{(s^3+8s^2+23s+28)} U(s)$$

If we write

$(s^3 + 8s^2 + 23s + 28)\ Y(s) = U(s)$, we can see that this is equivalent to the differential equation given by

$$\frac{d^3y(t)}{dt^3} + 8\frac{d^2y(t)}{dt^2} + 23\frac{dy}{dt} + 28\ y(t) = u(t)$$

We let the first state, $x_1(t)$ be equivalent to the output y(t) , the second state equal its derivative, the third state equal the next derivative and so on



Therefore we can write

$$x_1(t) = y(t)$$

$$\frac{dx_1}{dt} = x_2(t) = \frac{dy}{dt}$$

$$\frac{dx_2}{dt} = x_3(t) = \frac{d^2y}{dt^2}$$

The full differential equation above can be rewritten in terms of its highest derivative:

$$\frac{d^3y(t)}{dt^3} = -8\frac{d^2y(t)}{dt^2} - 23\frac{dy}{dt} - 28\,y(t) + u(t)$$

and the state variable notation introduced to give

$$\frac{dx_3(t)}{dt} = -8x_3(t) - 23x_2(t) - 28\,x_1(t) + u(t)$$

This gives us the following state variable system

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -28 & -23 & -8 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = [\ 1\ 0\ 0]\mathbf{x}(t)$$

In the above example, the numerator was simply a '1'. We would like to know how to deal with situations where the numerator is a polynomial in *s*.

For illustration we consider the same transfer function as above but add a lead term in the numerator; this gives,

$$Y(s) = \frac{28(2s+1)}{(s+4)(s^2+4s+7)} U(s) = \frac{(56s+28)}{(s^3+8s^2+23s+28)} U(s)$$

We can rewrite this expression as

$$\frac{Y(s)}{U(s)} = \frac{X(s)}{U(s)} \frac{Y(s)}{X(s)}$$

where

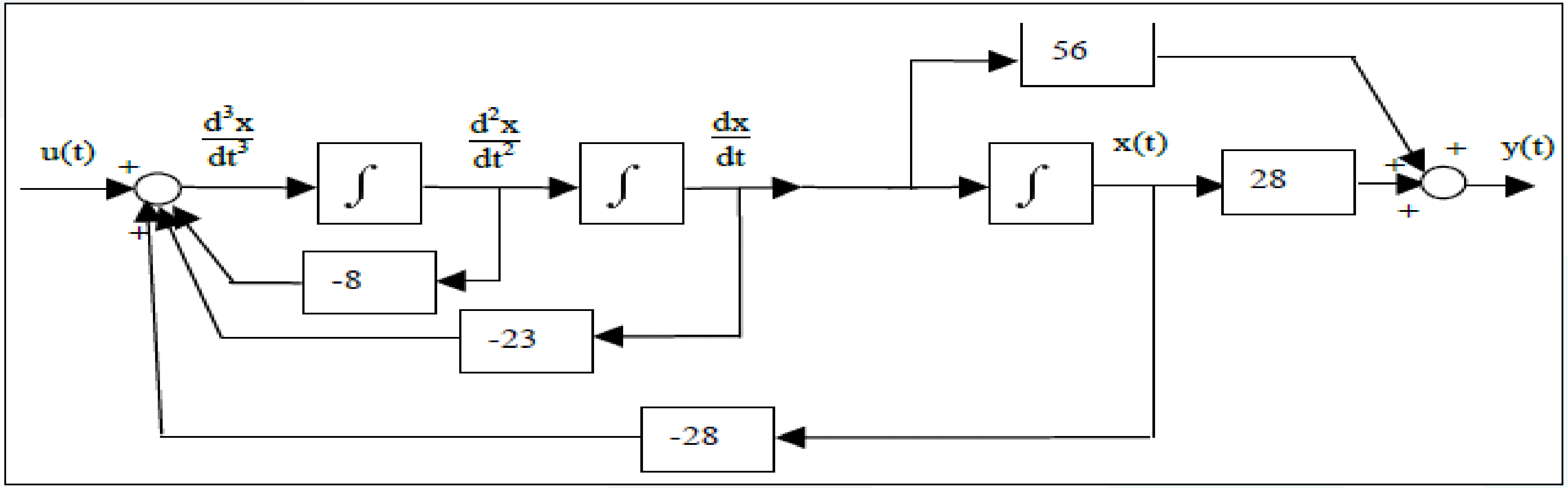
$$X(s) = \frac{1}{(s^3 + 8s^2 + 23s + 28)} U(s)$$

and

$$Y(s) = (56s + 28)\, X(s)$$

The transfer function from U(s) to X(s) is similar to the example above. However, the additional equation is for Y(s). This can be converted back to a differential equation to give

$$y(t) = 56\frac{dx}{dt} + 28\, x(t)$$



Using the state variable notation

$$y(t) = 56x_2(t) + 28\,x_1(t)$$

Therefore in matrix form we have the output equation as

$$y(t) = [\,28 \;\; 56 \;\; 0\,]\,\mathbf{x}(t)$$

with the state equations given as before

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -28 & -23 & -8 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

Example

$$Y_1(s) = \frac{1}{S^2 + 0.1S + 0.1} U(s) \qquad Y_2(s) = \frac{S}{S^2 + 0.1S + 0.1} U(s)$$

$(S^2 + 0.1S + 0.1)\, Y(s) = U(s)$, we can see that this is equivalent to the differential equation given by

$$\frac{d^2y(t)}{dt^2} + 0.1\frac{dy}{dt} + 0.1y(t) = u(t)$$

$$\frac{d^2y(t)}{dt^2} = -0.1\frac{dy}{dt} - 0.1y(t) + u(t)$$

Therefore we can write

$$x_1(t) = y(t)$$

$$\frac{dx_1}{dt} = x_2(t) = \frac{dy}{dt}$$

$$\frac{dx_2(t)}{dt} = -0.1x_2(t) - 0.1x_1(t) + u(t)$$

This gives us the following state variable system

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -0.1 & -0.1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = [\ 1 \ \ 0]\mathbf{x}(t)$$

$$Y_2(s) = \frac{S}{S^2 + 0.1S + 0.1} U(s)$$

We can rewrite this expression as

$$\frac{Y(s)}{U(s)} = \frac{X(s)}{U(s)} \frac{Y(s)}{X(s)}$$

$$X(s) = \frac{1}{S^2 + 0.1S + 0.1} U(s)$$

$$Y(s) = (\ s\ )\ X(s)$$

$$y(t) = \frac{dx}{dt}$$

Using the state variable notation

$y(t) = x_2(t)$

Therefore in matrix form we have the output equation as

$$y(t) = [0 \quad 1]\, \mathbf{x}(t)$$

Overall State Space Model

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -0.1 & -0.1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{x}(t)$$

Matlab Function tf2ss: Transfer Function to State Space

The form of the MATLAB expression: [A,B,C,D] = tf2ss(num,den)

The required inputs are

(i) num : a matrix which contains the numerator coefficients for each transfer function in a particular row of the transfer function matrix.

(ii) den is a vector containing the denominator polynomial coefficients

The resulting outputs are the state variable matrices A,B, C and D.

Using Matlab:

```
num = [1];
den=[1 0.1 0.1];
[A,B,C,D]=tf2ss(num,den)
```

```
A =

  -0.1000   -0.1000
   1.0000         0

B =

     1
     0

C =

     0     1

D =

     0
```

```
num = [1 0];
den=[1 0.1 0.1];
[A,B,C,D]=tf2ss(num,den)
```

```
A =

  -0.1000   -0.1000
   1.0000         0

B =

     1
     0

C =

     1     0

D =

     0
```

Example

Convert the following state variable model which has two inputs to transfer function format. Assume zero initial conditions.

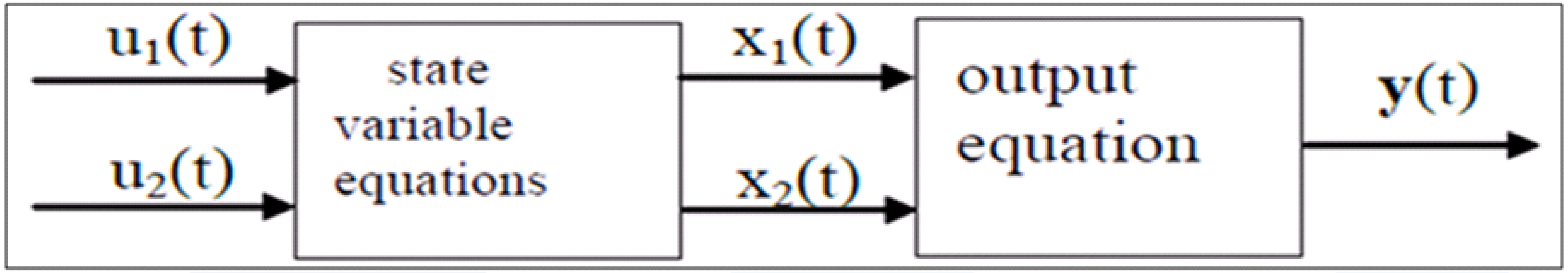
$$\dot{x}(t) = \begin{bmatrix} -6.3 & 3 \\ 0.5 & -5.4 \end{bmatrix} x(t) + \begin{bmatrix} 1 & 0.1 \\ 0.2 & 1 \end{bmatrix} u(t)$$

$$y(t) = [1 \ 0] \, x(t)$$

Solution

We enter the A,B,C and D matrices as follows:

```
A=[-6.3 3;0.5 -5.4];B=[1 0.1;0.2 1];C=[1 0];D=[0 0];
```



In our problem, we will find a transfer function model of the form:

$$\begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix} \begin{bmatrix} U_1(s) \\ U_2(s) \end{bmatrix}$$

$\mathbf{Y}(s) = \mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}\ \mathbf{U}(s) = \mathbf{G}(s)\mathbf{U}(s)$

$\mathbf{G}(s) = \mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}$

$$Y(s) = [1\ \ 0] \begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix}$$

$$Y(s) = [1\ \ 0] \begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} = X_1(s) = [G_{11}(s)\ \ G_{12}(s)] \begin{bmatrix} U_1(s) \\ U_2(s) \end{bmatrix}$$

The output Y(s) will depend on each input $U_1(s)$ and $U_2(s)$ through the transfer functions $G_{11}(s)$ and $G_{12}(s)$ respectively. The function ss2tf only provides one transfer function at each call of the function, so we must use it twice:

```
[num1,den1]=ss2tf(A,B,C,D,1)
```

```
num1 =
          0      1      6
den1 =
     1.0000   11.7000   32.5200
```

This gives $G_{11}(s) = \dfrac{s + 6}{s^2 + 11.7\,s + 35.52}$. Applying the function again for input 2 gives

```
[num2,den2]=ss2tf(A,B,C,D,2)
```

```
num2 =
          0    0.1000    3.5400
den2 =
     1.0000   11.7000   32.5200
```

We find that $G_{12}(s) = \dfrac{s + 3.54}{s^2 + 11.7\,s + 35.52}$

Therefore

$$Y(s) = \frac{s + 6}{s^2 + 11.7\,s + 35.52}\,U_1(s) + \frac{s + 3.54}{s^2 + 11.7\,s + 35.52}\,U_2(s)$$

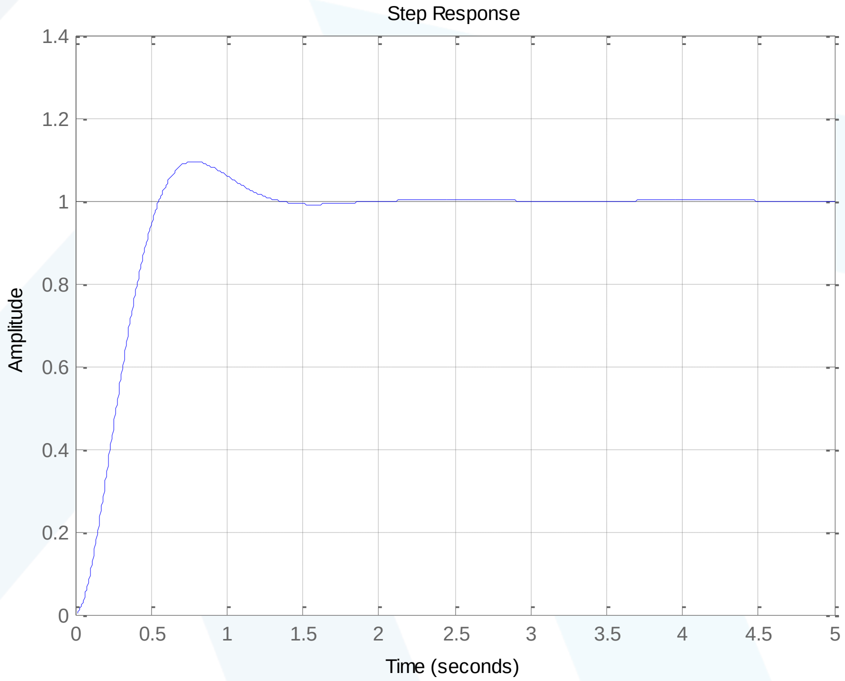
Example

```
num = [25];
den = [1 6 25];
t = 0:0.005:5;
step(num,den,t)
grid
[y,x,t] = step(num,den,t);
[ymax,tp] = max(y);
peak_time = tp*0.005
max_overshoot = ymax-y(end)
s = length(t);
while y(s) > 0.98*y(end)& y(s) < 1.02*y(end)
   s = s - 1;
end
settling_time = s*0.005
```

```
peak_time =
   0.7900

max_overshoot =
   0.0948

settling_time =
   1.1900
```



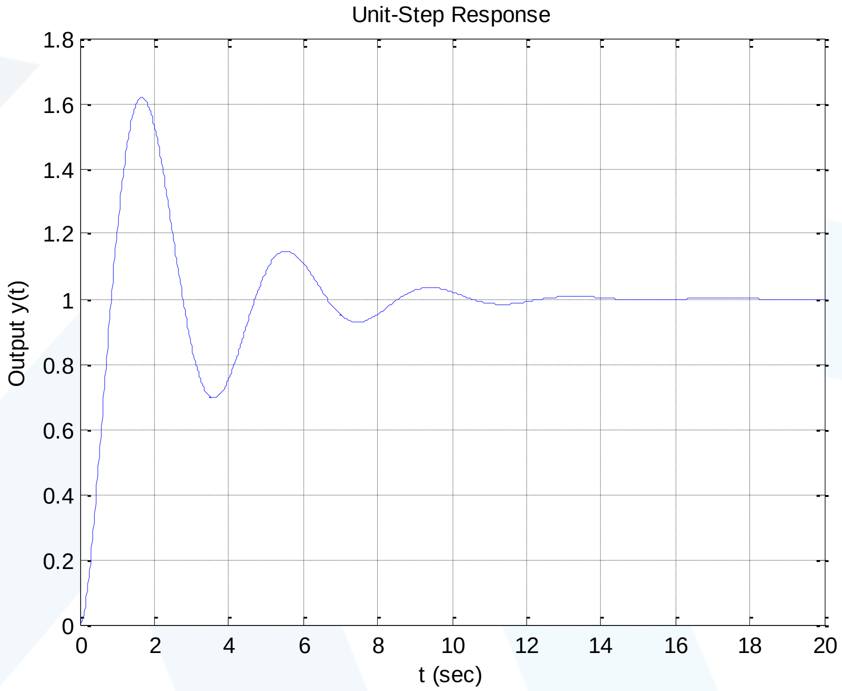
Example

```
num = [6.3223 18 12.811];
den = [1 6 11.3223 18 12.811];
t = 0:0.02:20;
[y,x,t] = step(num,den,t);
plot(t,y)
grid
[ymax,tp] = max(y);
peak_time = tp*0.02
max_overshoot = ymax-y(end)
s = length(t);
while y(s) > 0.98 *y(end)& y(s) < 1.02*y(end)
   s = s-1;
end
settling_time = s*0.02
```

peak_time =
1.6800

max_overshoot =
0.6184

settling_time =
10.06

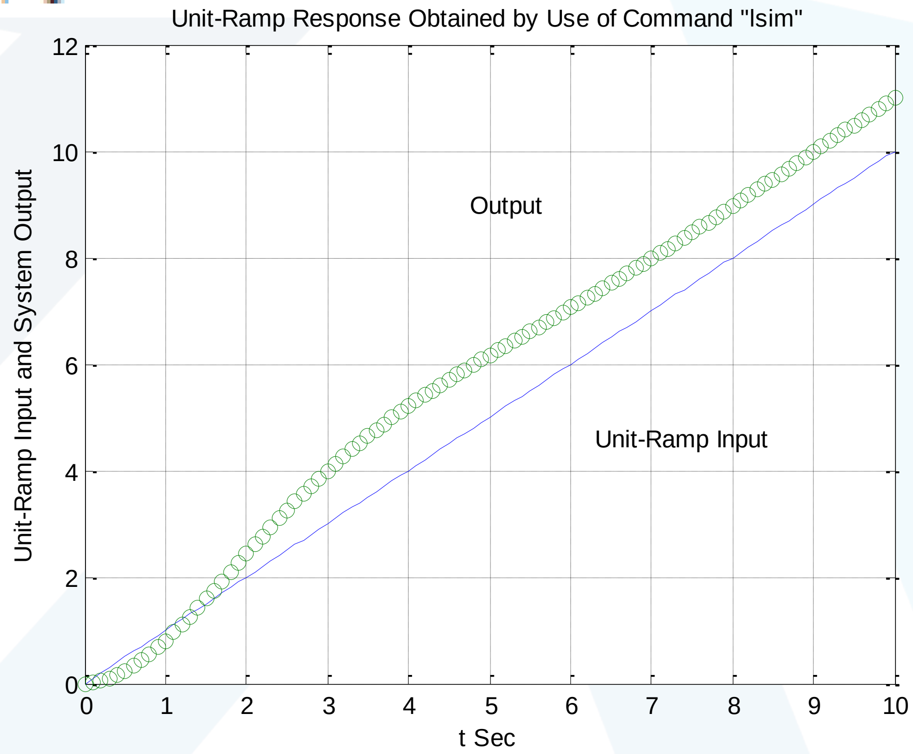


Example

Using the lsim command, obtain the unit-ramp response of the following system:

$$\frac{C(s)}{R(s)} = \frac{2s + 1}{s^2 + s + 1}$$

```
num = [2 1];
den = [1 1 1];
t = 0:0.1:10;
r = t;
y = lsim(num,den,r,t);
plot(t,r,'-',t,y,'o')
grid
```



Obtaining Response to Initial Condition by Use of Command Initial. If the system is given in the state-space form, then the following command

```
initial(A,B,C,D,[initial condition],t)
```

will produce the response to the initial condition.

Suppose that we have the system defined by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u, \qquad \mathbf{x}(0) = \mathbf{x}_0$$

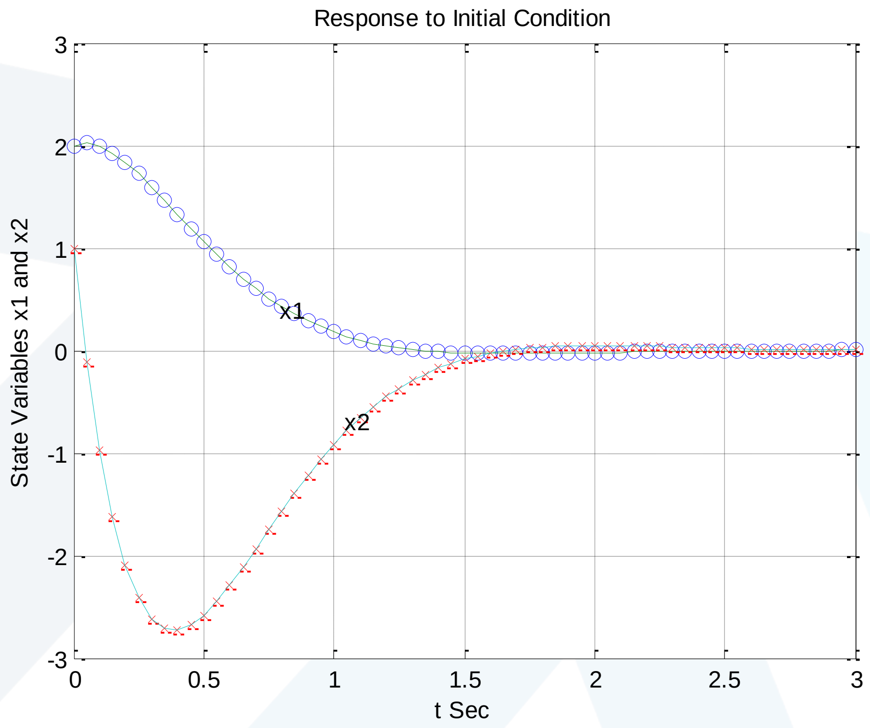
$$y = \mathbf{C}\mathbf{x} + Du$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -10 & -5 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad \mathbf{C} = [0 \quad 0], \qquad D = 0$$

$$\mathbf{x}_0 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

```
t = 0:0.05:3;
A = [0 1;-10 -5];
B = [0;0];
C = [0 0];
D = [0];
[y,x] = initial(A,B,C,D,[2;1],t);
x1 = [1 0]*x';
x2 = [0 1]*x';
plot(t,x1,'o',t,x1,t,x2,'x',t,x2)
grid
title('Response to Initial Condition')
xlabel('t Sec')
ylabel('State Variables x1 and x2')
gtext('x1')
gtext('x2')
```



Example

Consider the following system that is subjected to the initial condition. (No external forcing function is present.)

$$\dddot{y} + 8\ddot{y} + 17\dot{y} + 10y = 0$$

$$y(0) = 2, \qquad \dot{y}(0) = 1, \qquad \ddot{y}(0) = 0.5$$

Obtain the response $y(t)$ to the given initial condition.

By defining the state variables as

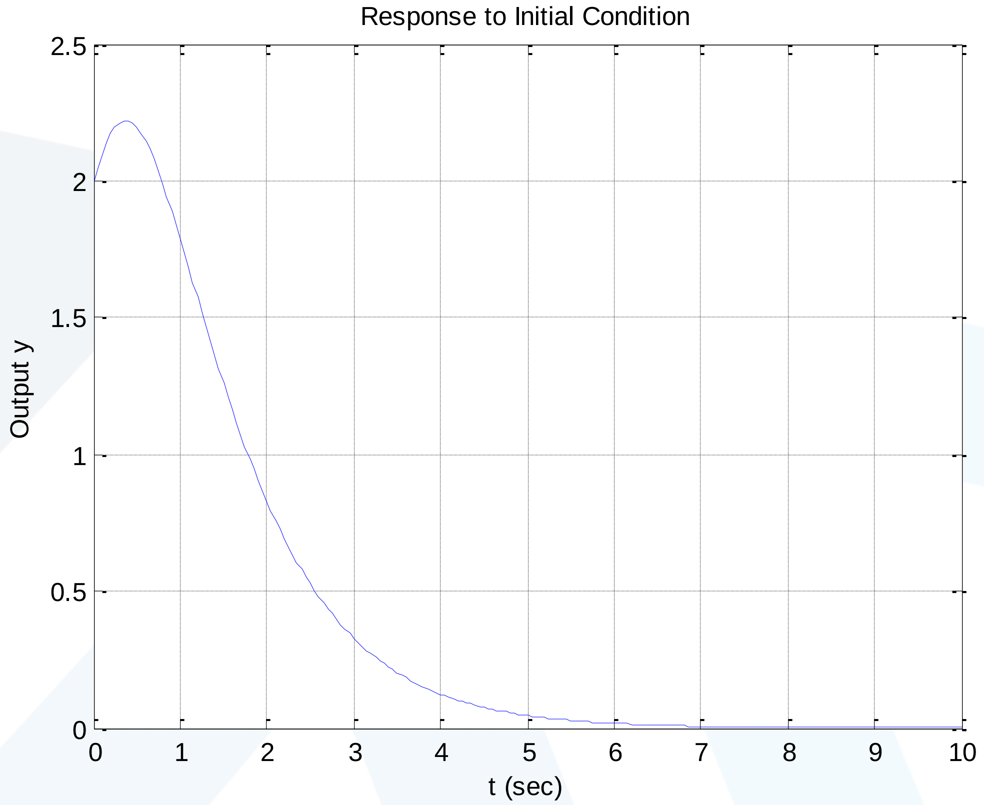
$$x_1 = y$$
$$x_2 = \dot{y}$$
$$x_3 = \ddot{y}$$

we obtain the following state-space representation for the system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -10 & -17 & -8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \qquad \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0.5 \end{bmatrix}$$

$$y = [1 \quad 0 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

```
t = 0:0.05:10;
A = [0 1 0;0 0 1;-10 -17 -8];
B = [0;0;0];
C = [1 0 0];
D = [0];
y = initial(A,B,C,D,[2;1;0.5],t);
plot(t,y)
grid
title('Response to Initial Condition')
xlabel('t (sec)')
ylabel('Output y')
```



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