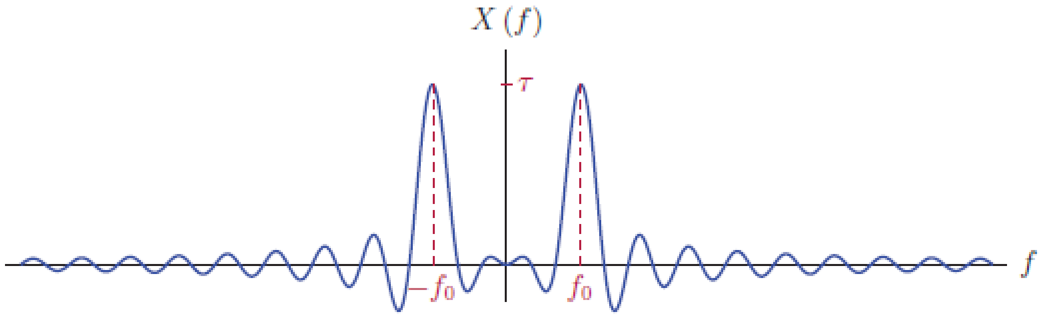
CEDC403: Signals and Systems

Lecture Notes 1 & 2: Signal Representation and Modeling



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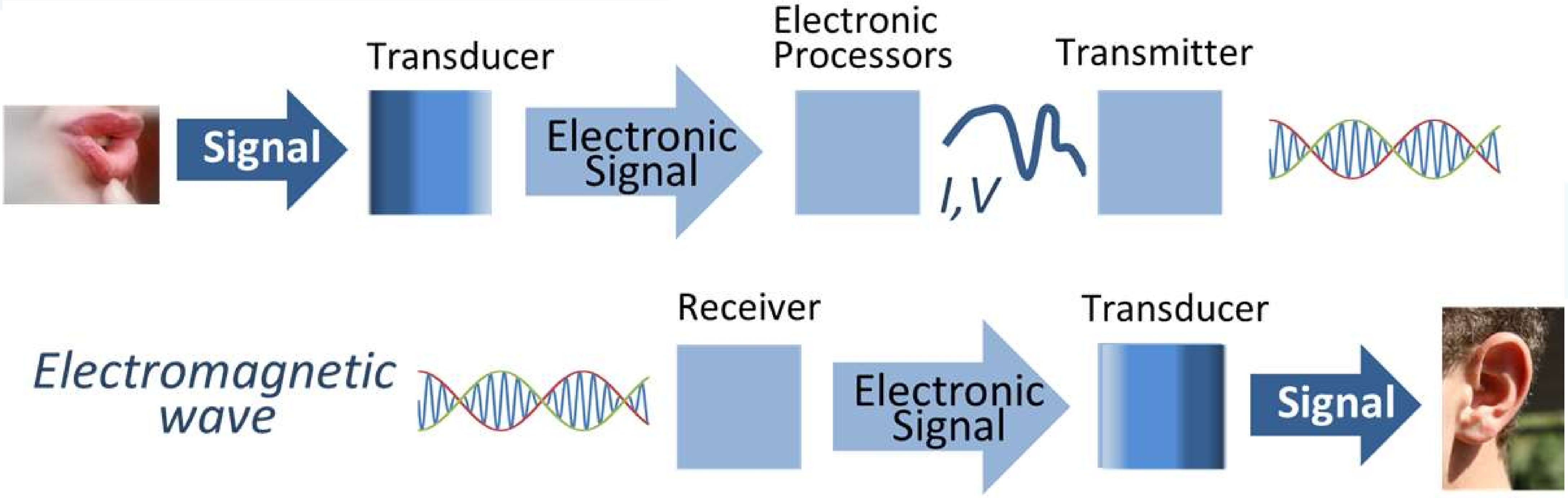
Chapter 1

Signal Representation and Modeling

1. Signals and Systems
2. Continuous-Time Signals
3. Basic building blocks for continuous-time signals
4. Discrete-Time Signals
5. Basic building blocks for discrete-time signals

Introduction



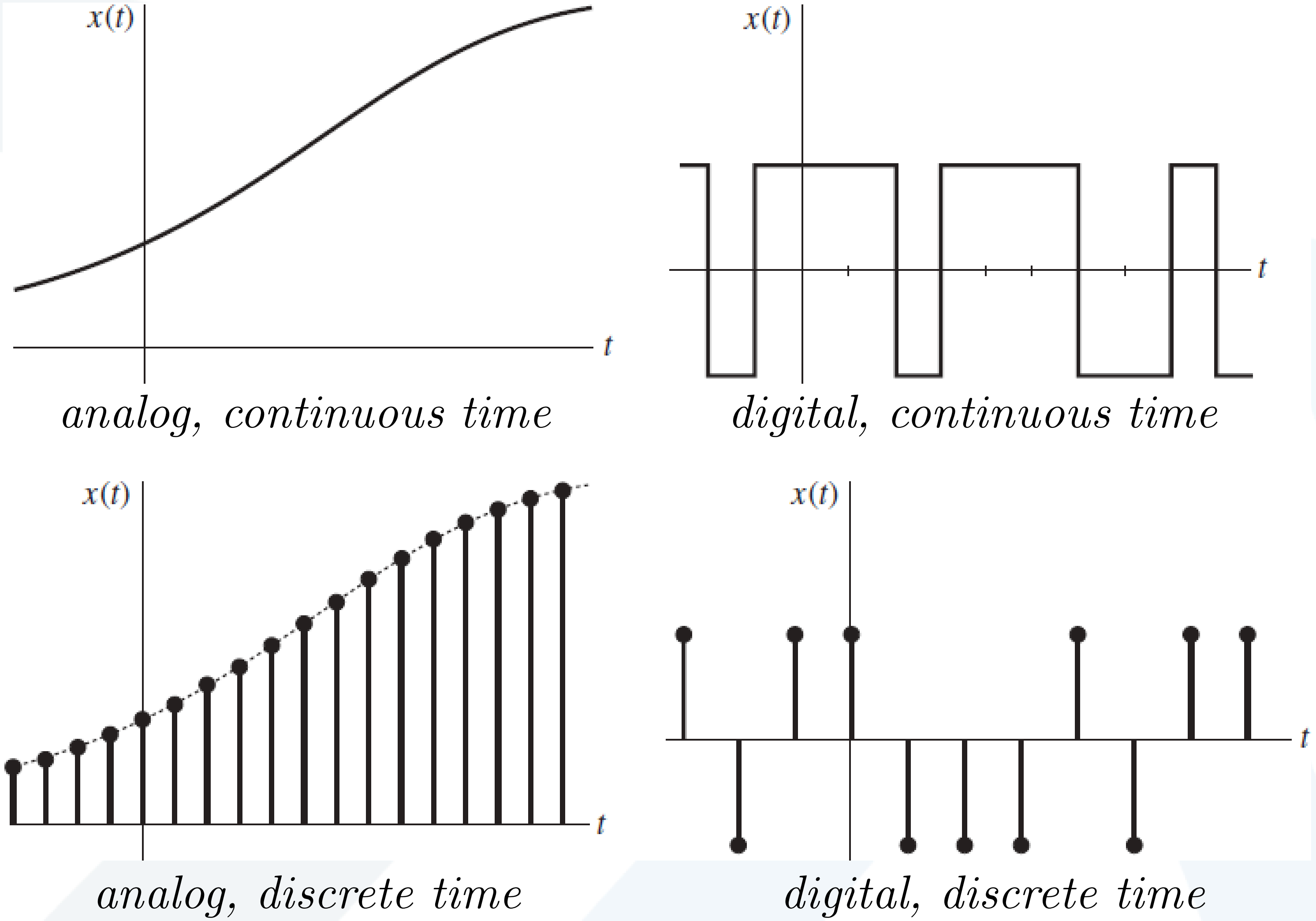
- The broadcast example (a commentator in a radio broadcast studio) includes acoustic, electrical and electromagnetic signals.

1. Signals and Systems

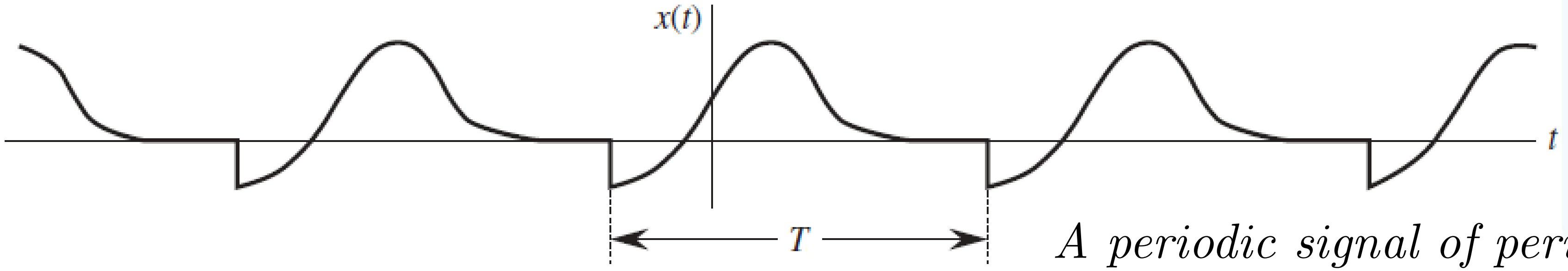
- A signal is a function of one or more variables that conveys information about some (usually physical) phenomenon.
- independent variable = time, space, …
- dependent variable = the function value itself.
- Some examples of signals include:
  - a voltage or current in an electronic circuit.
  - the position, velocity, or acceleration of an object.
  - a force or torque in a mechanical system.
  - a flow rate of a liquid or gas in a chemical process.
  - a digital image, digital video, or digital audio.

Classification of Signals

- **Continuous-time and discrete-time**
  - A continuous-time (CT) signal is a signal that is specified for every value of time t.
  - A discrete-time (DT) signal is a signal that is specified only at discrete values of t.
- **Analog and digital signals**
  - An Analog signal is a signal whose amplitude can take on any value in a continuous range.
  - A digital signal is a signal whose amplitude can take on only a finite number of values.


x(t)
t
analog, continuous time
x(t)
t
digital, continuous time
x(t)
t
analog, discrete time
x(t)
t
digital, discrete time

- **Periodic and Nonperiodic Signals**
  - A periodic signal is one that repeats itself. A CT signal $x(t)$ is said to be periodic with period T if $x(t) = x(t + T)$ for all $t \in R$. Likewise, a DT signal $x[n]$ is said to be periodic with period N if $x[n] = x[n + N]$ for all $n \in Z$.
  - A signal is aperiodic if it is not periodic.



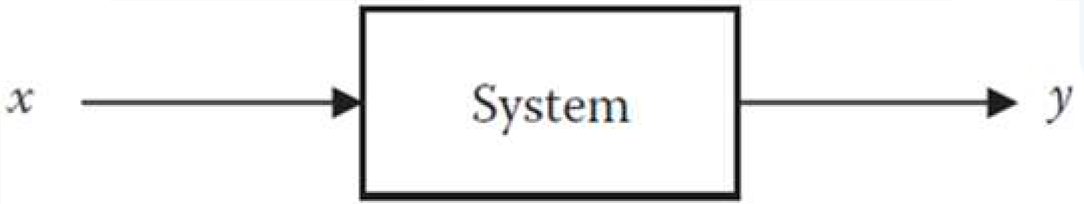
A periodic signal of period T

- **Deterministic or random signals**
  - A signal whose physical description is known completely, in either a mathematical form or a graphical form, is a deterministic signal.

- A signal whose values cannot be predicted precisely but are known only in terms of probabilistic description, such as mean value or mean-squared value, is a random signal.

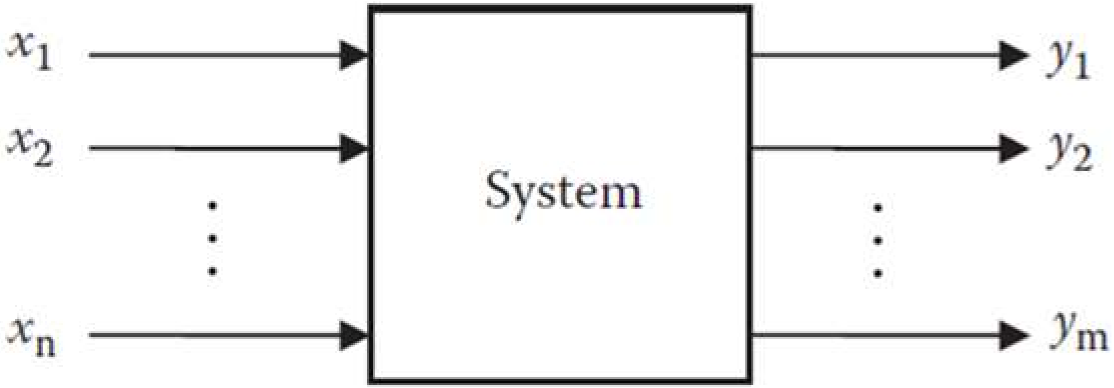
- **Energy and power signals**
  - A signal with finite energy is an energy signal, and a signal with finite and nonzero power is a power signal.

- A system is an entity that processes one or more input signals in order to produce one or more output signals.



system with single-input and single-output (SISO)

Input Signals



Output Signals

system with many inputs and outputs

Classification of Systems

- **Linear and nonlinear systems**
- **Time-Varying and Time-Invariant Systems**
  - A time-varying system is one whose parameters vary with time.
  - In a time-invariant system, a time shift (advance or delay) in the input signal leads to the time shift in the output signal.

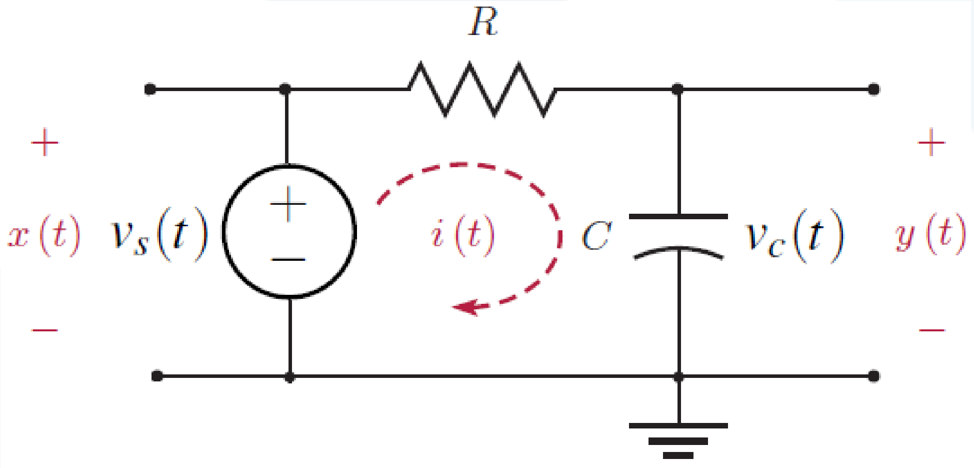
- **Memoryless (static) and with memory (dynamic) systems**
  - A memoryless system is one in which the current output depends only on the current input; it does not depend on the past or future inputs.
  - A system with memory is one in which the current output depends on the past and/or future input.
- **Causal and noncausal systems**
  - A causal system is one whose present response does not depend on the future values of the input.
- **Continuous-time and discrete-time systems**
  - CT system is a system whose inputs and outputs are CT signals.
  - DT system is a system whose inputs and outputs are DT signals.

- If a CT signal is sampled, the resulting signal is a DT signal. We can process a CT signal by processing its samples with a DT system.
- **Analog and digital systems**
  - Analog system is a system whose inputs and outputs are analog signals.
  - Digital system is a system whose inputs and outputs are digital signals.
- **Invertible and noninvertible systems**
  - An invertible system when we can obtain the input $x(t)$ back from the corresponding output $y(t)$ by some operation.
- **Stable and unstable systems**
  - A system is said to be stable if every bounded input applied at the input terminal results in a bounded output.

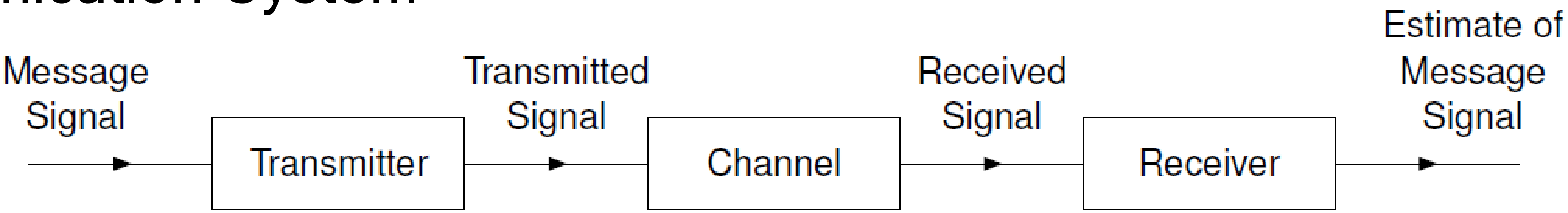
- This type of stability is also known as the stability in the BIBO (bounded-input/bounded-output) sense.

Examples of Systems:

- One very basic system is the resistor-capacitor (RC) network. Here, the input would be the source voltage v_s and the output would be the capacitor voltage v_c.

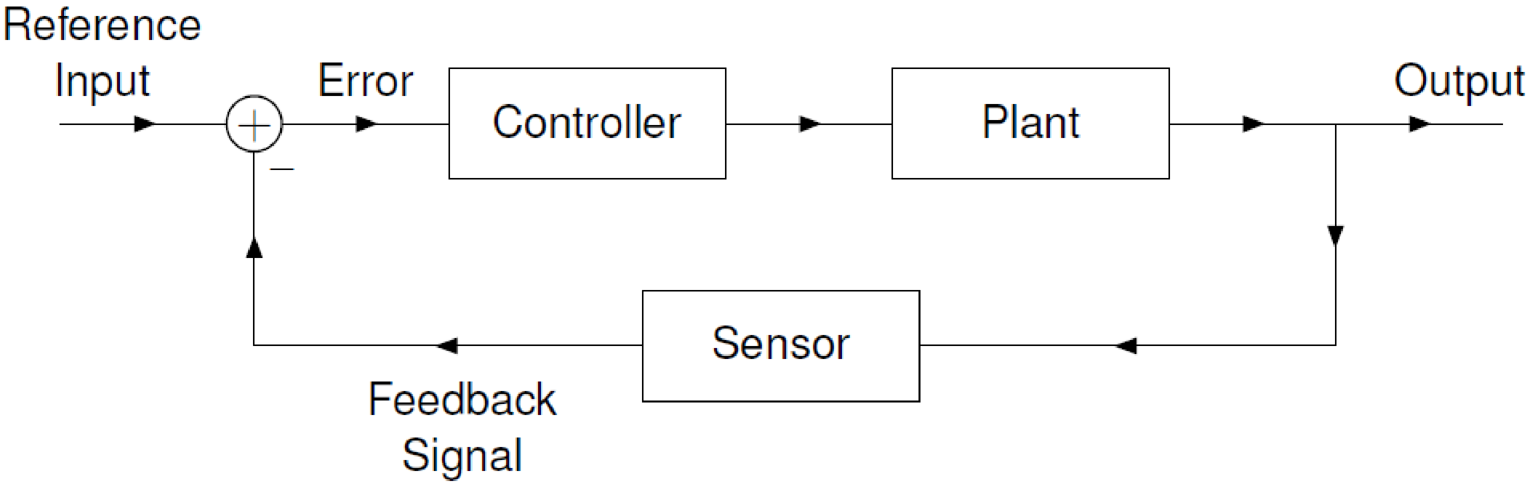


- Communication System



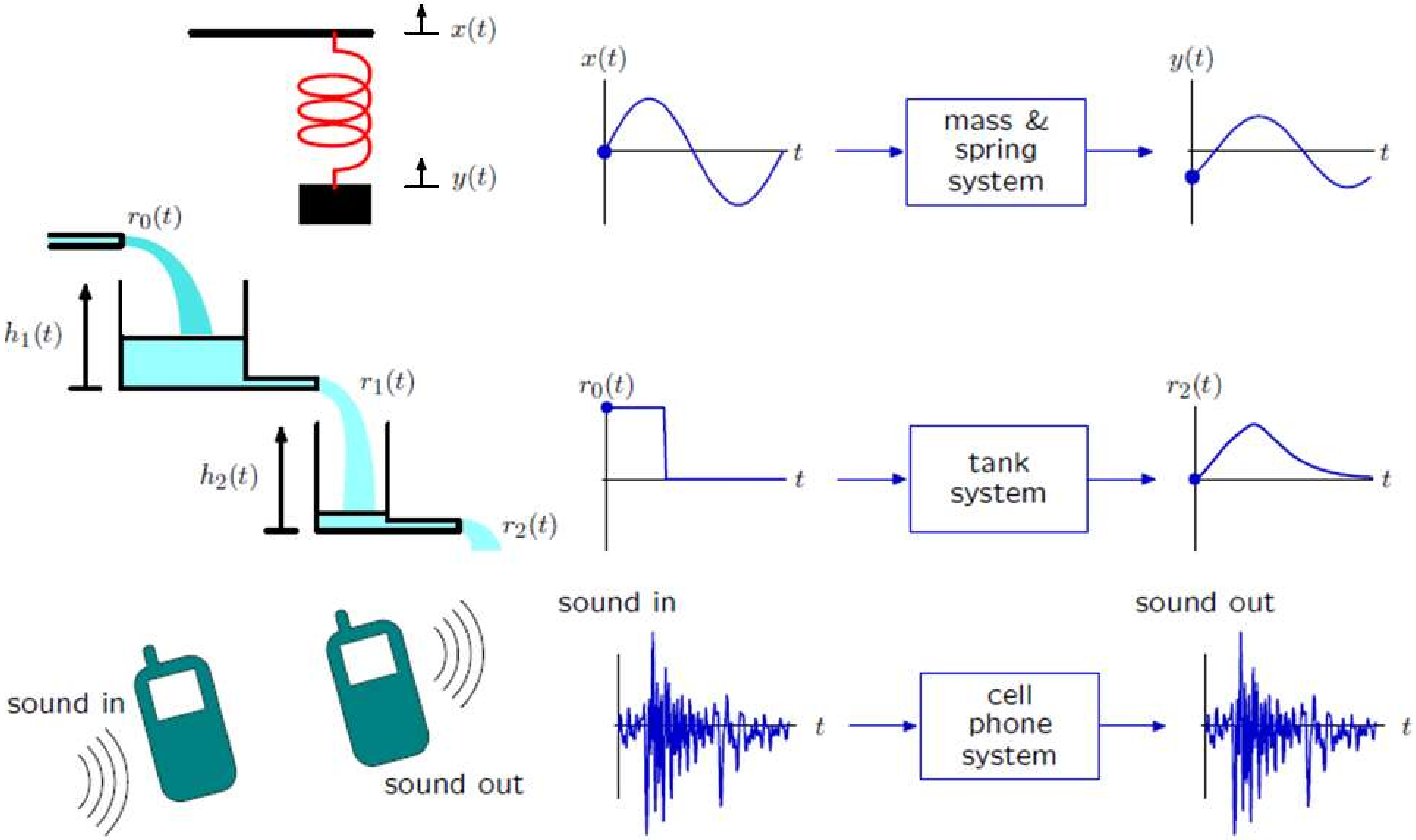
General Structure of a Communication System

- Feedback Control System



General Structure of a Feedback Control System

- The Signals and Systems approach has broad application: electrical, mechanical, optical, acoustic, biological, financial, ...

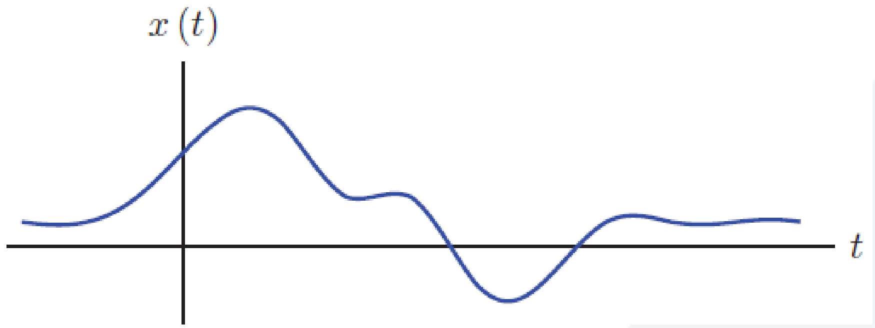


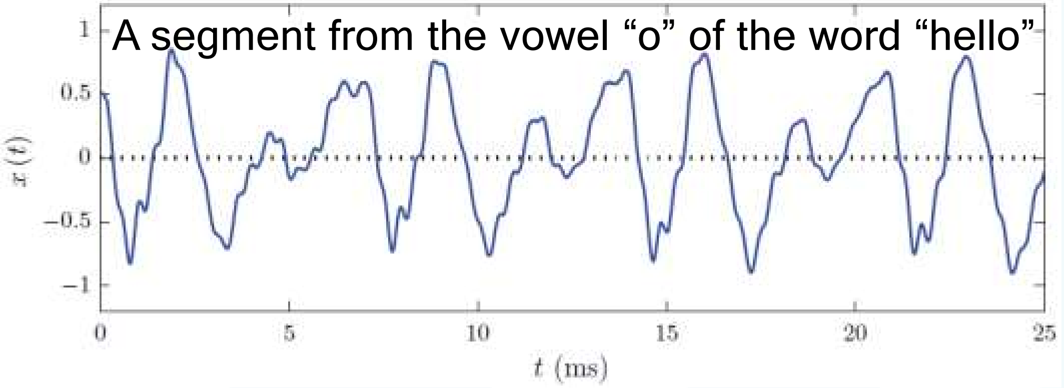
Mathematical Modeling of Signals

- Understand the characteristics of the signal in terms of its behavior in time and in terms of the frequencies it contains (signal analysis).
- Develop methods of creating signals with desired characteristics (signal synthesis).
- Understand how a system responds to a signal and why (system analysis).
- Develop methods of constructing a system that responds to a signal in some prescribed way (system synthesis).
- The mathematical model for a signal is in the form of a formula, function, algorithm or a graph that approximately describes the time variations of the physical signal.

2. Continuous-Time Signals

- Consider $x(t)$, a mathematical function of time chosen to approximate the strength of the physical quantity at the time instant t.
- The signal $x(t)$, is referred to as a continuous-time signal or an analog signal. t is the independent variable, and x is the dependent variable.



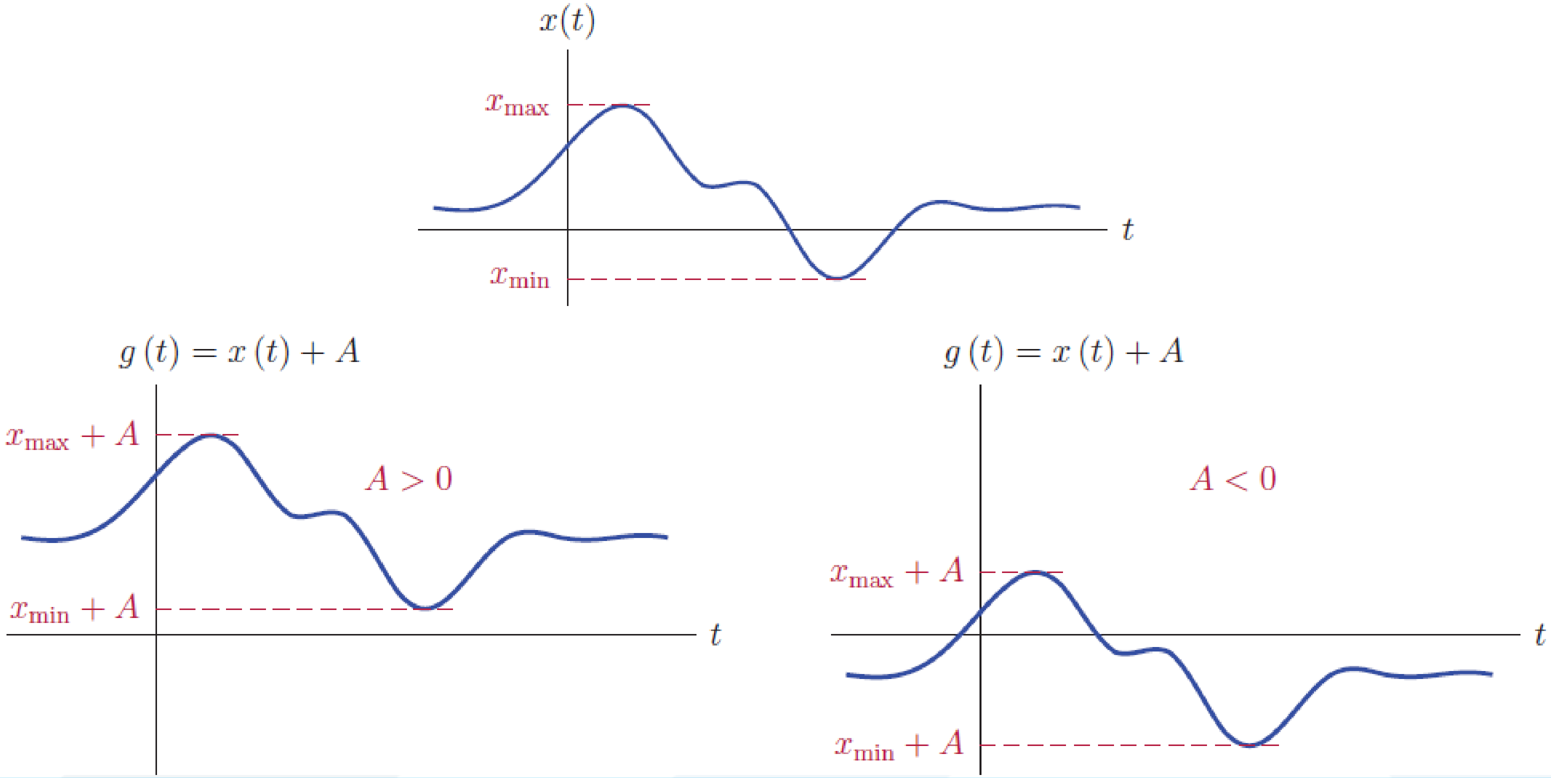


- Some signals can be described analytically. For ex., the function $x(t) = 5\sin(12t)$, or by segments as:

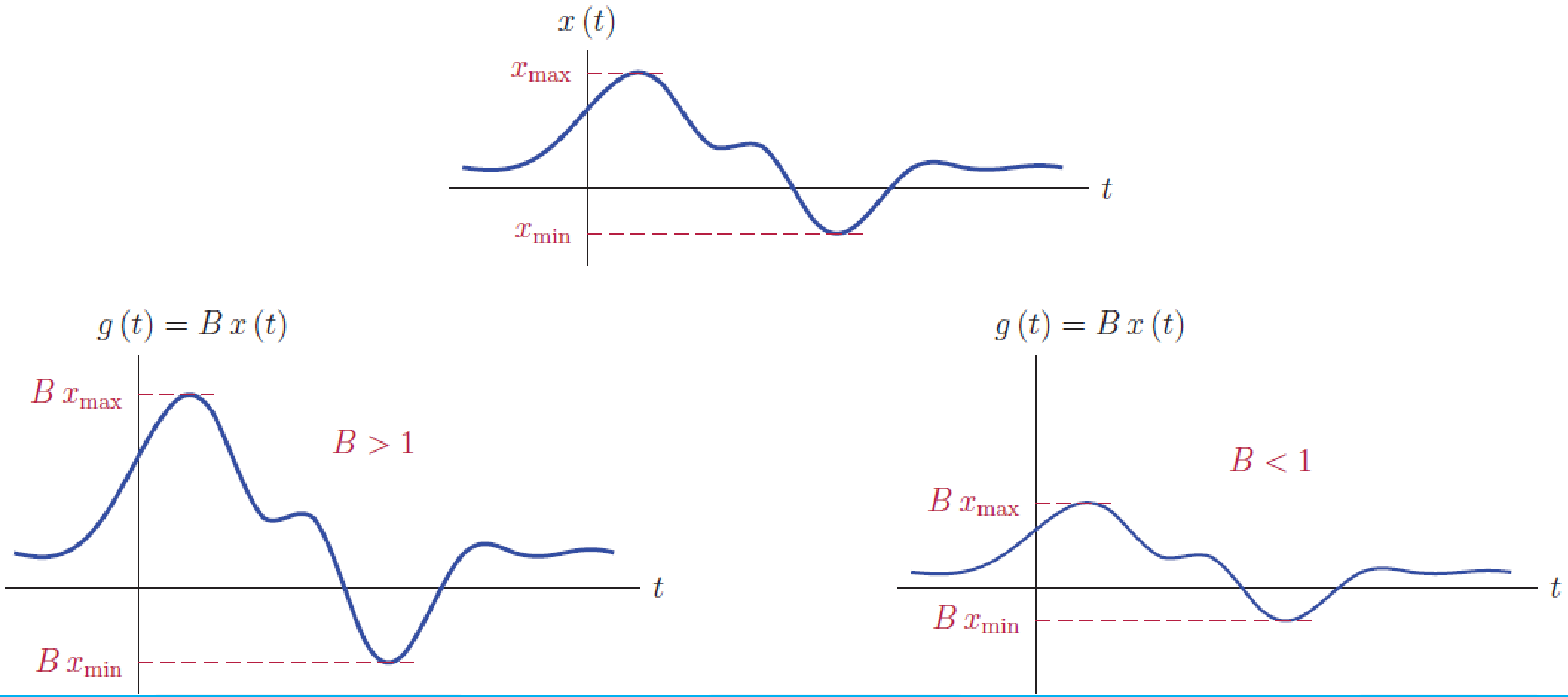
$$x(t) = \begin{cases} e^{-3t} - e^{-5t}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

Signal operations

- **Amplitude shifting** maps the input signal x to the output signal g as given by $g(t) = x(t) + A$, where A is a real number.

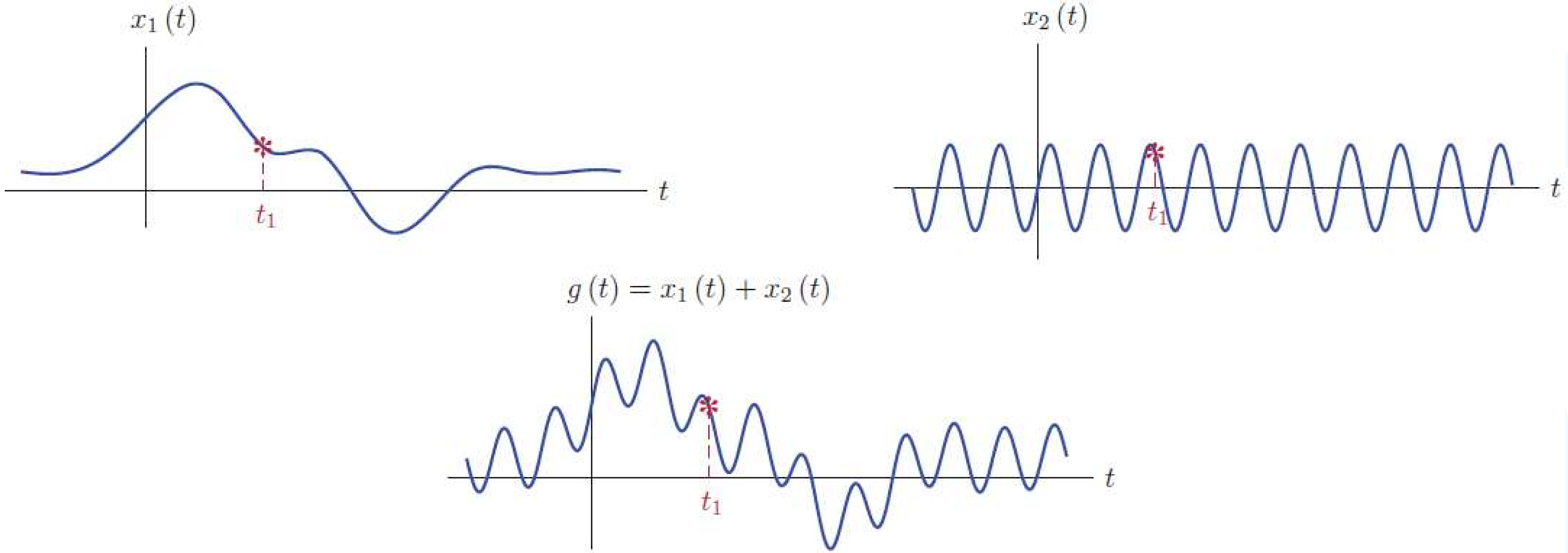


- Amplitude scaling maps the input signal x to the output signal g as given by $g(t) = Bx(t)$, where B is a real number.
- Geometrically, the output signal g is expanded/compressed in amplitude.

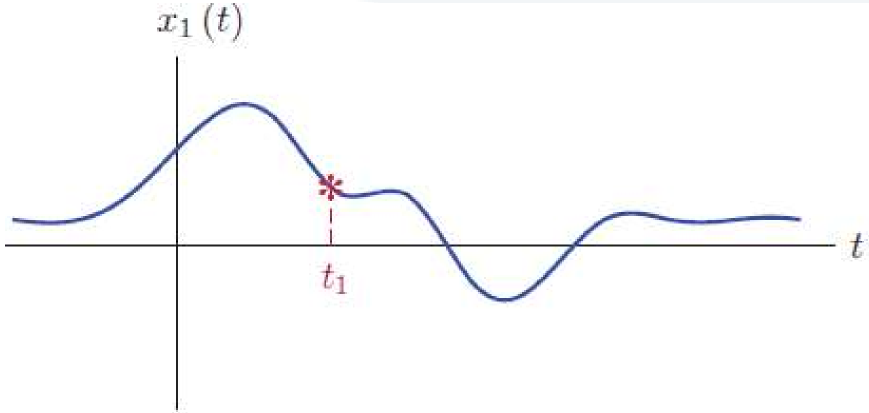


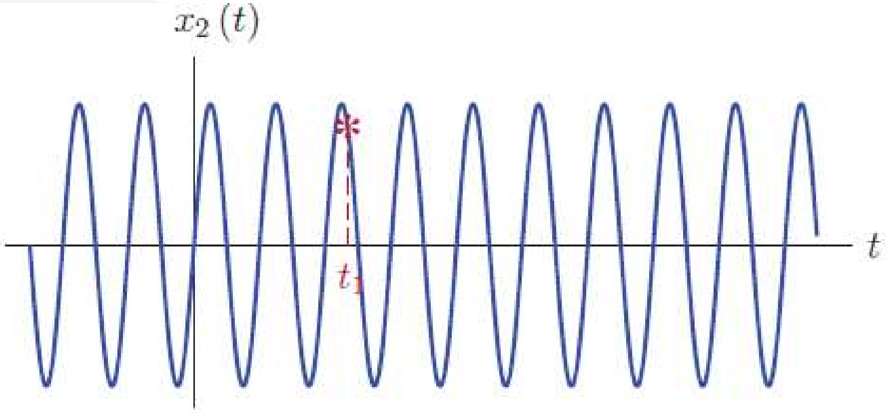
- Addition and Multiplication of two signals

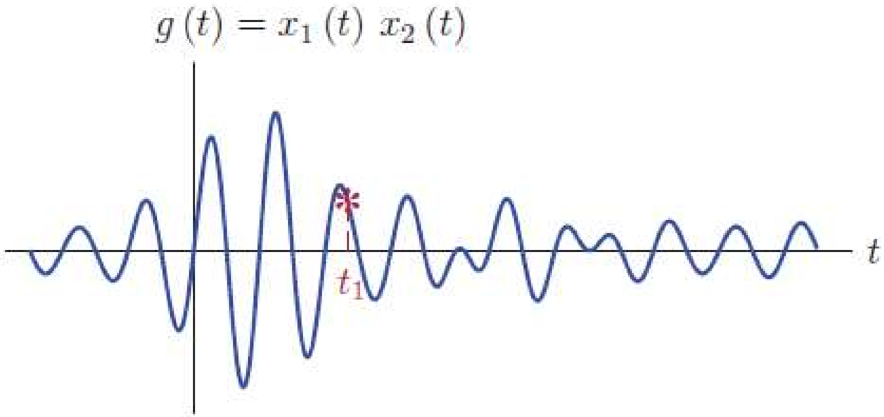
  Addition of two signals is accomplished by adding the amplitudes of the two signals at each time instant. $g(t) = x_1(t) + x_2(t)$.



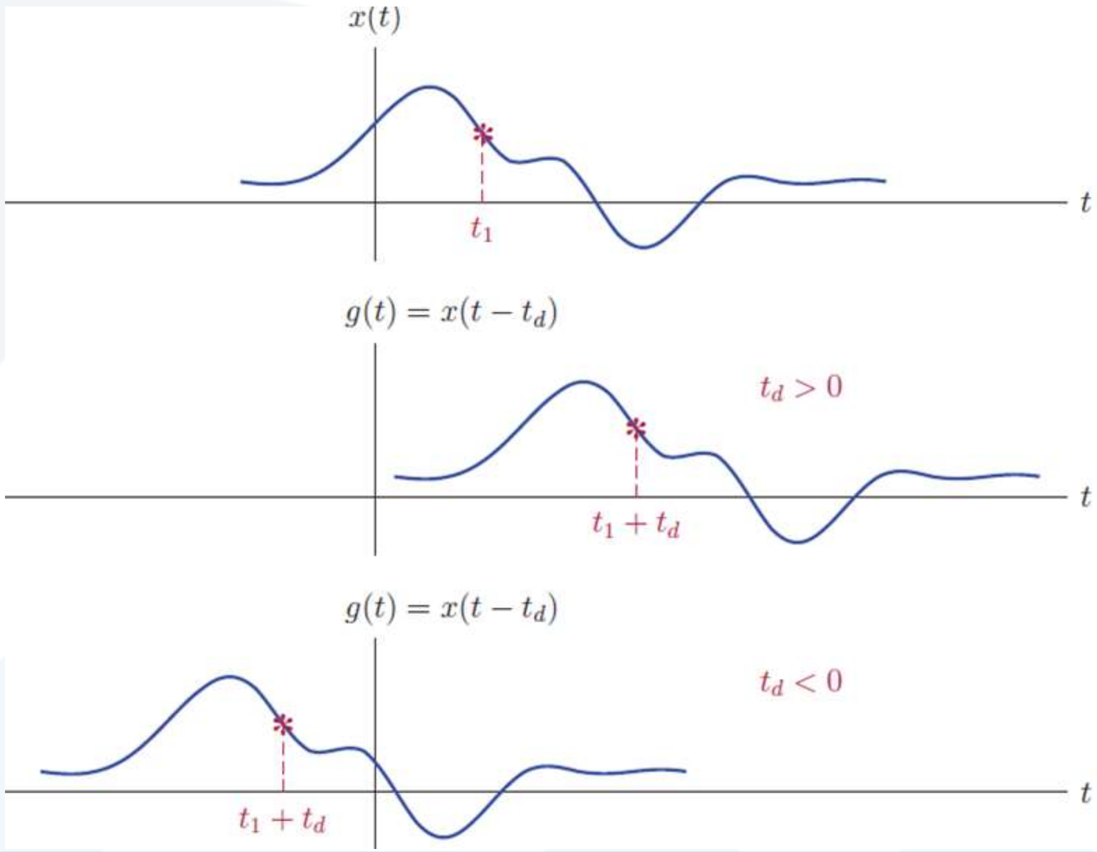
Multiplication of two signals is accomplished by multiplying the amplitudes of the two signals at each time instant. $g(t) = x_1(t).x_2(t)$.



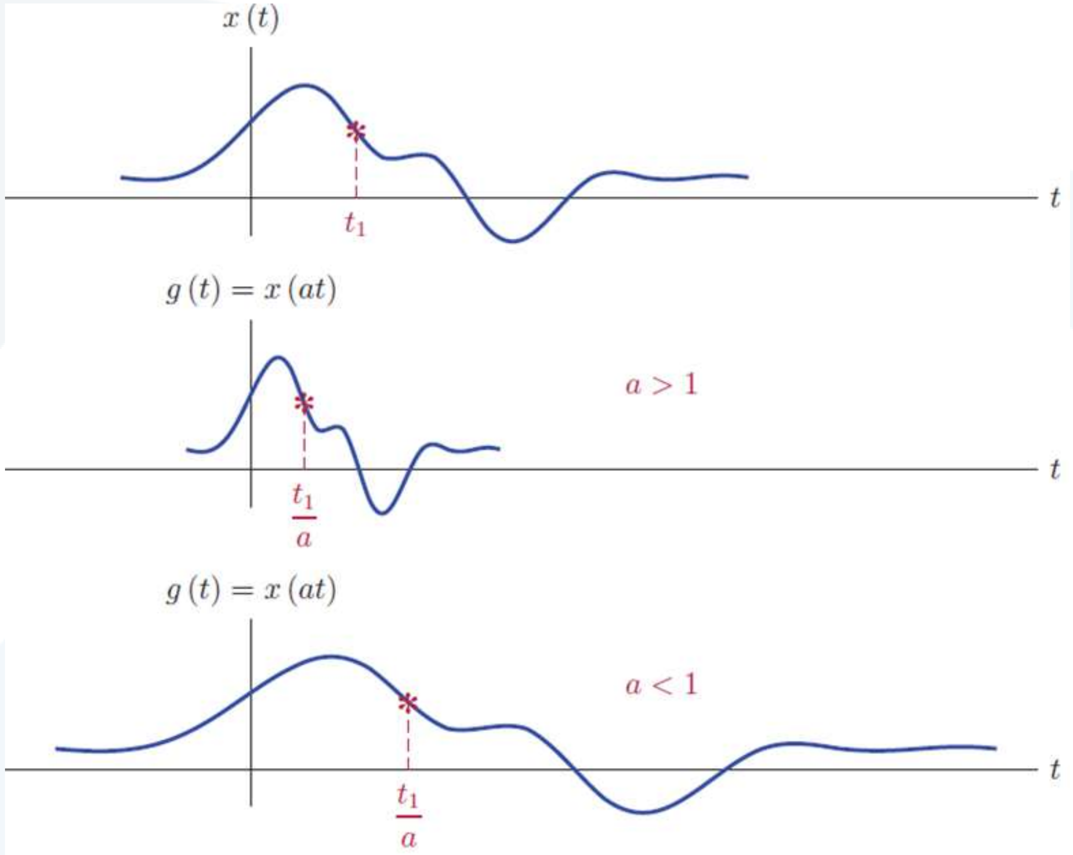




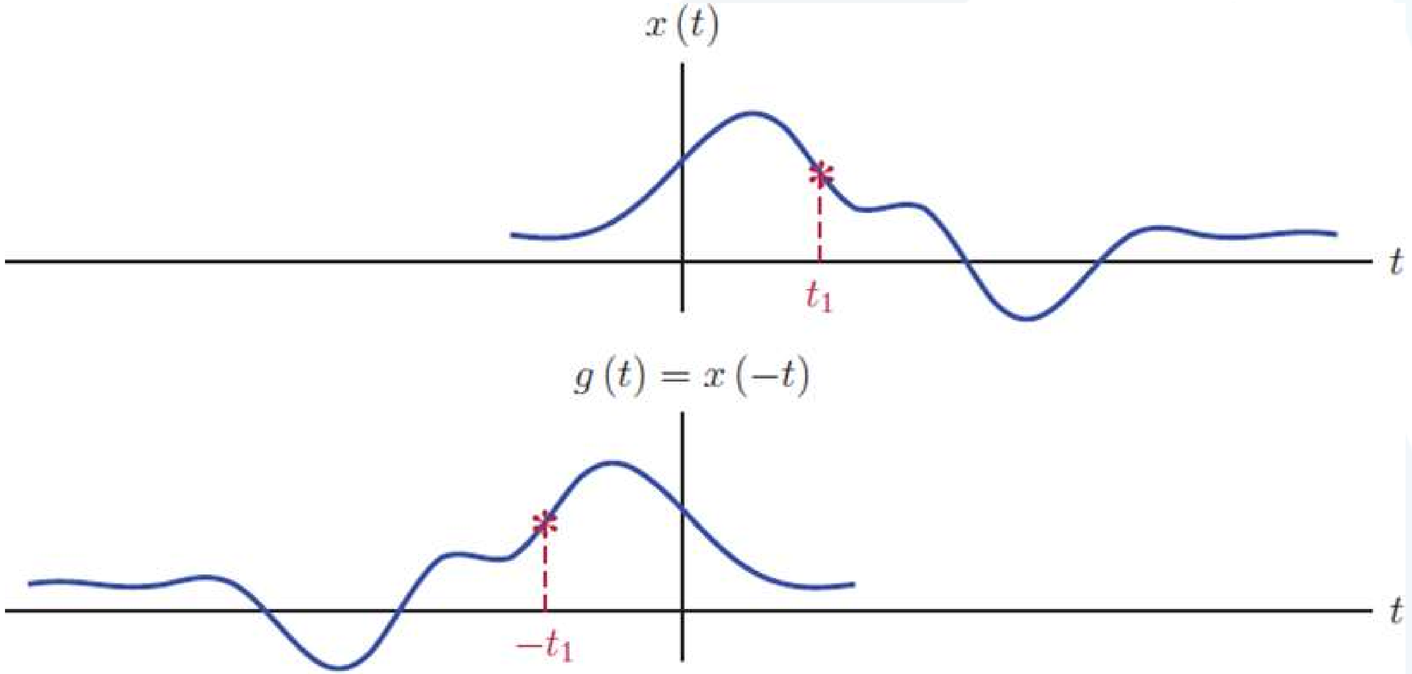
- Time shifting (also called translation) maps the input signal x to the output signal g as given by: $g(t) = x(t - t_d)$; where t_d is a real number.
- Such a transformation shifts the signal (to the left or right) along the time axis.
- If $t_d > 0$, g is shifted to the right by $|t_d|$, relative to x (i.e., delayed in time).
- If $t_d < 0$, g is shifted to the left by $|t_d|$, relative to x (i.e., advanced in time).



- Time scaling (also called dilation) maps the input signal x to the output signal g as given by: $g(t) = x(at)$; where a is a strictly positive real number.
- Such a transformation is associated with a compression/expansion along the time axis.
- If $a > 1$, g is compressed along the horizontal axis by a factor of a, relative to x.
- If $a < 1$, g is expanded (stretched) along the horizontal axis by a factor of $1/a$, relative to x.



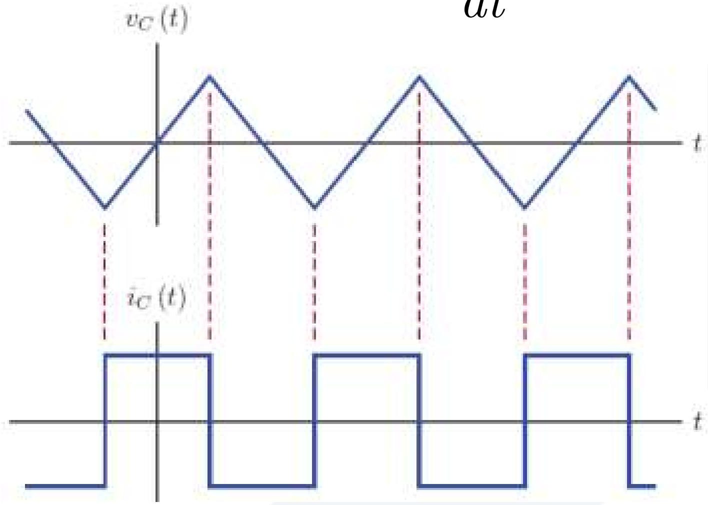
- Time reversal (also known as reflection) maps the input signal x to the output signal g as given by $g(t) = x(-t)$.
- Geometrically, the output signal g is a reflection of the input signal x about the (vertical) line $t = 0$.



- Integration and differentiation

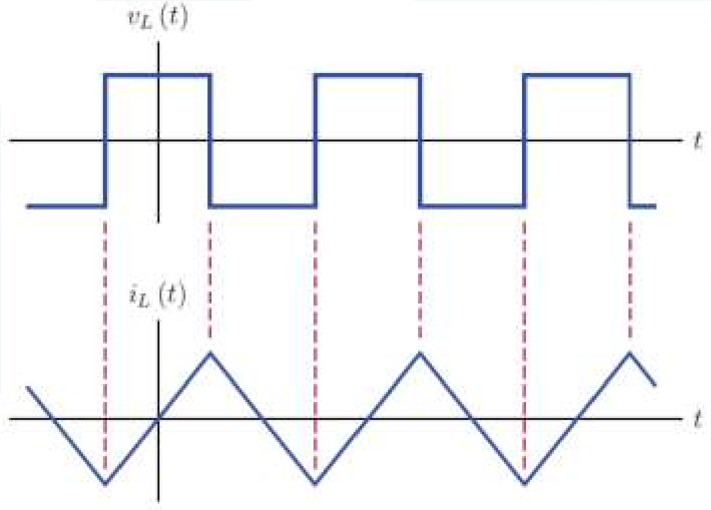
Given a continuous-time signal $x(t)$, a new signal $g(t)$ may be defined as its time derivative in the form: $g(t) = dx(t)/dt.$ Similarly, a signal can be defined as the integral of another signal in the form: $g(t) = \int_{-\infty}^{t} x(\tau)d\tau$

$$i_C(t) = C\frac{dv_C(t)}{dt}$$



ideal capacitor

$$i_L(t) = \frac{1}{L}\int_{-\infty}^{t} v_L(\tau)d\tau$$



ideal inductor

- Sum of periodic signals

For two periodic signals x_1 and x_2 with fundamental periods T_1 and T_2, respectively, and the sum $y = x_1 + x_2$:

- The sum y is periodic if and only if the ratio T_1/T_2 is a rational number (i.e., the quotient of two integers).
- If y is periodic, its fundamental period is rT_1 (or equivalently, qT_2, since $rT_1 = qT_2$), where $T_1/T_2 = q/r$ and q and r are integers and coprime. (Note that rT_1 is simply the least common multiple of T_1 and T_2).

For example $x(t) = \sin(2\pi 1.5\, t) + \sin(2\pi 2.5\, t)$

$T_1 = 1/1.5 = 2/3\ s,\ T_2 = 1/2.5 = 2/5\ s \Rightarrow T_1/T_2 = 5/3$

$T = 5T_2 = 3T_1 = 2\ s.$

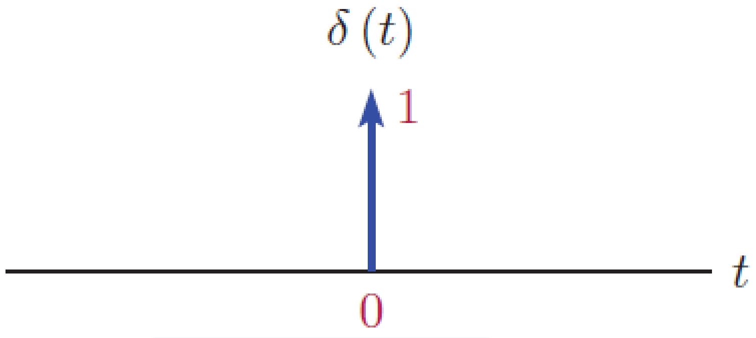
3. Basic building blocks for continuous-time signals

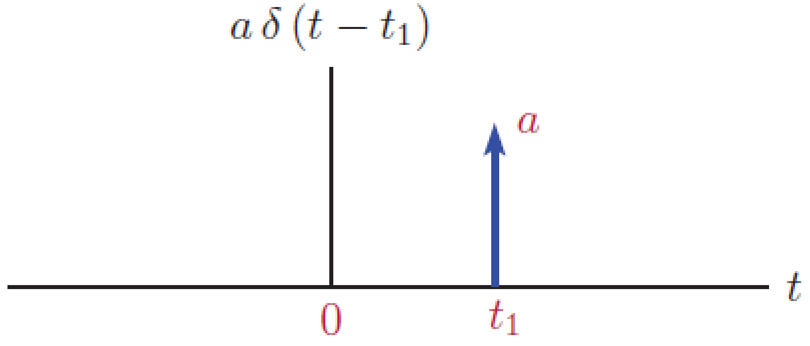
Unit-impulse function

- The unit-impulse function (Dirac delta function or delta function), denoted δ, is defined by:

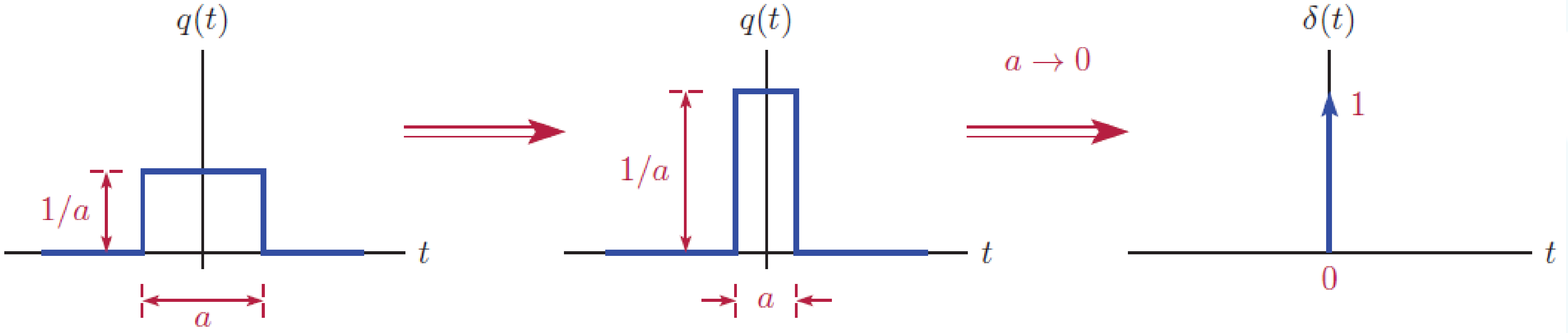
$$\delta(t) = \begin{cases} 0, & \text{if } t \neq 0 \\ \text{undefined}, & \text{if } t = 0 \end{cases} \quad \text{and} \quad \int_{-\infty}^{\infty} \delta(t)\,dt = 1$$

- Technically, δ is not a function in the ordinary sense. Rather, it is what is known as a generalized function.





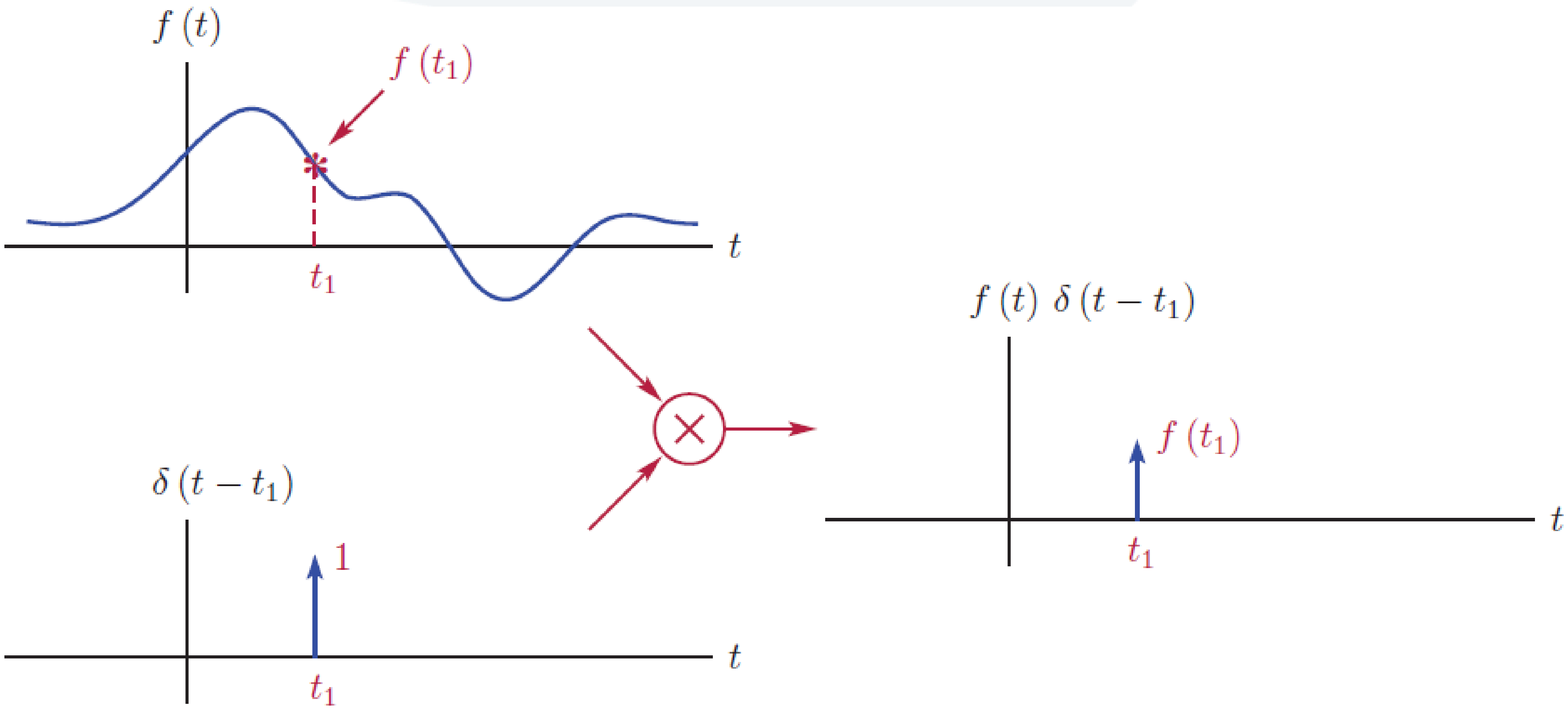
- Define $q(t) = \begin{cases} 1/a, & |t| < a/2 \\ 0, & |t| > a/2 \end{cases}$
- Clearly, for any choice of a, $\int_{-\infty}^{\infty} q(t)\,dt = 1$
- The function δ can be obtained as the following limit: $\delta(t) = \lim_{a \to 0} q(t)$



- Sampling property. For any continuous function f and any real constant t_1, $f(t)\delta(t - t_1) = f(t_1)\delta(t - t_1)$.

- Sifting property. For any continuous function f and any real constant t_1:

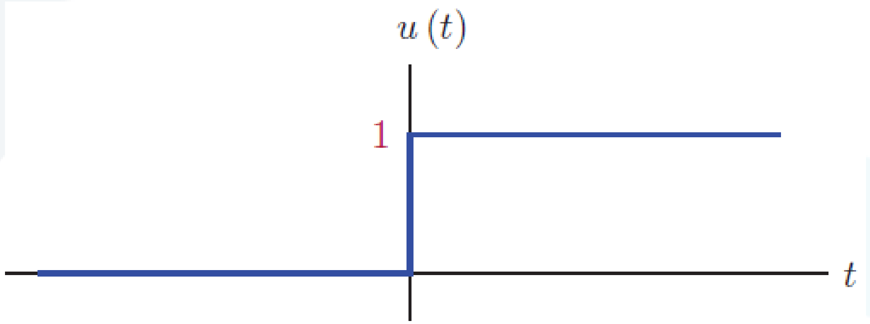
$$\int_{-\infty}^{\infty} f(t)\delta(t - t_1)dt = f(t_1)$$



Unit-Step Function

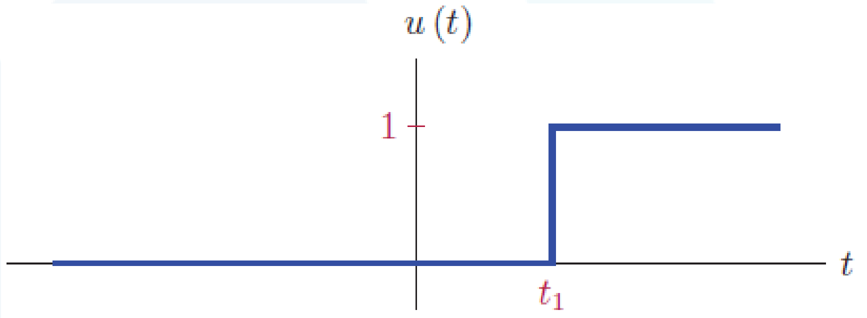
- The unit-step function (also known as the Heaviside function), denoted u, is defined as:

$$u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$



- A time shifted version of the unit-step function:

$$u(t - t_1) = \begin{cases} 1, & t \geq t_1 \\ 0, & t < t_1 \end{cases}$$

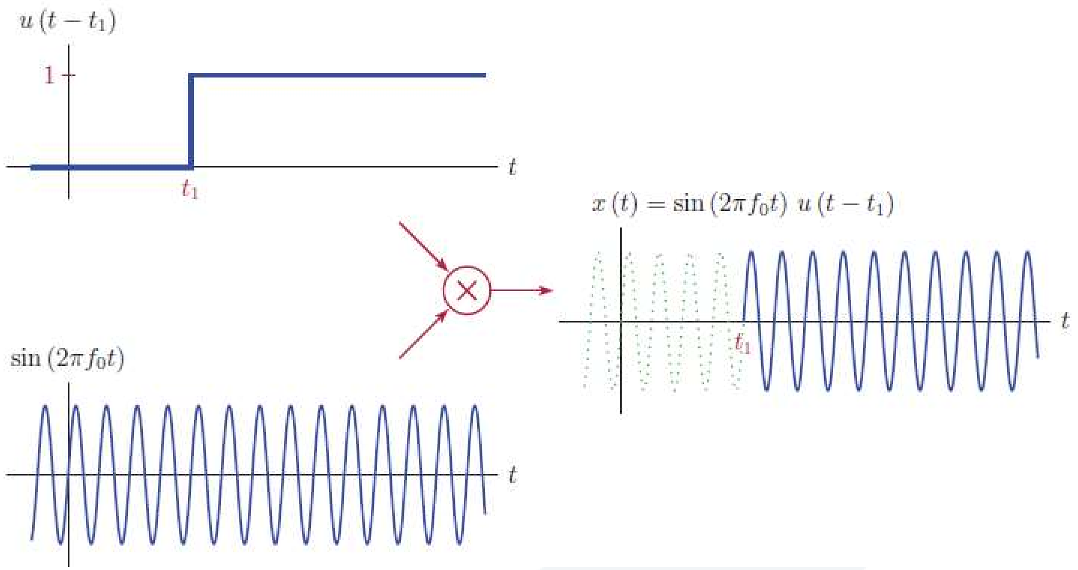


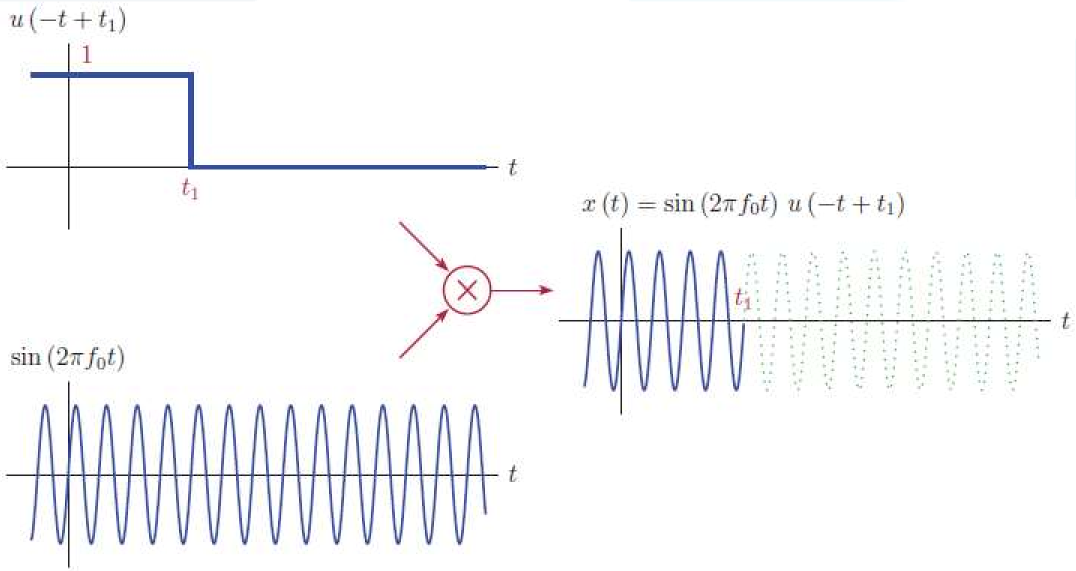
- Signals begin at $t = 0$ (causal signals) can be described in terms of $u(t)$.

- Using the unit-step function to turn a signal on/off at a specified time instant:

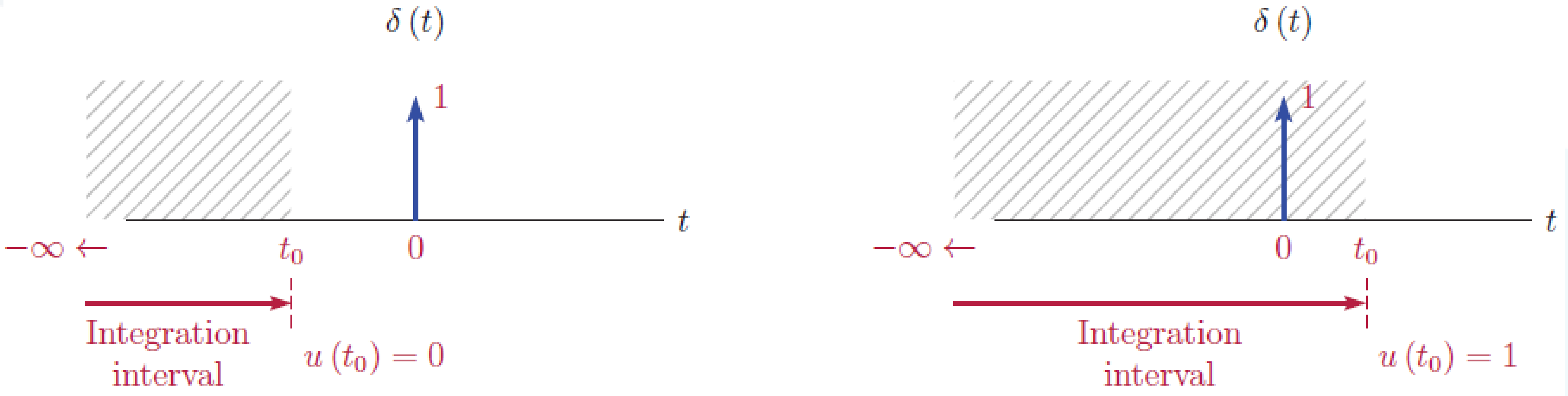
$$x(t)u(t-t_1) = \begin{cases} \sin(2\pi f_0 t), & t \geq t_1 \\ 0, & t < t_1 \end{cases}$$

$$x(t)u(-t+t_1) = \begin{cases} \sin(2\pi f_0 t), & t \leq t_1 \\ 0, & t > t_1 \end{cases}$$



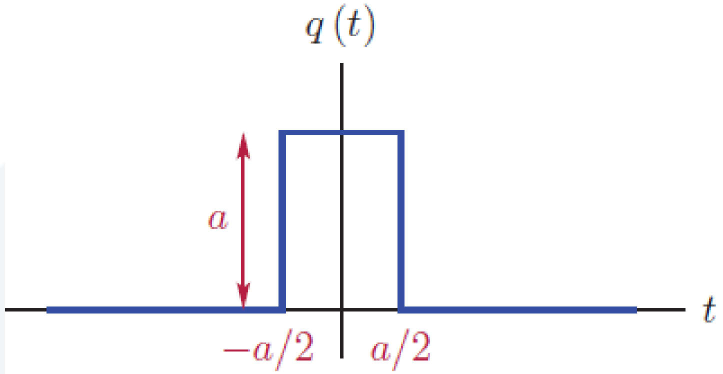


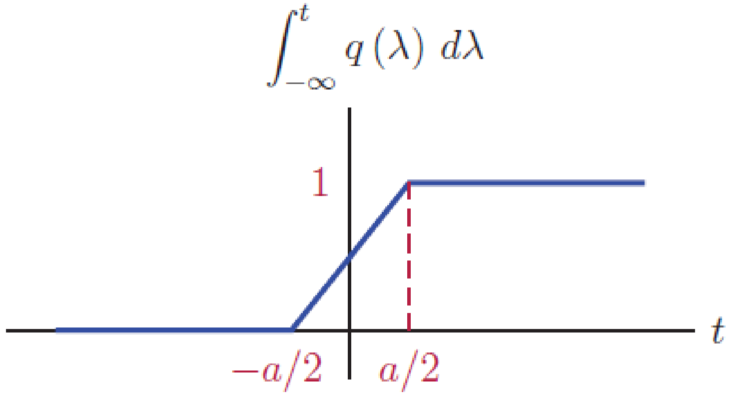
- The Relationship between the unit-step function and the unit-impulse function:



$$u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau$$

$$\delta(t) = \frac{du(t)}{dt}$$



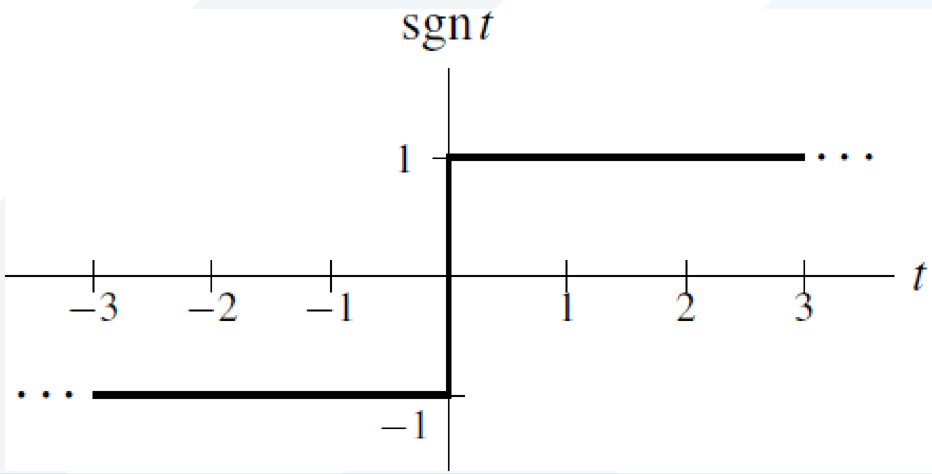


Signum Function

- The signum function, denoted sgn, is defined as:

$$\operatorname{sgn} t = \begin{cases} 1 & \text{if } t > 0 \\ 0 & \text{if } t = 0 \\ -1 & \text{if } t < 0 \end{cases}$$

- From its definition, one can see that the signum function simply computes the sign of a number.

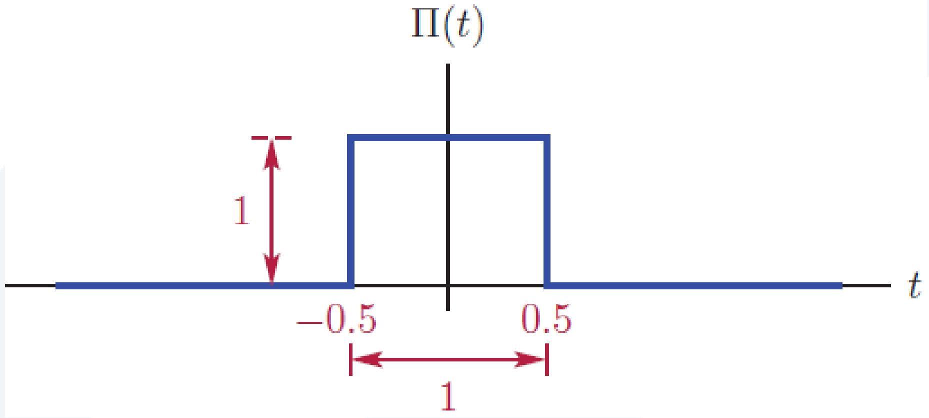


Unit-pulse function

- The unit-pulse function (also called the unit-rectangular pulse function), denoted rect, is given by:

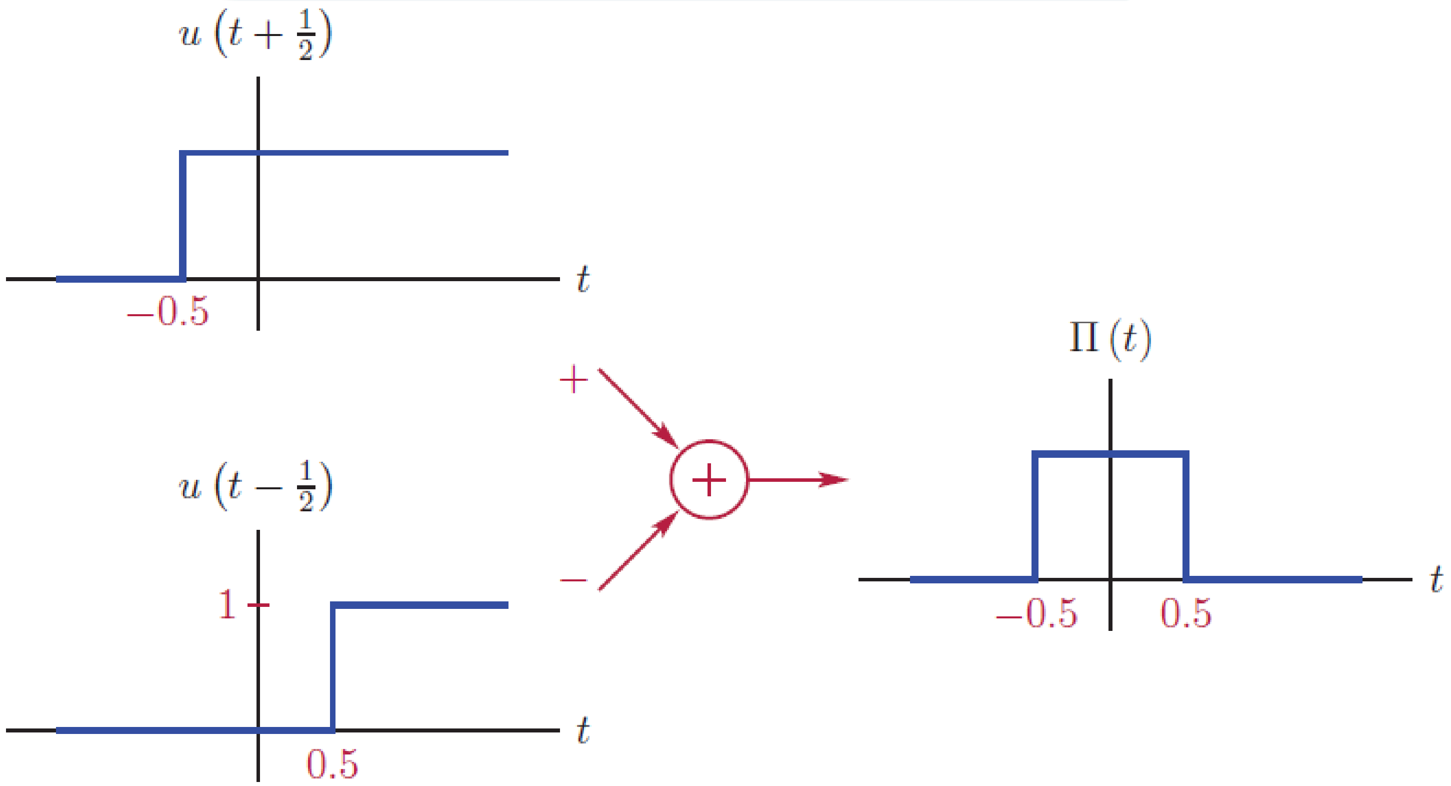
$$\operatorname{rect} t = \prod(t) = \begin{cases} 1, & \text{if } -\frac{1}{2} < t < \frac{1}{2} \\ 0, & \text{otherwise} \end{cases}$$

- Due to the manner in which the rect function is used in practice, the actual value of $\operatorname{rect} t$ at $t = \pm ½$ is unimportant. Sometimes ≠ values are used.



- Constructing a unit-pulse function from unit-step functions:

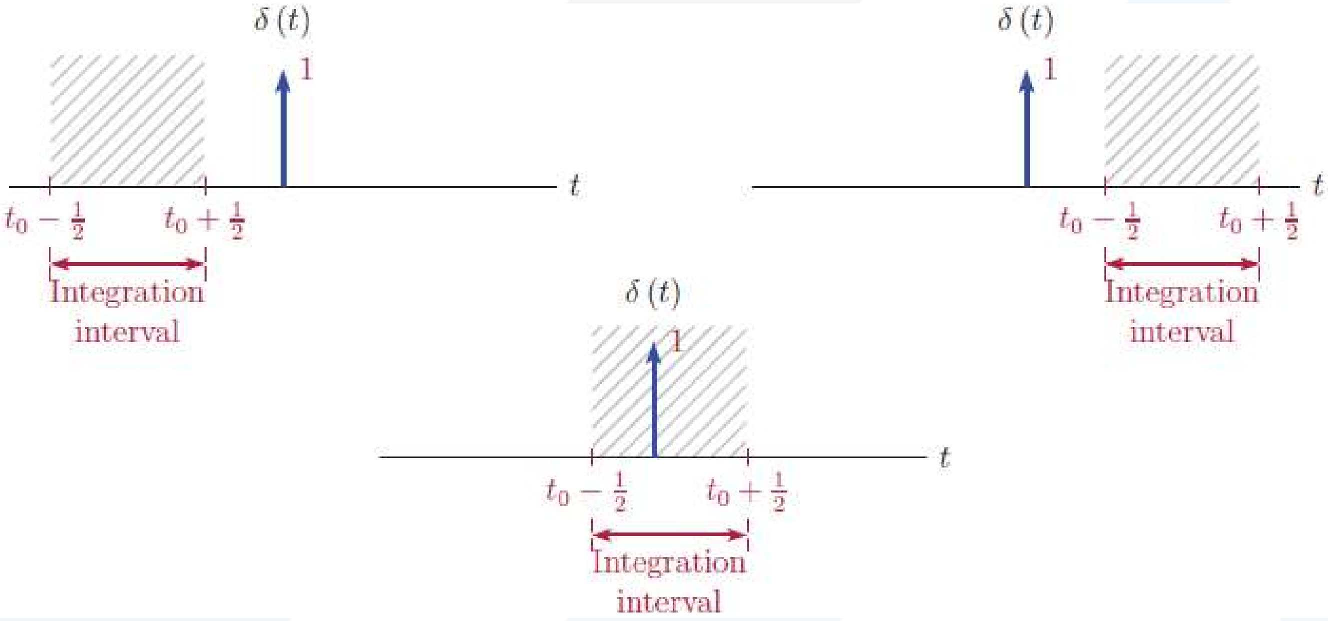
$$\prod(t) = u(t+\tfrac{1}{2}) - u(t-\tfrac{1}{2})$$



- Constructing a unit-pulse function from unit- impulse functions:

$$\prod(t) = u(t+\tfrac{1}{2}) - u(t-\tfrac{1}{2}) = \int_{-\infty}^{t+1/2} \delta(\tau)d\tau - \int_{-\infty}^{t-1/2} \delta(\tau)d\tau = \int_{t-1/2}^{t+1/2} \delta(\tau)d\tau$$

$$\int_{t-1/2}^{t+1/2} \delta(\tau)d\tau = \begin{cases} 1, & t-\frac{1}{2} < 0 \text{ and } t+\frac{1}{2} > 0 \\ 0, & \text{otherwise} \end{cases} = \begin{cases} 1, & -\frac{1}{2} < t < \frac{1}{2} \\ 0, & \text{otherwise} \end{cases}$$

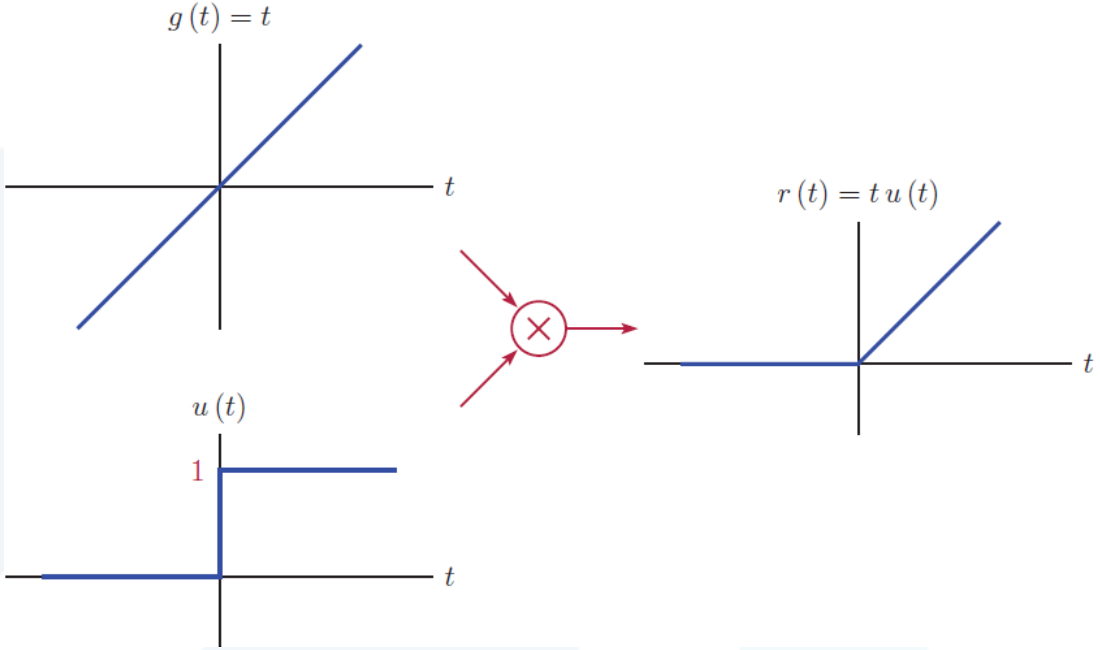


Unit-Ramp Function

- The unit-ramp function, denoted r, is defined as:

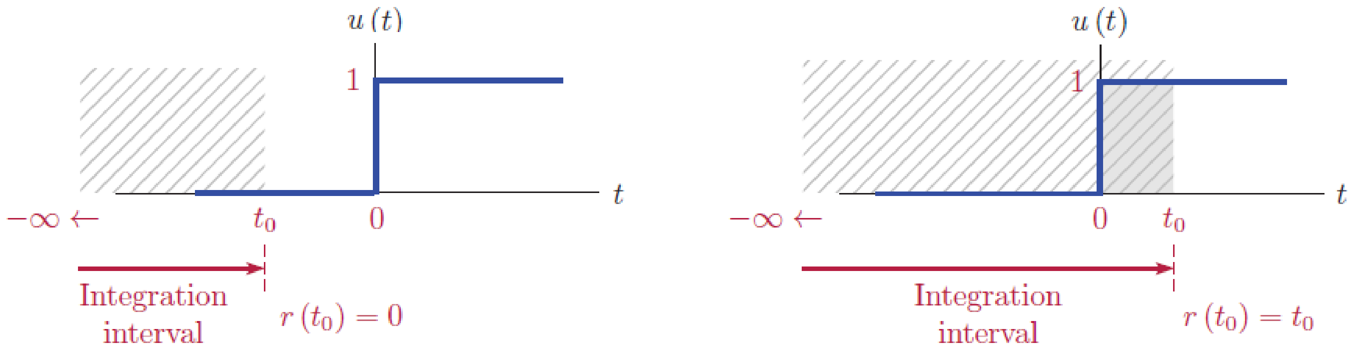
$$r(t) = \begin{cases} t, & \text{if } t \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

or, equivalently: $r(t) = tu(t)$.



- Constructing a unit-ramp function from a unit-step:

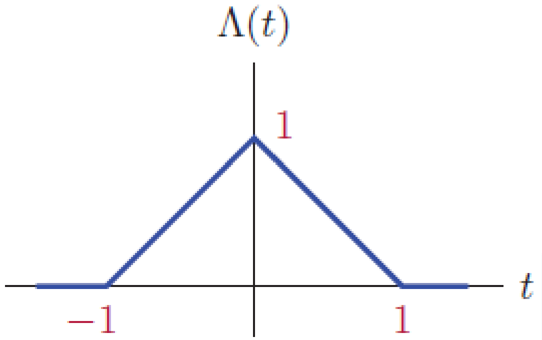
$$r(t) = \int_{-\infty}^{t} u(\tau) d\tau$$



Unit Triangular Function

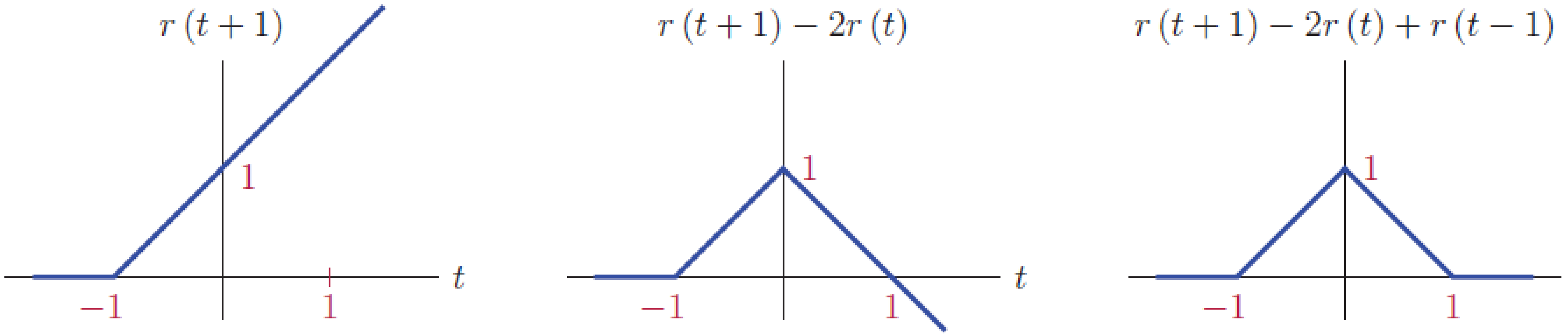
- The unit triangular function (unit-triangular pulse function), denoted tri, is defined as:

$$\mathrm{tri}\,t = \Lambda(t) = \begin{cases} 1 - |t|, & \text{if } |t| \le 1 \\ 0, & \text{otherwise} \end{cases}$$



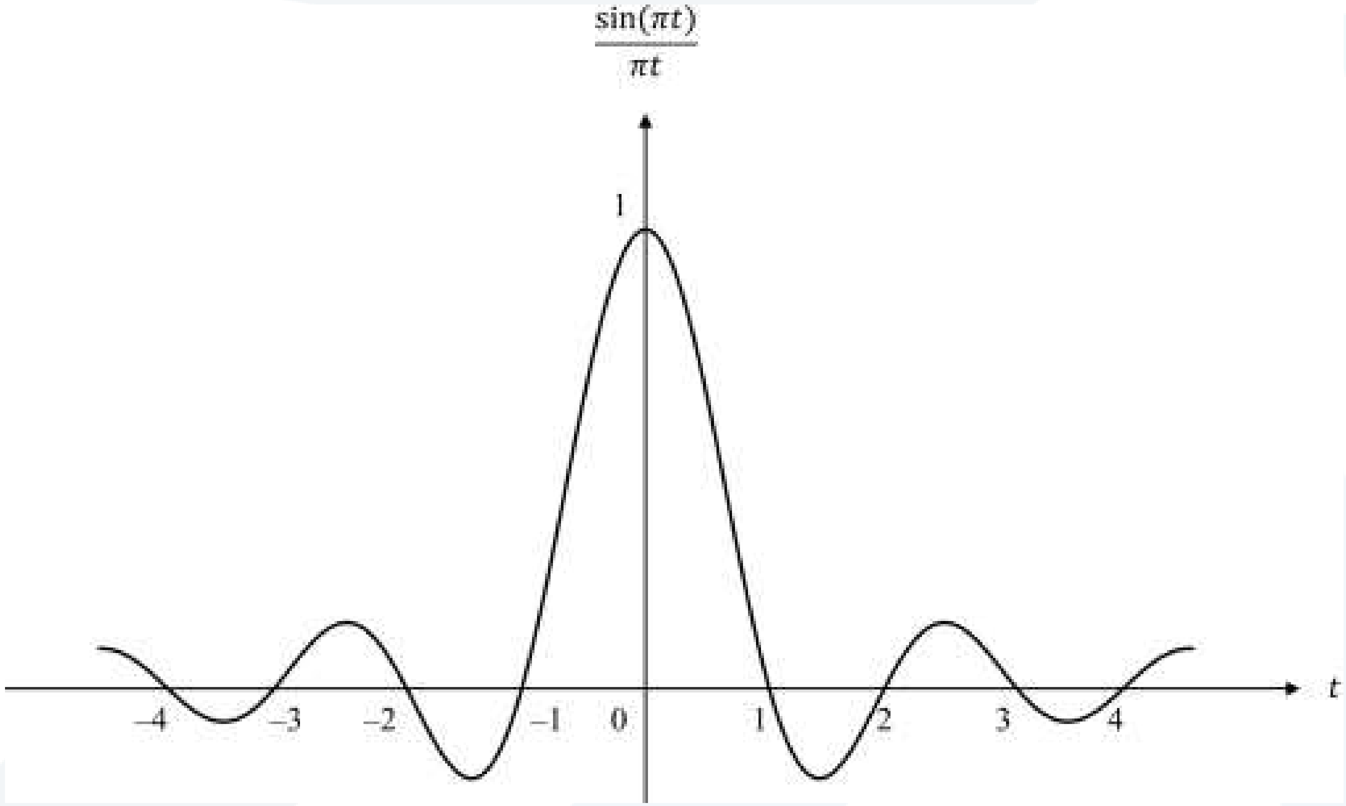
- Constructing a unit-triangle using unit-ramp functions:

$$\Lambda(t) = r(t+1) - 2r(t) + r(t-1)$$



Cardinal Sine Function

- The cardinal sine function, denoted sinc, is given by $\operatorname{sinc} t = \dfrac{\sin(\pi t)}{\pi t}$



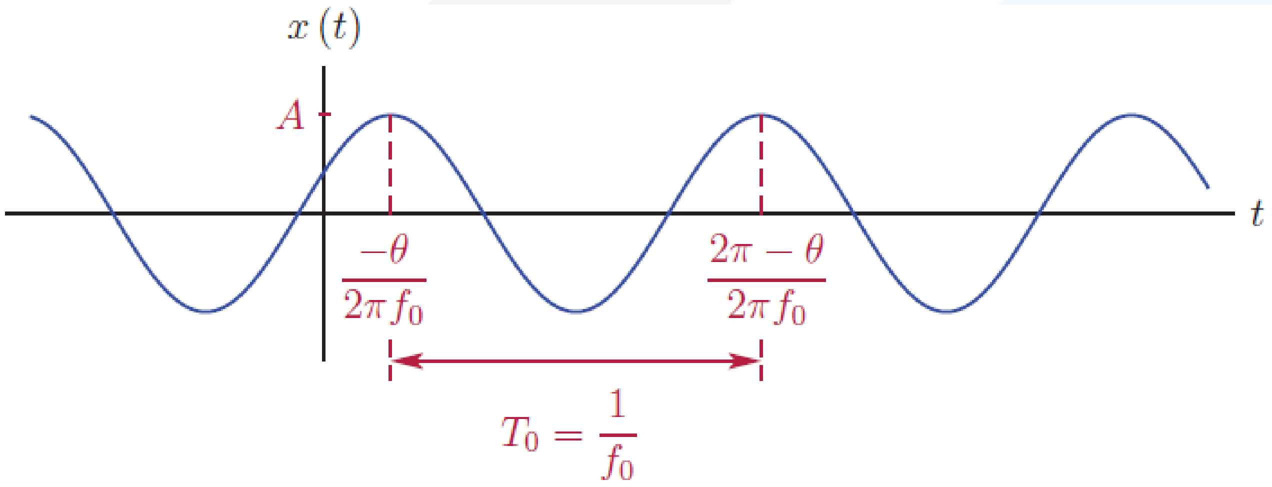
Sinusoidal Signal

- A real sinusoidal function is a function of the form:

$$x(t) = A\cos(\omega_0 t + \theta)$$

where A is the amplitude of the signal, ω_0 is the radian frequency (rad/s), and θ is the initial phase angle (rad), all are real constants.

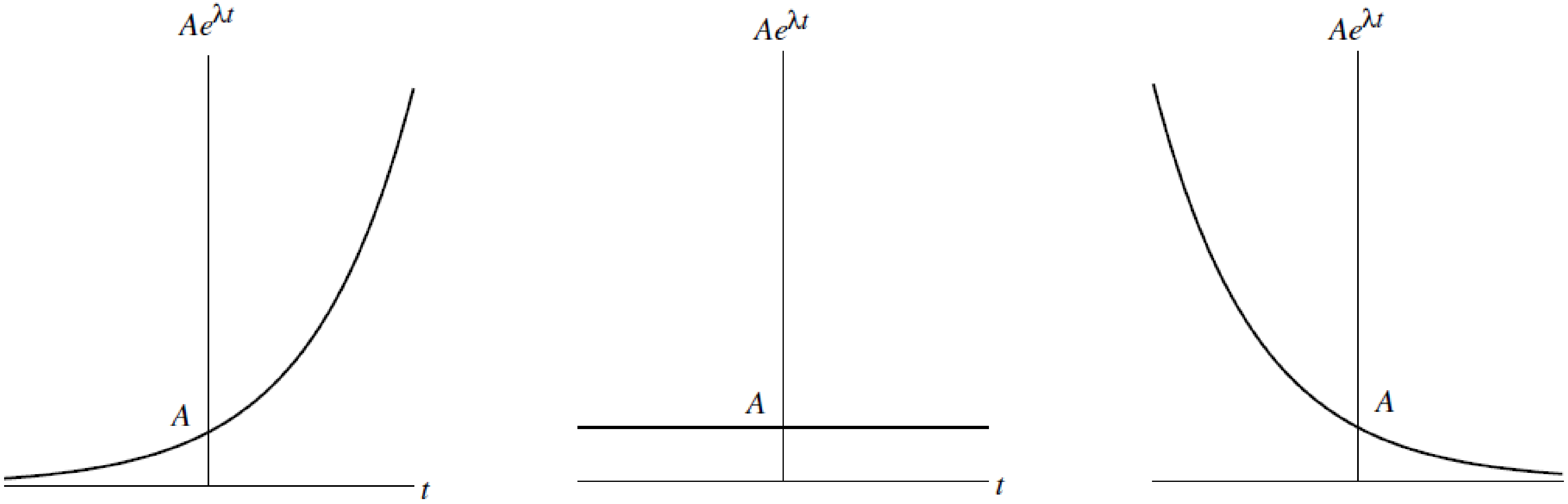
$\omega_0 = 2\pi f_0$ where f_0 is the frequency (Hz), $T_0 = 1/f_0$ is the period (s).



Complex Exponential Function

- A complex exponential function is a function of the form $x(t) = Ae^{\lambda t}$, where A and λ are complex constants.
- A complex exponential can exhibit one of a number of distinct modes of behavior, depending on the values of A and λ.
- For example, as special cases, complex exponentials include real exponentials and complex sinusoids.
- A real exponential function is a special case of a complex exponential $x(t) = Ae^{\lambda t}$, where A and λ are restricted to be real numbers.
- A real exponential can exhibit one of three distinct modes of behavior, depending on the value of λ, as illustrated below.

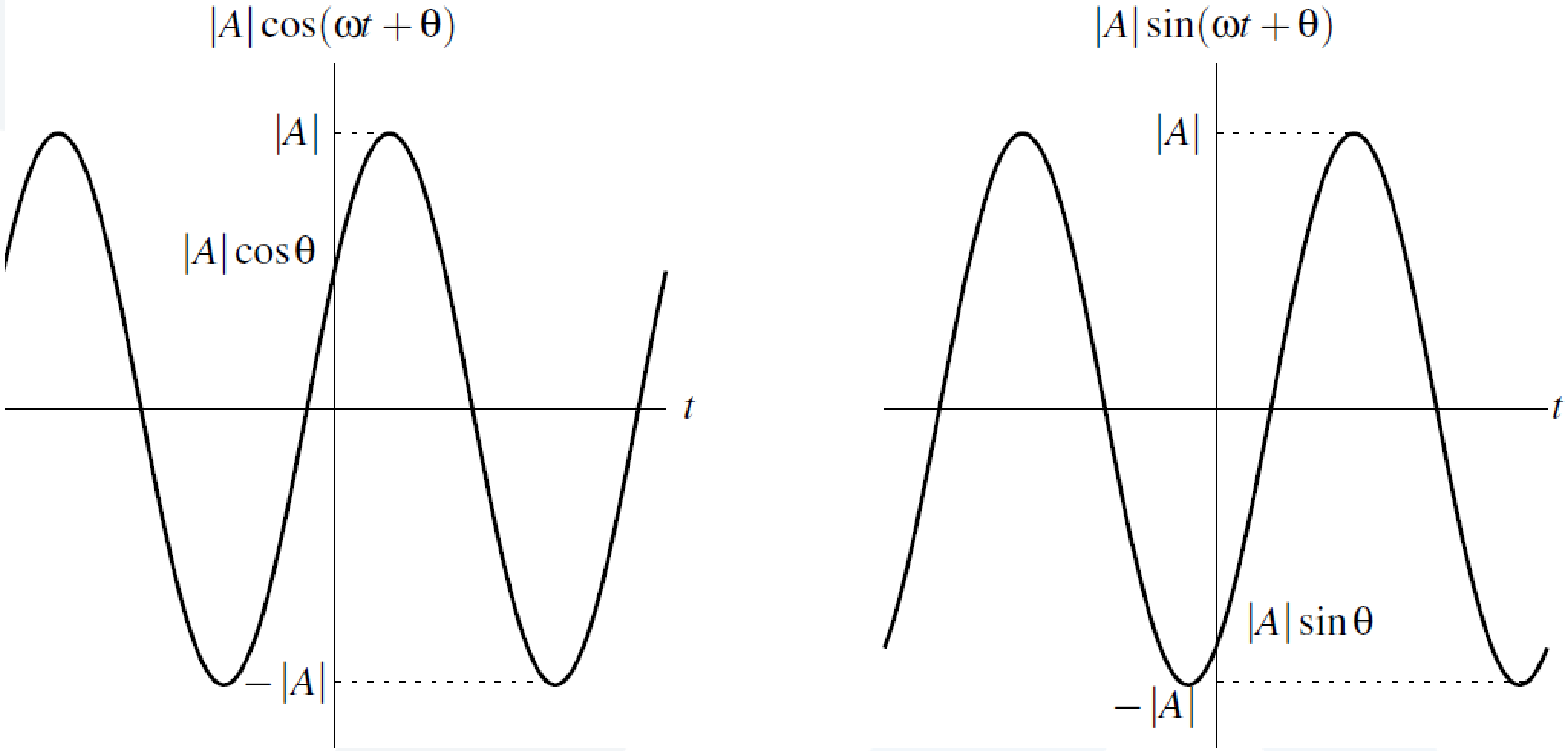
- If $\lambda > 0$, $x(t)$ increases exponentially as t increases (growing exponential).
- If $\lambda < 0$, $x(t)$ decreases exponentially as t increases (decaying exponential).
- If $\lambda = 0$, $x(t)$ simply equals the constant A.



$\lambda > 0$ $\lambda = 0$ $\lambda < 0$

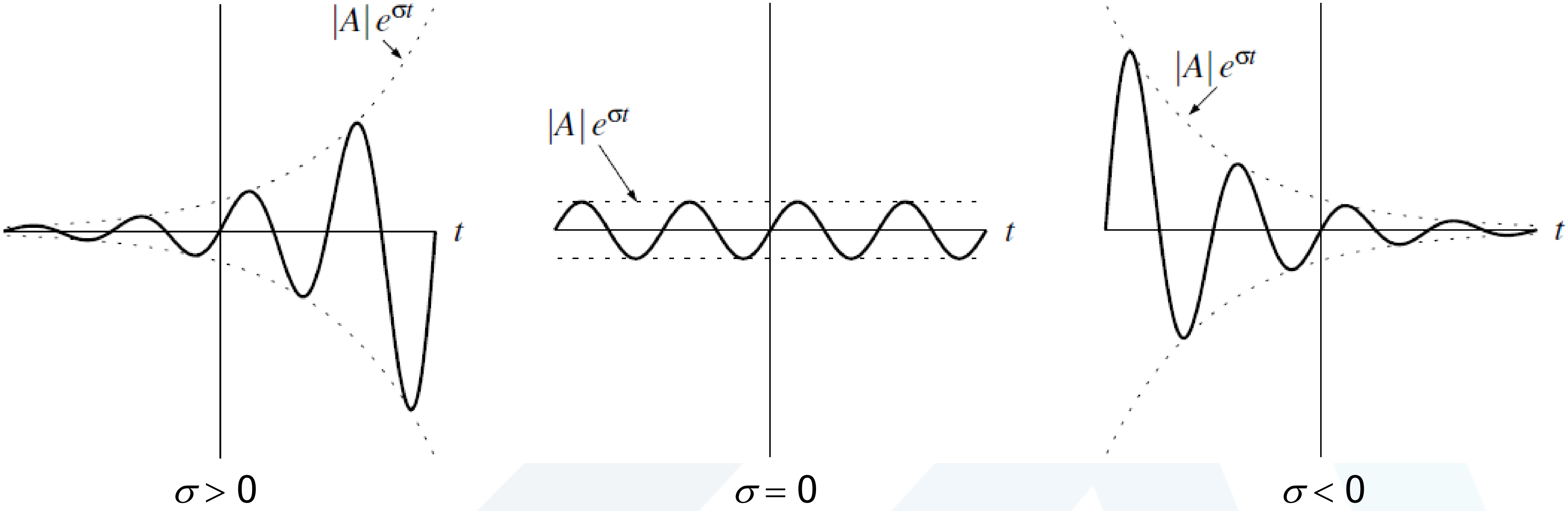
Complex Sinusoidal Function

- A complex sinusoidal function is a special case of a complex exponential $x(t) = Ae^{\lambda t}$, where A is complex and λ is purely imaginary (i.e., Re$\{\lambda\} = 0$).
- That is, a complex sinusoidal function is a function of the form $x(t) = Ae^{j\omega t}$, where A is complex and ω is real.
- By expressing A in polar form as $A = |A|e^{j\theta}$ (where θ is real) and using Euler's relation, we can rewrite $x(t)$ as: $x(t) = \underbrace{|A|\cos(\omega t + \theta)}_{\text{Re}\{x(t)\}} + j\underbrace{|A|\sin(\omega t + \theta)}_{\text{Im}\{x(t)\}}$
- Thus, Re$\{x\}$ and Im$\{x\}$ are the same except for a time shift.
- Also, x is periodic with fundamental period $T = 2\pi/|\omega|$ and fundamental frequency $|\omega|$.



- In the most general case of a complex exponential function $x(t) = Ae^{\lambda t}$, A and λ are both complex.

- Letting $A = |A|e^{j\theta}$ and $\lambda = \sigma + j\omega$ (where θ, σ, and ω are real), and using Euler's relation, we can rewrite $x(t)$ as: $x(t) = \underbrace{|A|e^{\sigma t}\cos(\omega t + \theta)}_{\mathrm{Re}\{x(t)\}} + j\underbrace{|A|e^{\sigma t}\sin(\omega t + \theta)}_{\mathrm{Im}\{x(t)\}}$
- Three distinct modes depending on the value of σ:
  - If $\sigma = 0$, Re{x} and Im{x} are real sinusoids.
  - If $\sigma > 0$, Re{x} and Im{x} are each the product of a real sinusoid and a growing real exponential.
  - If $\sigma < 0$, Re{x} and Im{x} are each the product of a real sinusoid and a decaying real exponential.
- From Euler's relation, a complex sinusoid can be expressed as the sum of two real sinusoids as:



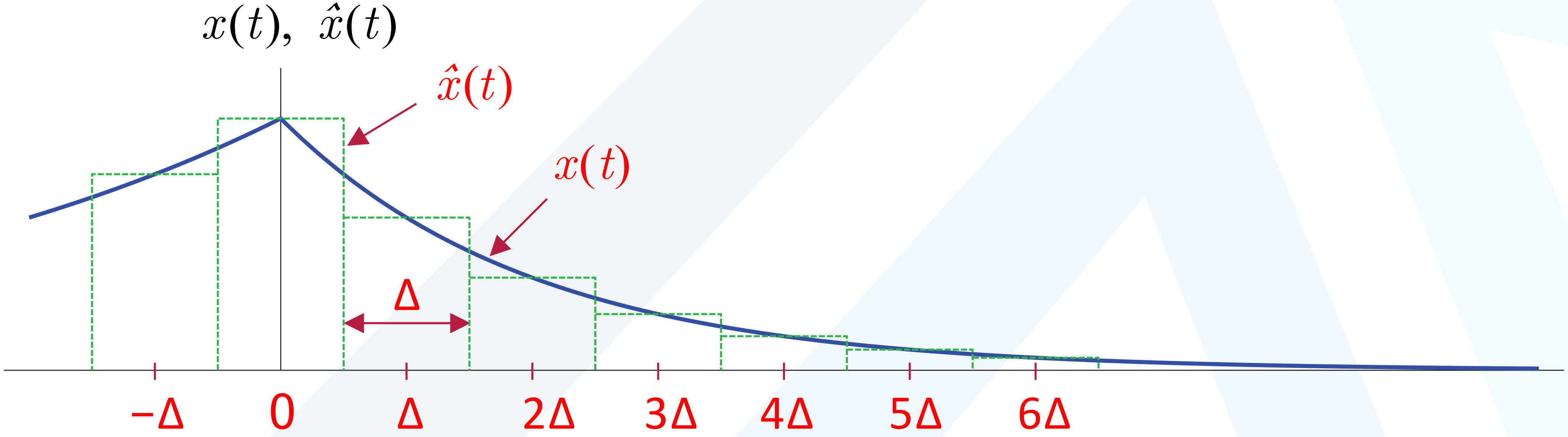
- Moreover, a real sinusoid can be expressed as the sum of two complex sinusoids using the identities:

$A\cos(\omega t+\theta)=\frac{A}{2}[e^{j(\omega t+\theta)}+e^{-j(\omega t+\theta)}]$ and $A\sin(\omega t+\theta)=\frac{A}{2}[e^{j(\omega t+\theta)}-e^{-j(\omega t+\theta)}]$

Impulse decomposition for continuous-time signals

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$$

$$\hat{x}(t) = \sum_{n=-\infty}^{\infty} x(n\Delta)\prod\left(\frac{t-n\Delta}{\Delta}\right)$$



Energy and power definitions

- The energy of a continuous time signal $x(t)$ is given by: $E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt$
- The average power of a continuous time signal $x(t)$ is given by:

  periodic complex signal: $P_x = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} |x(t)|^2 dt$

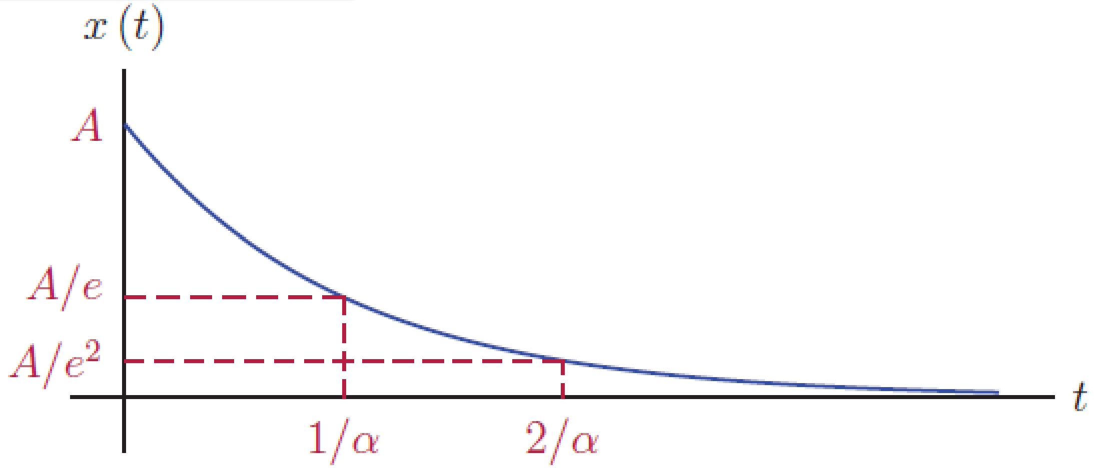
  non-periodic complex signal: $P_x = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt$
- Energy signals are those that have finite energy and zero power, i.e., $E_x < \infty$, and $P_x = 0$.
- Power signals are those that have finite power and infinite energy, i.e., $E_x \to \infty$, and $P_x < \infty$.

- Example 1: Energy of exponential signal

  Compute the energy of the exponential signal (where $\alpha > 0$).

$$x(t) = \begin{cases} A e^{-\alpha t} & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$E_x = \int_0^{\infty} A^2 e^{-2\alpha t} dt = \frac{A^2}{2\alpha}$$



- Example 2: Power of a sinusoidal signal

$$x(t) = A \sin(2\pi f_0 t + \theta)$$

$$P_x = f_0 \int_{-1/2f_0}^{1/2f_0} A^2 \sin^2(2\pi f_0 t + \theta) dt = \frac{A^2}{2}$$

Symmetry properties

Even and odd symmetry

- A real-valued signal is said to have even symmetry if it has the property: $x(-t) = x(t)$ for all values of t.
- A real-valued signal is said to have odd symmetry if it has the property: $x(-t) = -x(t)$ for all values of t.

Decomposition into even and odd components

- Every real-valued signal $x(t)$ has a unique representation of the form: $x(t) = x_e(t) + x_o(t)$; where the signals x_e and x_o are even and odd, respectively.
- In particular, the signals x_e and x_o are given by:

$$x_e(t) = ½[x(t) + x(-t)] \text{ and } x_o(t) = ½[x(t) - x(-t)]$$

Symmetry properties for complex signals

- A complex-valued signal is said to have conjugate symmetric if it has the property: $x(-t) = x^*(t)$ for all values of t.
- A complex-valued signal is said to have conjugate antisymmetric if it has the property: $x(-t) = -x^*(t)$ for all values of t.

Decomposition of complex signals

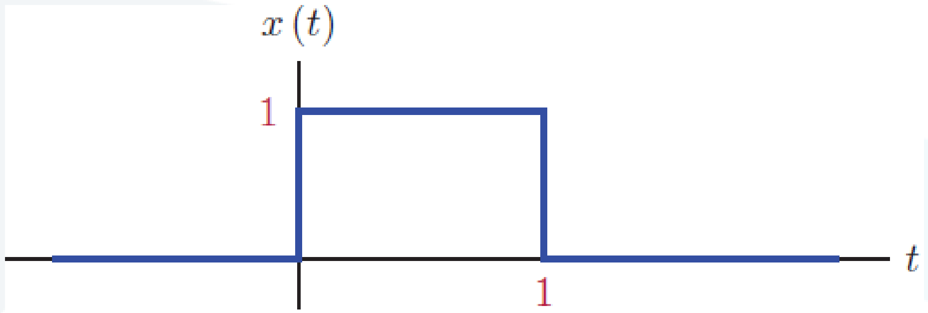
- Every complex-valued signal $x(t)$ has a unique representation of the form: $x(t) = x_E(t) + x_O(t)$; where the signals x_E and x_O are conjugate symmetric and conjugate antisymmetric, respectively.
- In particular, the signals x_E and x_O are given by:

$$x_E(t) = ½[x(t) + x^*(-t)] \text{ and } x_O(t) = ½[x(t) - x^*(-t)]$$

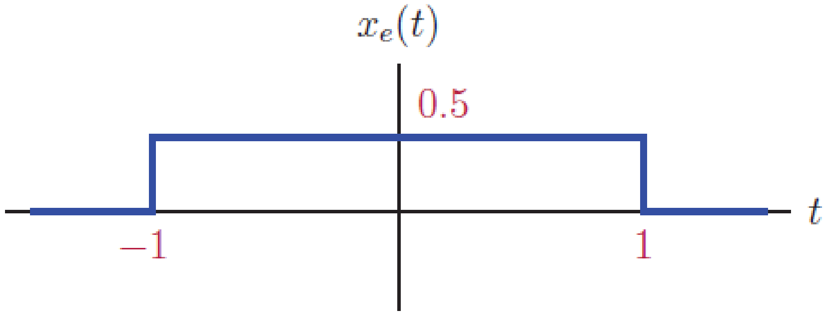
- Example 3: Even and odd components of a rectangular pulse

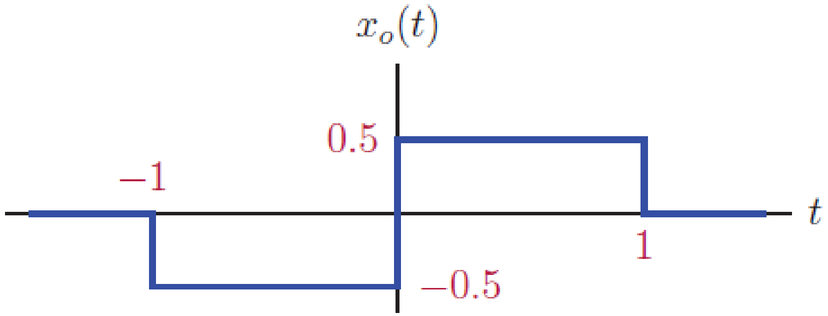
Determine the even and the odd components of the rectangular pulse signal.

$$\Pi\left(t-\tfrac{1}{2}\right)=\begin{cases}1 & \text{if } 0<t<1\\ 0 & \text{otherwise}\end{cases}$$



$$x_e(t)=\frac{\Pi\left(t-\frac{1}{2}\right)+\Pi\left(-t-\frac{1}{2}\right)}{2}=\frac{\Pi(t/2)}{2},\qquad x_o(t)=\frac{\Pi\left(t-\frac{1}{2}\right)-\Pi\left(-t-\frac{1}{2}\right)}{2}$$



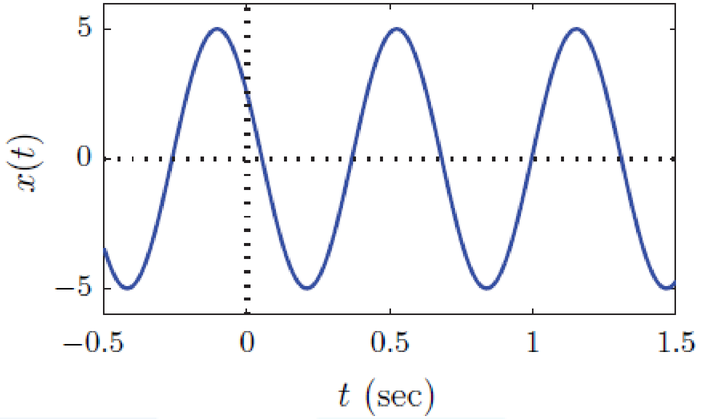


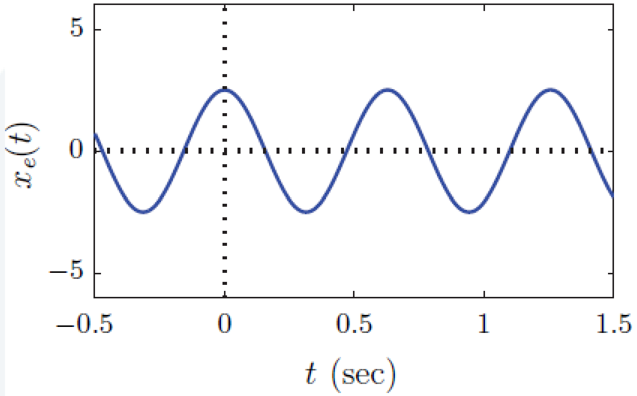
- Example 4: Even and odd components of a sinusoidal signal

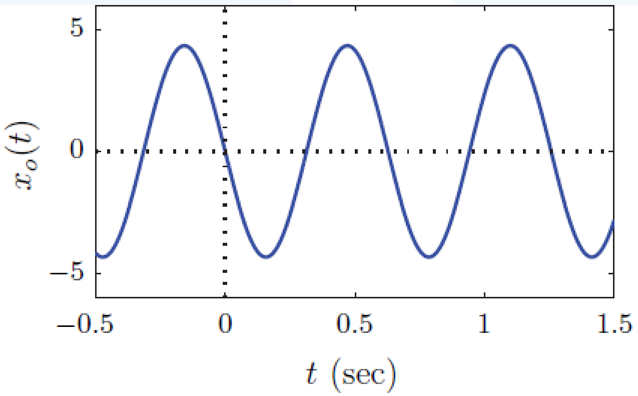
  Determine the even and the odd components of the sinusoidal signal $x(t) = 5\cos(10t + \pi/3)$.

$$x_e(t) = \frac{5}{2}\cos(10t + \pi/3) + \frac{5}{2}\cos(-10t + \pi/3)$$
$$= \frac{5}{2}\cos(10t)$$

$$x_o(t) = \frac{5}{2}\cos(10t + \pi/3) - \frac{5}{2}\cos(-10t + \pi/3)$$
$$= -\frac{5\sqrt{3}}{2}\sin(10t)$$







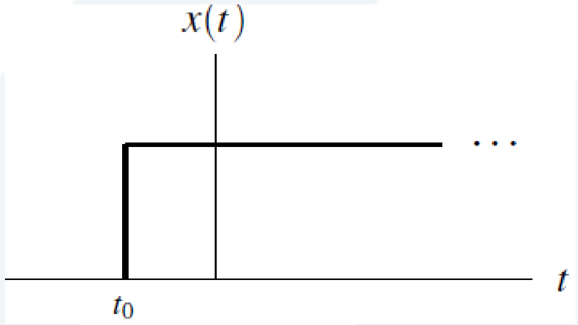
- Example 5: Symmetry of a complex exponential signal

  Consider the complex exponential signal $x(t) = Ae^{j\omega t}$, A: real

  $$x(-t) = Ae^{-j\omega t}) = (Ae^{j\omega t})^* = x^*(t) \Rightarrow \text{the signal } x(t) \text{ is conjugate symmetric.}$$

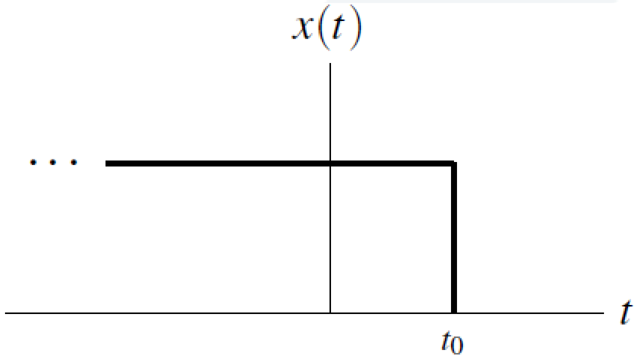
Right and Left-Sided Signals

- A signal x is said to be right sided if, for some (finite) real constant t_0, the following condition holds: $x(t) = 0$ for all $t < t_0$ (i.e., x is only potentially nonzero to the right of t_0).



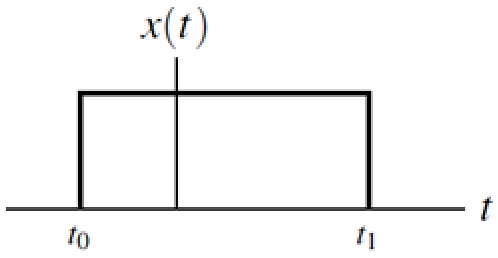
- A signal x is said to be causal if $x(t) = 0$ for all $t < 0$.

- A causal signal is a special case of a right-sided signal.
- A causal signal is not to be confused with a causal system.
- A signal x is said to be left sided if, for some (finite) real constant t_0, the following condition holds: $x(t) = 0$ for all $t > t_0$ (i.e., x is only potentially nonzero to the left of t_0).

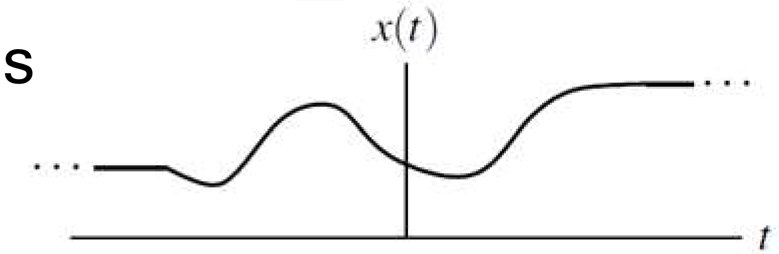


- A signal x is said to be anticausal if $x(t) = 0$ for all $t > 0$.
- An anticausal signal is a special case of a left-sided signal.
- An anticausal signal is not to be confused with a anticausal system.

Finite-Duration and Two-Sided Signals



- A signal that is both left sided and right sided is said to be finite duration (or finite support).
- A signal that is neither left sided nor right sided is said to be two sided.

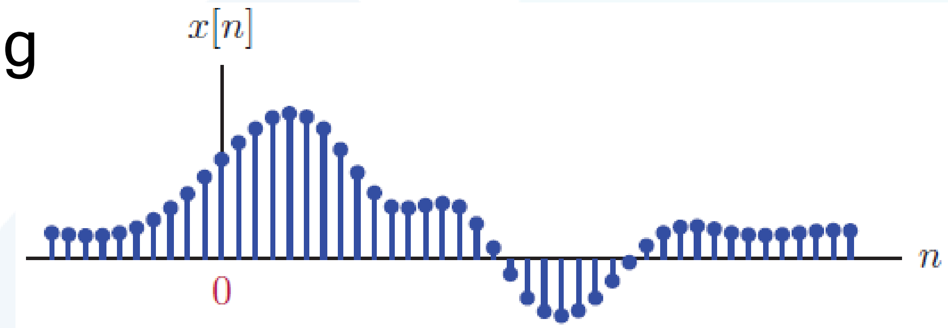


Bounded Signals

- A signal x is said to be bounded if there exists some (finite) positive real constant A such that $|x(t)| \le A$ for all t (i.e., $x(t)$ is finite for all t).
- For ex., sine and cosine signals are bounded, since $|\sin t| \le 1$ and $|\cos t| \le 1$ $\forall t$.
- In contrast, the tangent signal and any nonconstant polynomial function p (e.g., $p(t) = t^2$) are unbounded, since $\lim_{t \to \pi/2} |\tan t| = \infty$ and $\lim_{t \to \infty} |p(t)| = \infty$

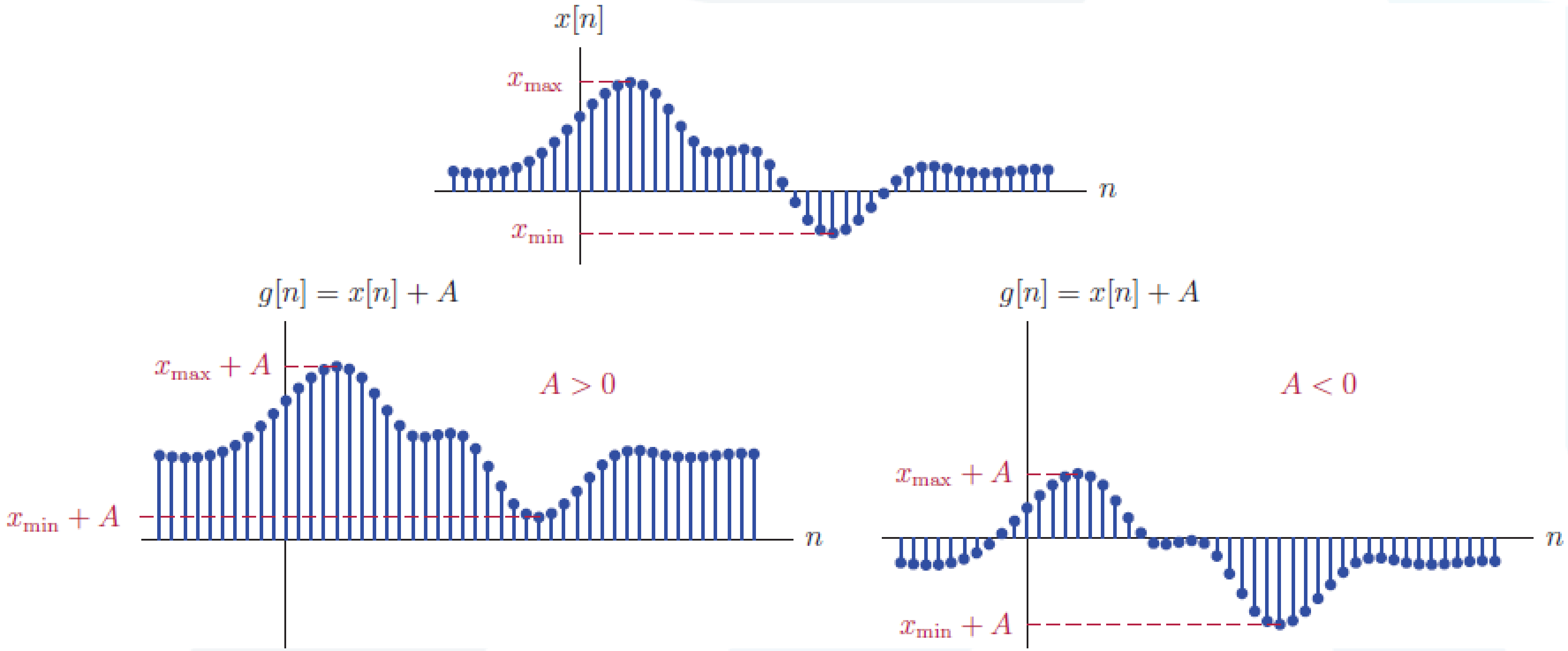
4. Discrete-Time Signals

- DT signals are not defined at all time instants. they are defined only at time instants that are integer multiples of a fixed time increment T, that is, at $t = nT$.
- Consequently, the mathematical model for a DT signal is a function $x[n]$ in which independent variable n is an integer, and is referred to as sample index.
- Sometimes DT signals are also modeled using mathematical functions: $x[n] = 3\sin[0.2n]$.
- In a DT signal the time variable is discrete, yet the amplitude is continuous.
- If, In a DT signal, we limit the amplitude values to a discrete set, the resulting signal is called a digital signal (two possible values is called binary signal).

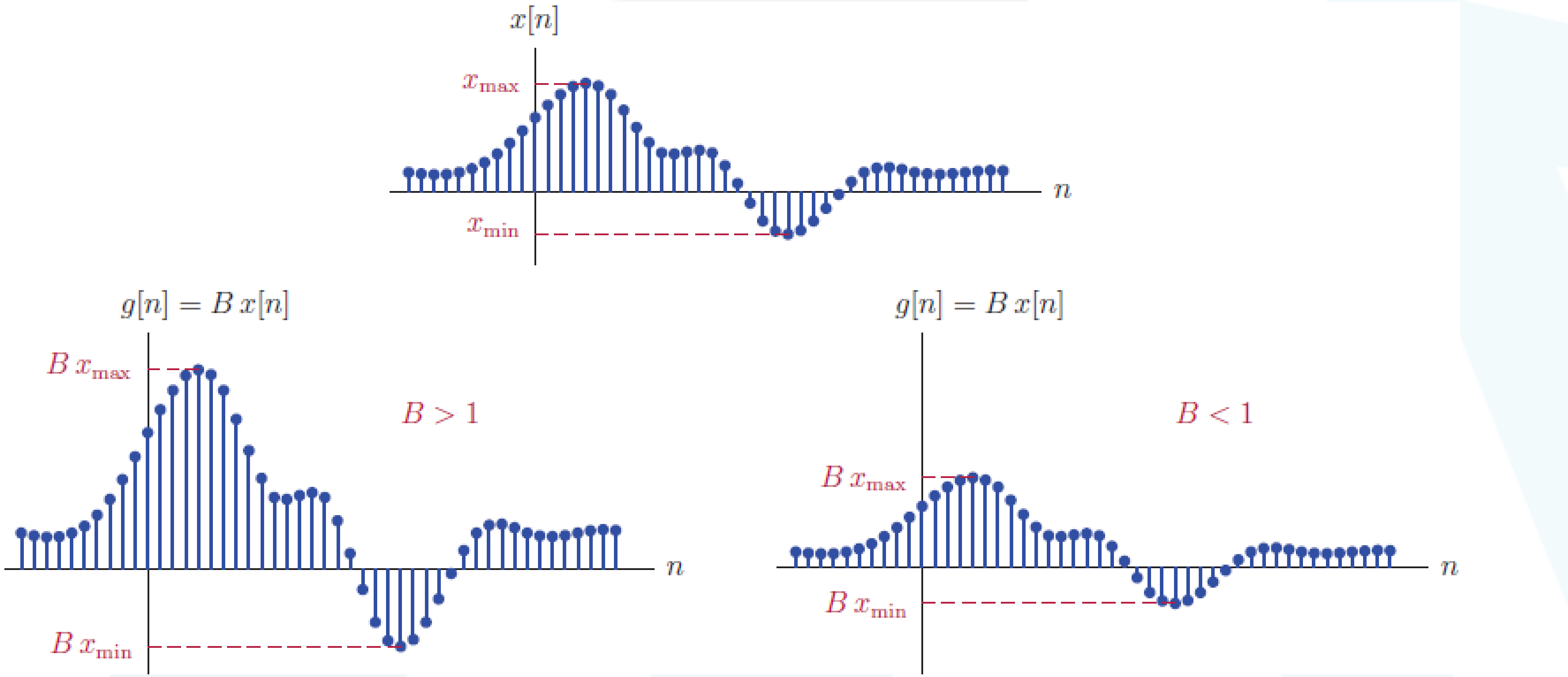


Signal operations

- Amplitude shifting maps the input signal $x[n]$ to the output signal g as given by $g[n] = x[n] + A$, where A is a real number.

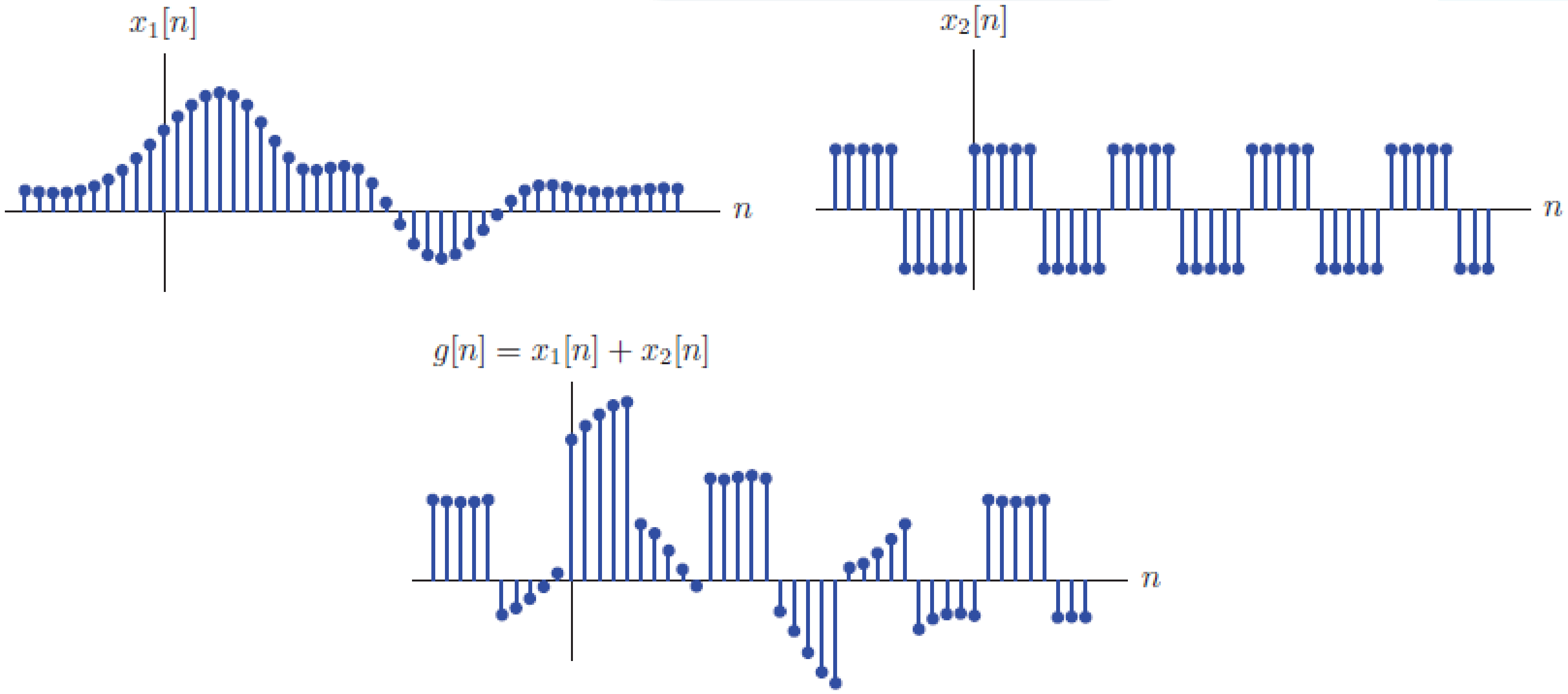


- Amplitude scaling maps the input signal x to the output signal g as given by $g[n] = Bx[n]$, where B is a real number.
- Geometrically, the output signal g is expanded/compressed in amplitude.

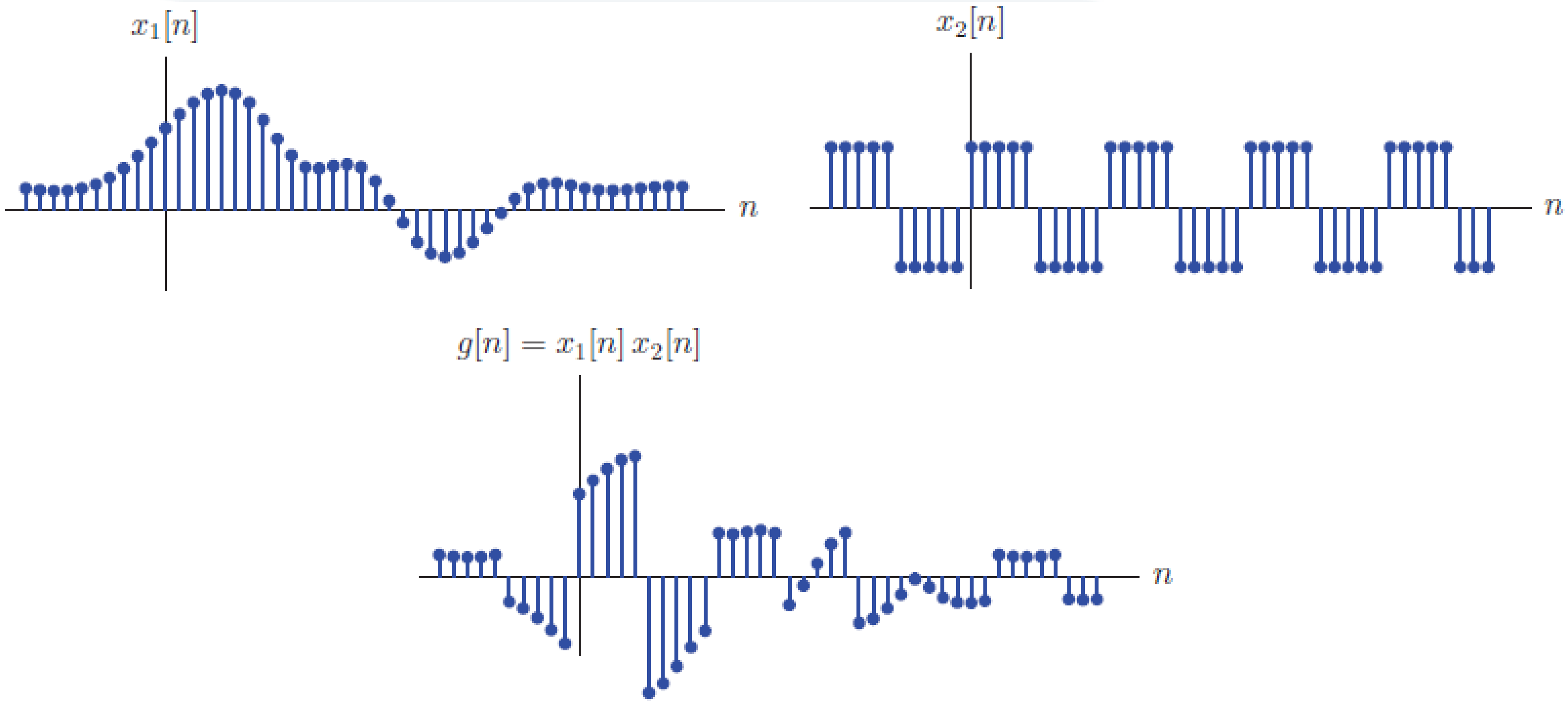


- Addition and Multiplication of two signals

  Addition of two signals is accomplished by adding the amplitudes of the two signals at each time instant. $g[n] = x_1[n] + x_2[n]$.

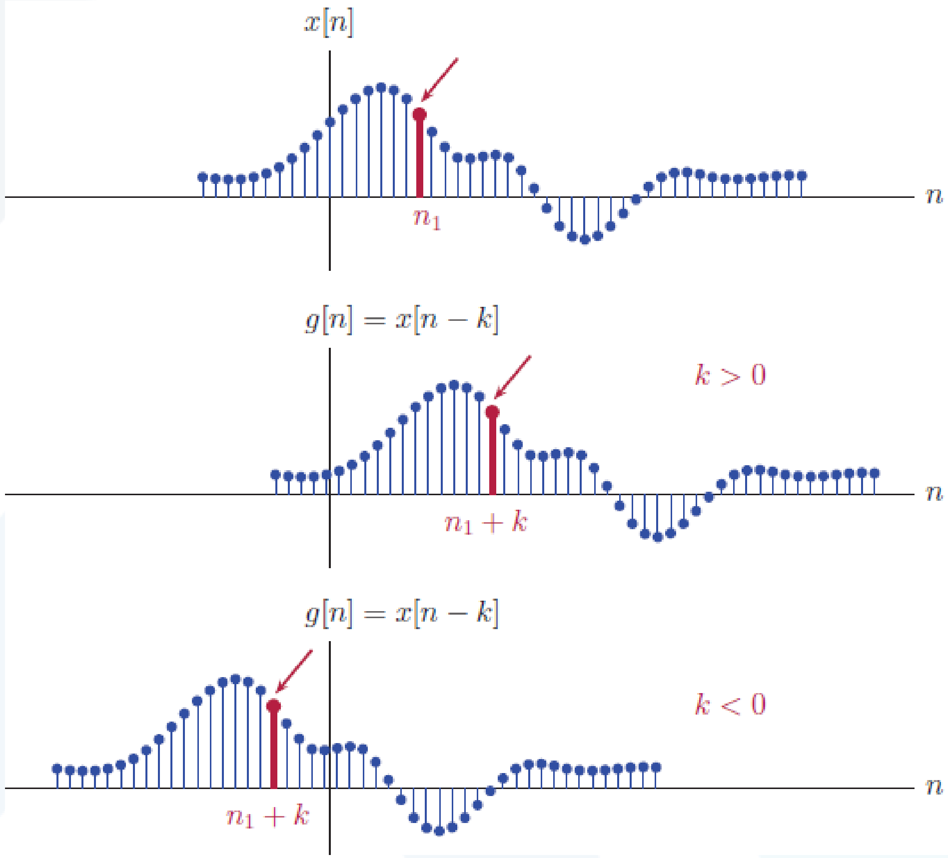


Multiplication of two signals is accomplished by multiplying the amplitudes of the two signals at each time instant. $g[n] = x_1[n]\, x_2[n]$.



- Time shifting (also called translation) maps the input signal x to the output signal g as given by: $g[n] = x[n - k]$; where k is an integer.

- Such a transformation shifts the signal (to the left or right) along the time axis.
- If $k > 0$, g is shifted to the right by $|k|$, relative to x (i.e., delayed in time).
- If $k < 0$, g is shifted to the left by $|k|$, relative to x (i.e., advanced in time).



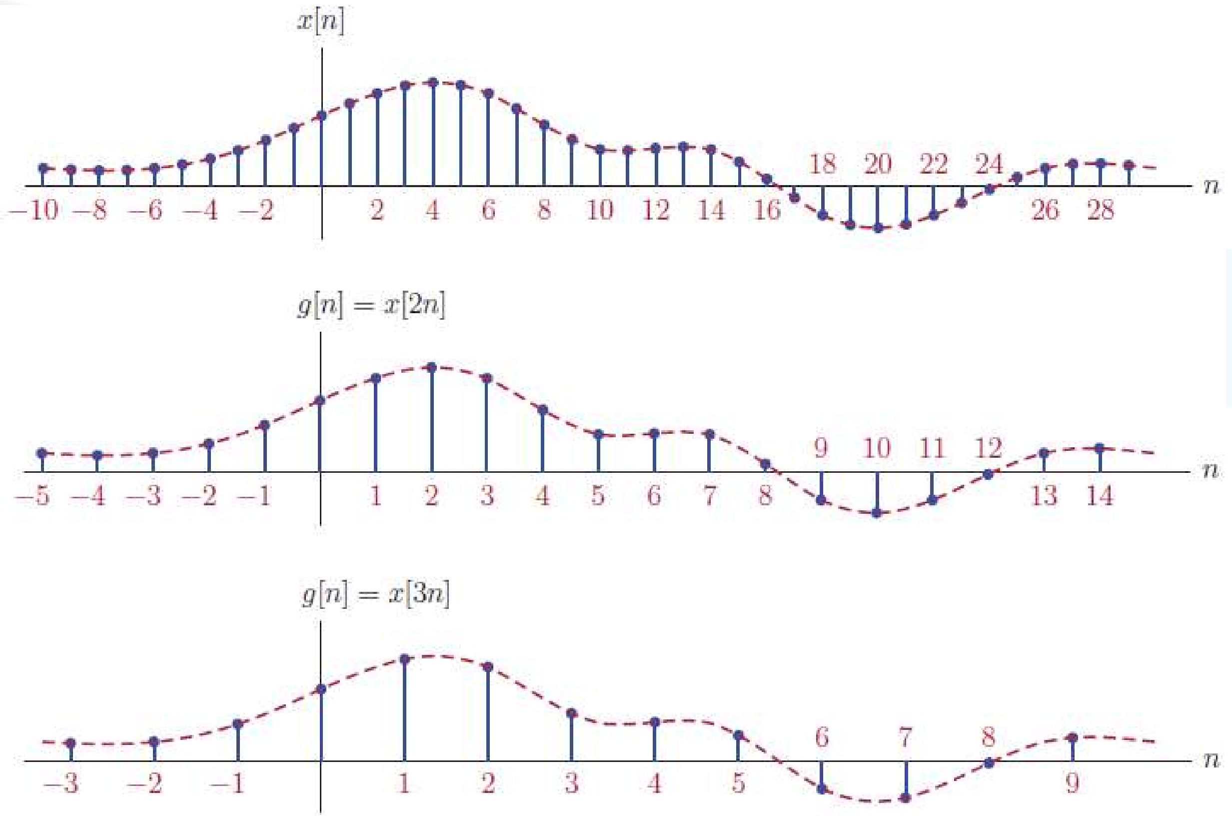
- Time scaling maps the input signal x to the output signal g as given by:

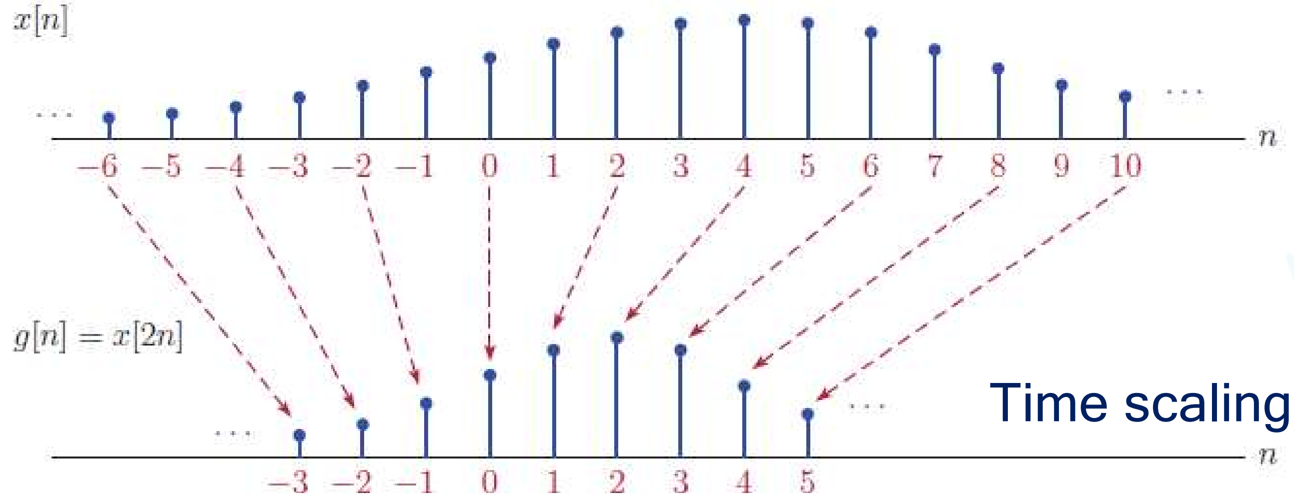
$$g[n] = x[kn]; \quad \text{downsampling}$$

and

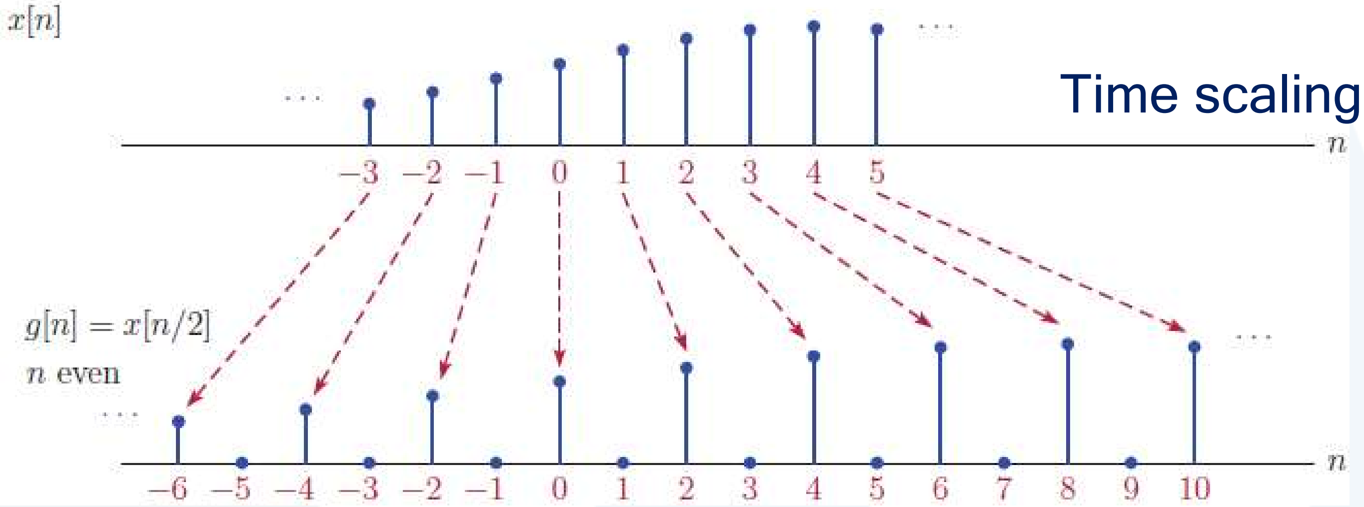
$$g[n] = x[n/k]; \text{ upsampling}$$

where k is a strictly positive integer.



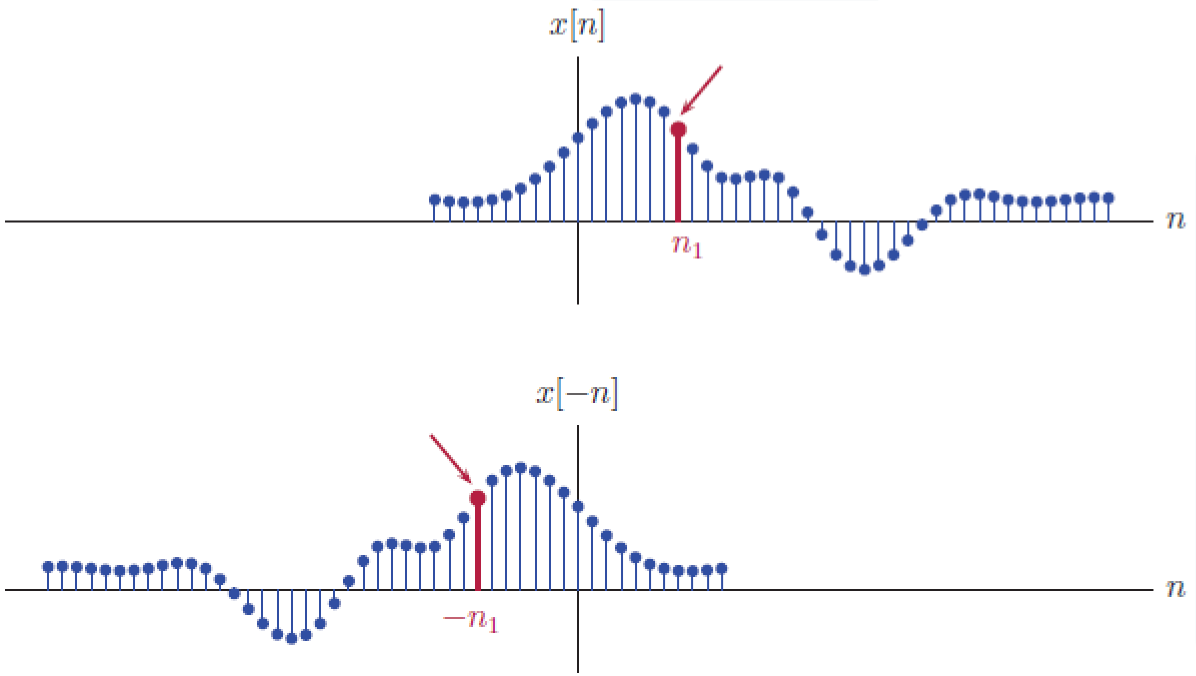


Time scaling (downsampling)



Time scaling (upsampling)

- Time reversal (also known as reflection) maps the input signal x to the output signal g as given by $g[n] = x[-n]$.
- Geometrically, the output signal g is a reflection of the input signal x about the (vertical) line $n = 0$.

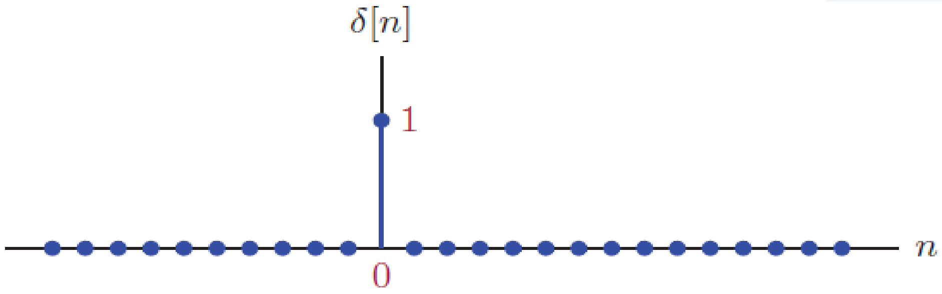


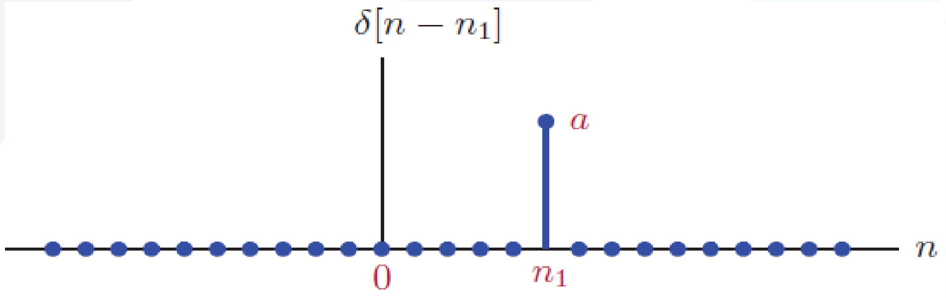
5. Basic building blocks for discrete-time signals

Unit-impulse Signal

- The unit-impulse signal, denoted δ, is defined by:

$$\delta[n] = \begin{cases} 1, & \text{if } n = 0 \\ 0, & \text{if } n \neq 0 \end{cases} \qquad a\delta[n - n_1] = \begin{cases} a, & \text{if } n = n_1 \\ 0, & \text{if } n \neq n_1 \end{cases}$$



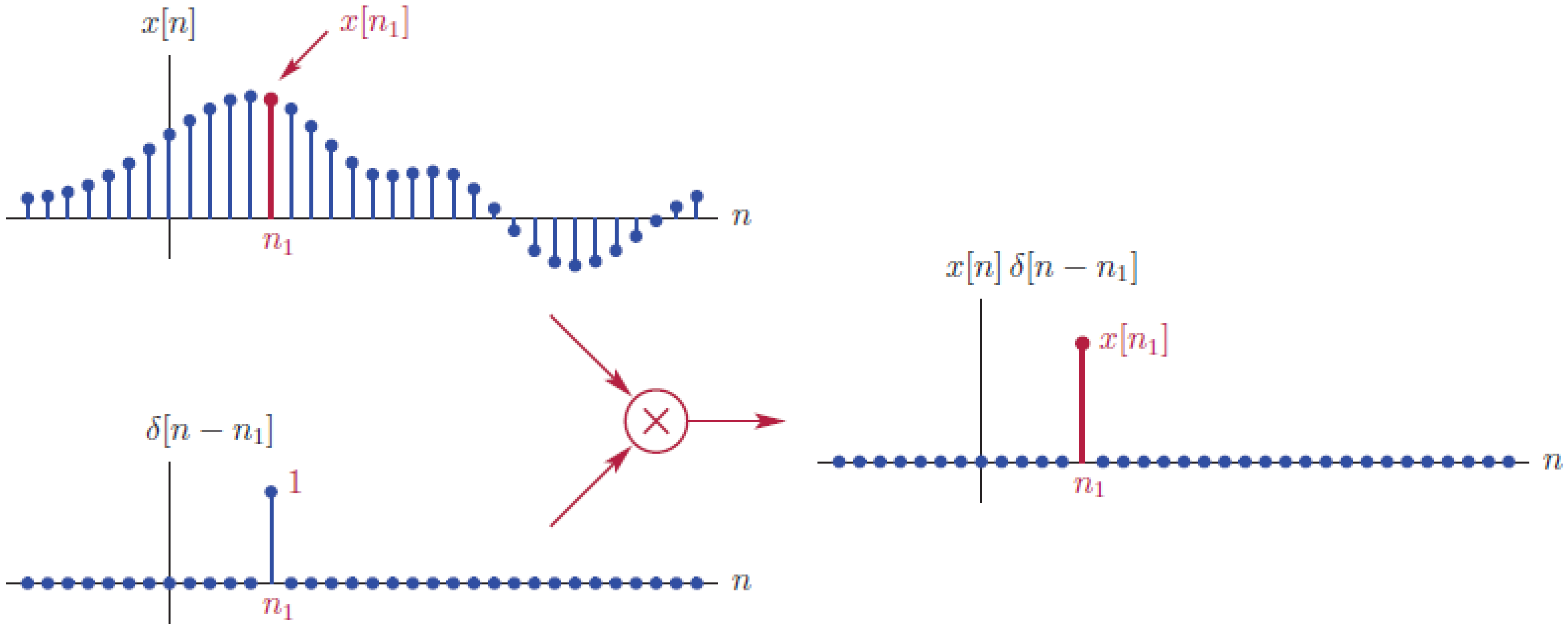


- Sampling property of the unit-impulse signal:

$$x[n]\delta[n - n_1] = x[n_1]\delta[n - n_1] = \begin{cases} x[n_1], & n = n_1 \\ 0, & n \neq n_1 \end{cases}$$

- Sifting property of the unit-impulse signal

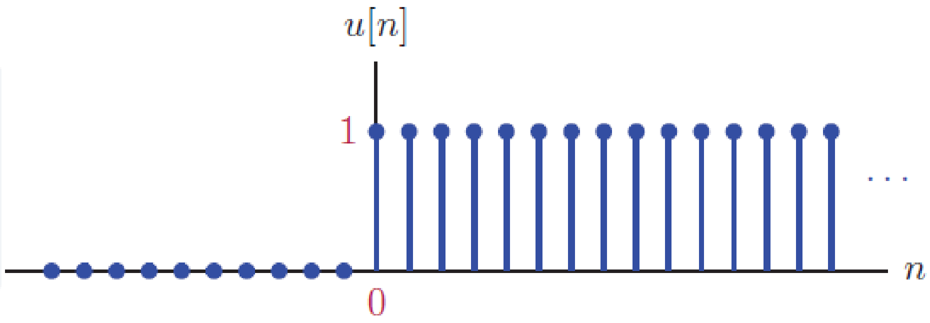
$$\sum_{n=-\infty}^{\infty} x[n]\delta[n-n_1] = x[n_1]$$



Unit-Step Signal

- The unit-step signal, denoted u, is defined as:

$$u[n] = \begin{cases} 1, & \text{if } n \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

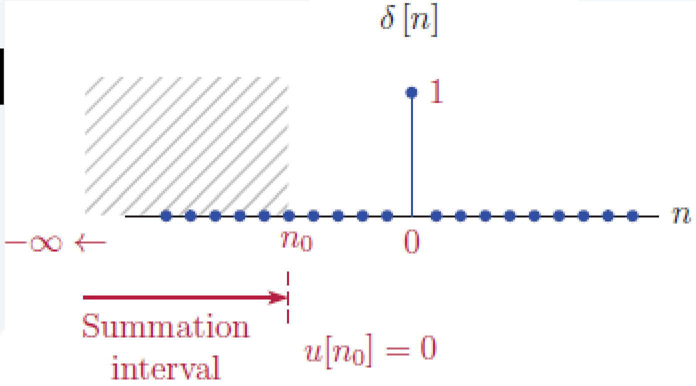


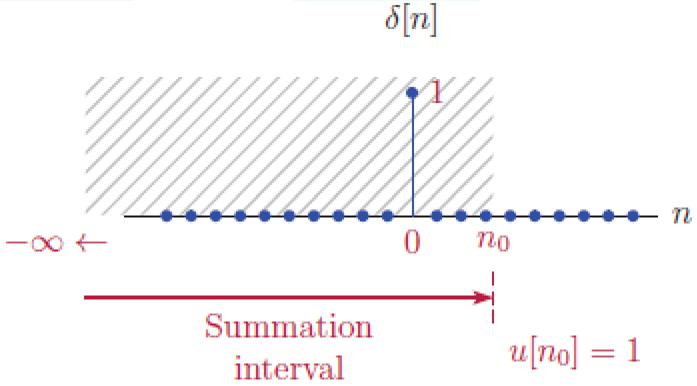
- Relationship between the unit-step signal and the unit-impulse signal:

$$\delta[n] = u[n] - u[n-1]$$

- Conversely, $u[n] = \sum_{k=-\infty}^{n} \delta[k]$

or, $u[n] = \sum_{k=0}^{\infty} \delta[n-k]$





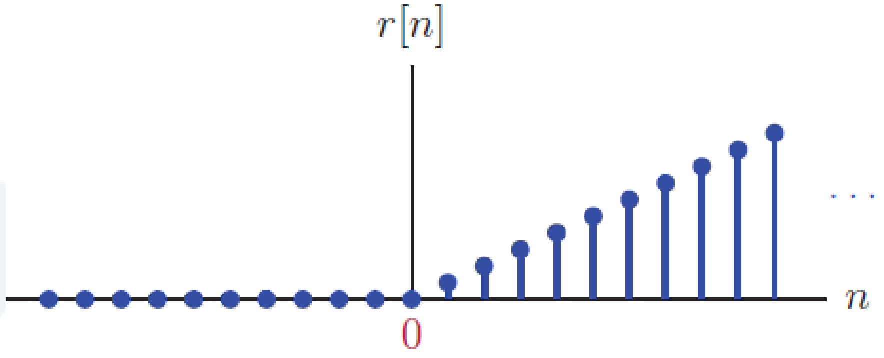
Unit-Ramp Signal

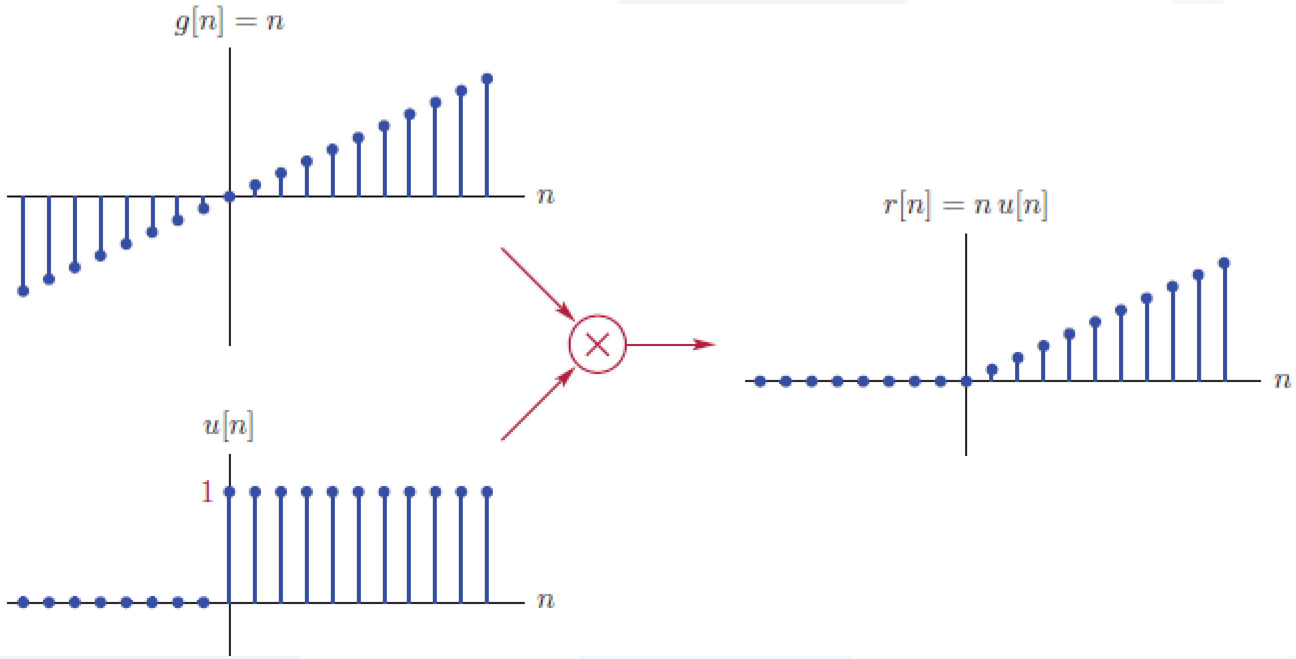
- The unit-ramp signal, denoted r, is defined as:

$$r[n] = \begin{cases} n, & \text{if } n \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

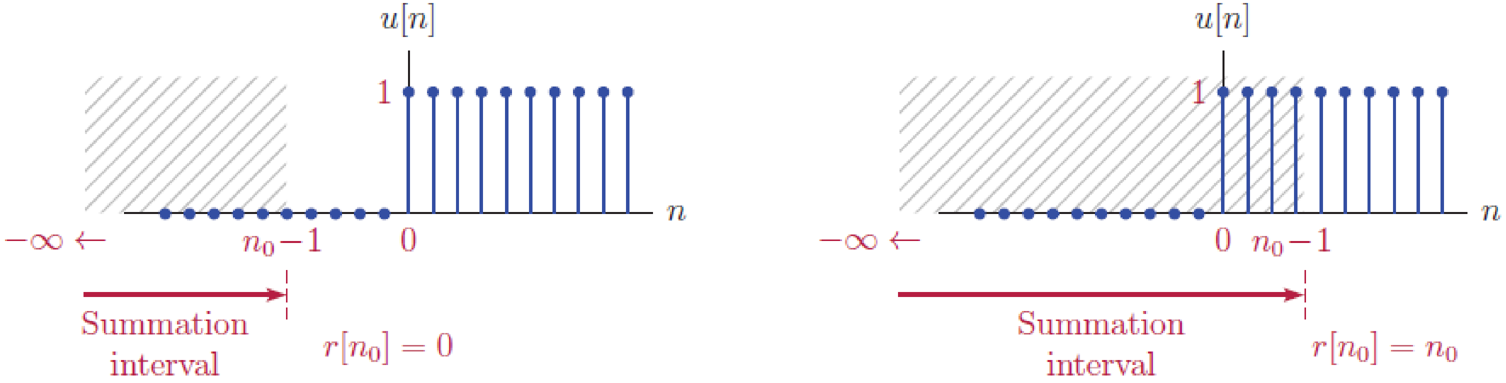
or, equivalently:

$$r[n] = nu[n]$$



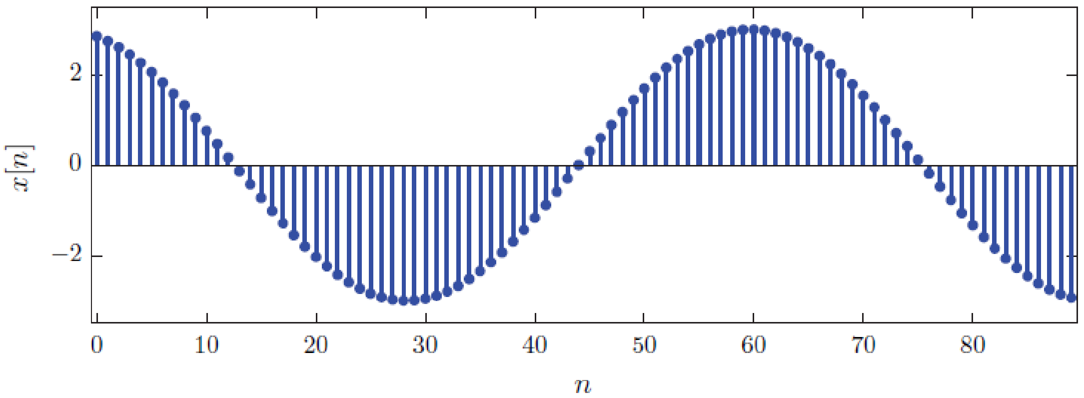


- Constructing a unit-ramp from a unit-step $r[n] = \sum_{n=-\infty}^{n-1} u[k]$

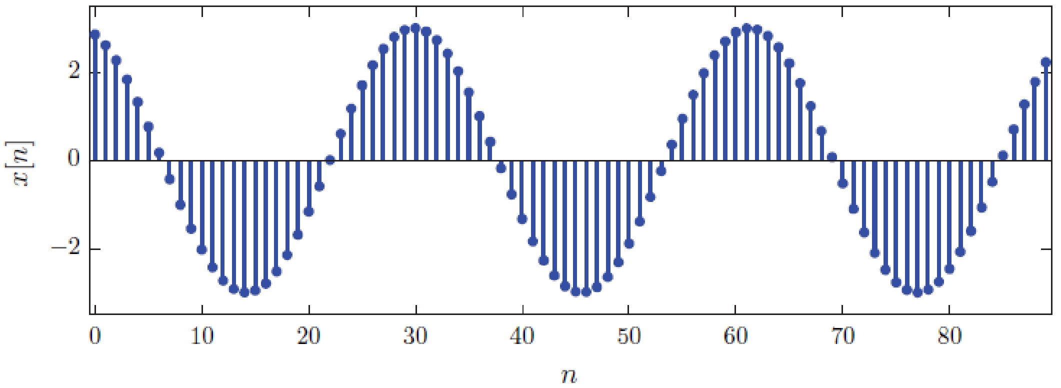


Sinusoidal Signal

- A discrete-time sinusoidal signal is a signal of the form: $x[n] = A\cos(\Omega_0 n + \theta)$ where A is the amplitude of the signal, Ω_0 is the angular frequency (rad), and θ is the initial phase angle (rad). $\Omega_0 = 2\pi F_0$ where F_0 is the normalized frequency (a dimensionless quantity).



$x[n] = 3\cos(0.1n + \pi/10)$



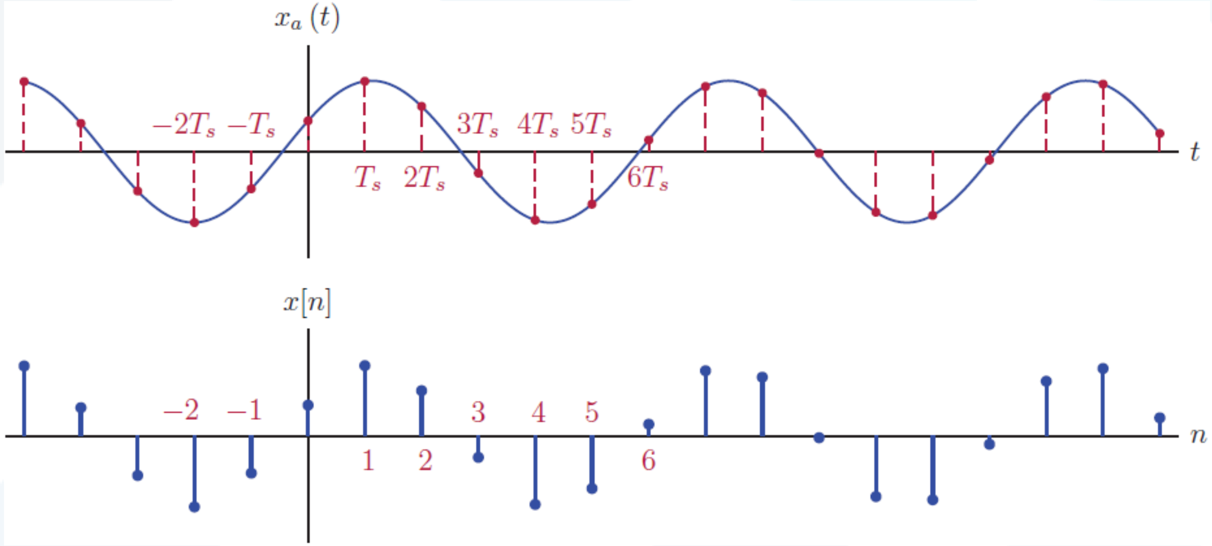
$x[n] = 3\cos(0.2n + \pi/10)$

A fundamental difference between a DT sinusoidal signal and its CT:

- For continuous-time sinusoidal signal $x_a(t) = A\cos(\omega_0 t + \theta)$: ω_0 is in rad/s.
- For discrete-time sinusoidal signal $x[n] = A\cos(\Omega_0 n + \theta)$: Ω_0 is in rad.
- Let us evaluate the amplitude of $x_a(t)$ at time instants that are integer multiples of T_s, and construct a discrete-time signal:

$$x[n] = x_a(nT_s) = A\cos(\omega_0 T_s n + \theta) = A\cos(2\pi f_0 T_s n + \theta)$$

- Since the signal $x_a(t)$ is evaluated at intervals of T_s, the number of samples taken per unit time is $1/T_s$. $x[n] = A\cos\left(2\pi\left[f_0/f_s\right]n + \theta\right) = A\cos(2\pi F_0 n + \theta)$
- The act of constructing a discrete-time signal by evaluating a continuous-time signal at uniform intervals is called sampling.
- The parameters f_s and T_s are referred to as the sampling rate and the sampling interval respectively.



Impulse decomposition for discrete-time signals

- Consider an arbitrary discrete-time signal $x[n]$. Let us define a new signal $x_k[n]$ by:

$$x_k[n] = x[k]\delta[n-k] = \begin{cases} x[k], & n = k \\ 0, & n \neq k \end{cases}$$

- The signal $x[n]$ can be reconstructed by: $x[n] = \sum_{k=-\infty}^{\infty} x_k[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$

Periodic discrete-time signals

- A discrete-time signal is said to be periodic if it satisfies: $x[n] = x[n+N]$

  for all values of the integer index n and for a specific value of $N \neq 0$. The parameter N is referred to as the period of the signal.

- The period of a periodic signal is not unique. That is, a signal that is periodic with period N is also periodic with period kN, for every (strictly) positive integer k, $x[n] = x[n + kN]$.
- The smallest period with which a signal is periodic is called the fundamental period.
- The normalized fundamental frequency of a discrete-time periodic signal is $F_0 = 1/N$.

Periodicity of discrete-time sinusoidal signals

$$A\cos(2\pi F_0 n + \theta) = A\cos(2\pi F_0[n + N] + \theta)$$
$$= A\cos(2\pi F_0 n + 2\pi F_0 N + \theta)$$

$2\pi F_0 N = 2\pi k \Rightarrow N = k/F_0$ $\quad$ N must be an integer value

- Example 6: Periodicity of a discrete-time sinusoidal signal

  Check the periodicity of the following discrete-time signals:

  a. $x[n] = \cos(0.2n)$ b. $x[n] = \cos(0.2\pi n + \pi/5)$

  c. $x[n] = \cos(0.3\pi n - \pi/10)$

  a. $x[n] = \cos(0.2n)$

  $\Omega_0 = 0.2 \Rightarrow F_0 = \Omega_0/2\pi = 0.2/2\pi = 0.1/\pi \Rightarrow N = k/F_0 = 10\pi k$

  Since no value of k would produce an integer value for N, the signal is not periodic.

  b. $x[n] = \cos(0.2\pi n + \pi/5)$

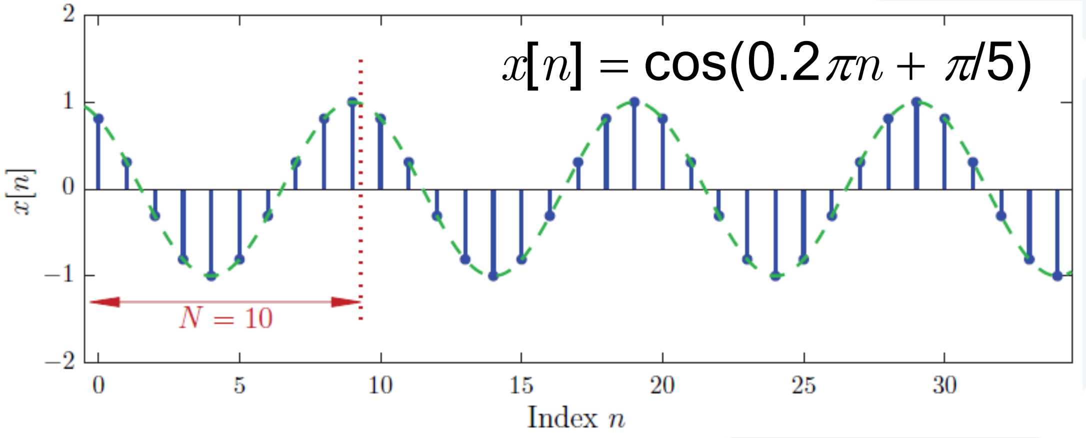
  $\Omega_0 = 0.2\pi \Rightarrow F_0 = \Omega_0/2\pi = 0.2\pi/2\pi = 0.1 \Rightarrow N = k/F_0 = 10k$

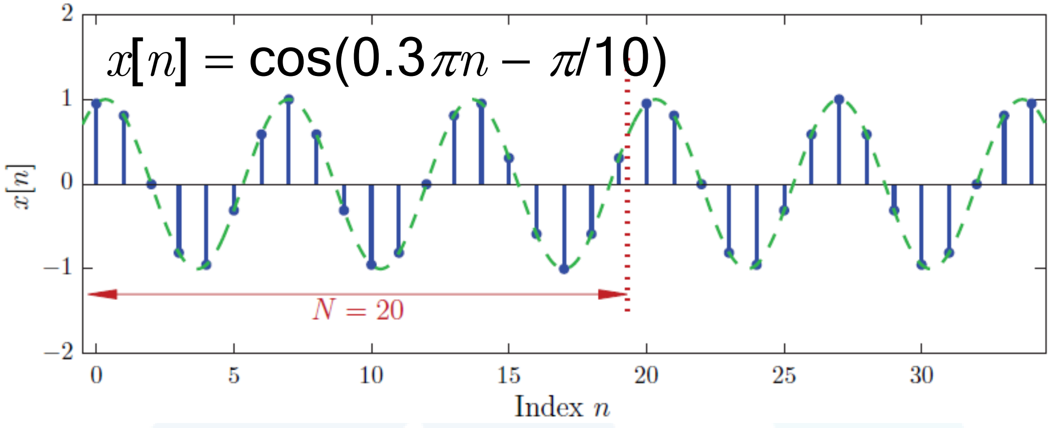
  For $k = 1$ we have $N = 10$ samples as the fundamental period.

c. $x[n] = \cos(0.3\pi n - \pi/10)$

$\Omega_0 = 0.3\pi \Rightarrow F_0 = \Omega_0/2\pi = 0.3\pi/2\pi = 0.15 \Rightarrow N = k/F_0 = k/0.15$

For $k = 3$ we have $N = 20$ samples as the fundamental period.





- Example 7: Periodicity of a multi-tone discrete-time sinusoidal signal

Comment on the periodicity of the two-tone discrete-time signal:

$$x[n] = 2\cos(0.4\pi n) + 1.5\sin(0.48\pi n)$$

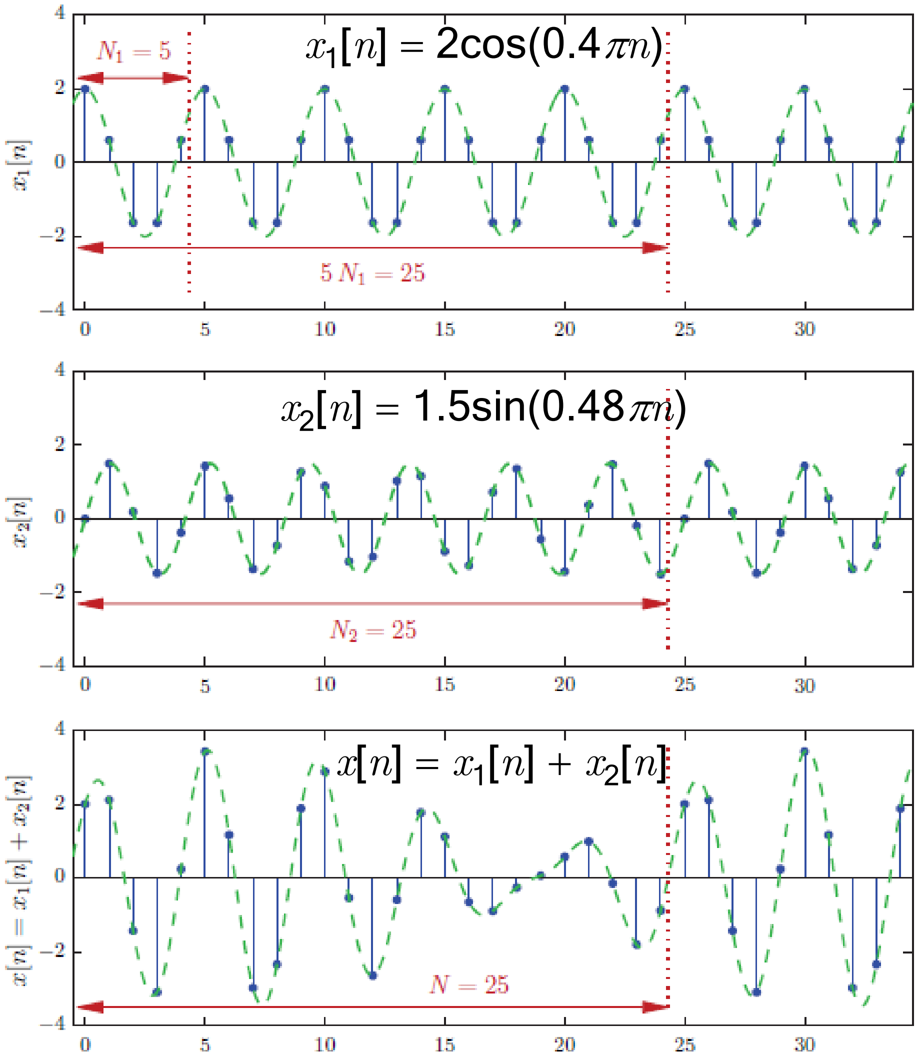
$$x[n] = x_1[n] + x_2[n]$$
$$x_1[n] = 2\cos(\Omega_1 n)$$
$$\Omega_1 = 0.4\pi \Rightarrow F_1 = \Omega_1/2\pi = 0.4\pi/2\pi = 0.2$$
$$\Rightarrow N = k_1/F_1 = 5k_1$$

For $k_1 = 1$ we have $N_1 = 5$ samples as the fundamental period.

$$x_2[n] = 1.5\cos(\Omega_2 n)$$
$$\Omega_2 = 0.48\pi \Rightarrow F_2 = \Omega_2/2\pi = 0.48\pi/2\pi = 0.24$$
$$\Rightarrow N_2 = k_2/F_2 = k_2/0.24$$

For $k_2 = 6$ we have $N_2 = 25$ samples as the fundamental period.

$$\Rightarrow N = 25$$



Energy and power definitions

- The energy of a discrete time signal $x[n]$ is given by $E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2$
- The average power of a discrete time signal $x[n]$ is given by:

  periodic complex signal $P_x = \frac{1}{N}\sum_{n=0}^{N-1} |x[n]|^2$

  non-periodic complex signal $P_x = \lim_{M\to\infty} \frac{1}{2M+1}\sum_{n=-M}^{M} |x[n]|^2$
- Energy signals are those that have finite energy and zero power, i.e., $E_x < \infty$, and $P_x = 0$.
- Power signals are those that have finite power and infinite energy, i.e., $E_x \to \infty$, and $P_x < \infty$.

- Example 8: Energy and power signals

  Determine whether the sequence $x[n] = a^n u[n]$ is an energy signal or a power signal or neither for the following cases: (a) $|a| < 1$, (b) $|a| = 1$, (c) $|a| > 1$.

$$E_x = \sum_{n=-\infty}^{\infty} \left|x[n]\right|^2 = \sum_{n=0}^{\infty} \left|a^{2n}\right|, \quad P_x = \lim_{M\to\infty} \frac{1}{2M+1} \sum_{n=-M}^{M} \left|x[n]\right|^2 = \lim_{M\to\infty} \frac{1}{2M+1} \sum_{n=0}^{M} \left|a^{2n}\right|$$

(a) $$E_x = \sum_{n=0}^{\infty} \left|a^{2n}\right| = \frac{1}{1-|a|^2} < \infty,$$

$$P_x = \lim_{M\to\infty} \frac{1}{2M+1} \sum_{n=0}^{M} \left|a^{2n}\right| = \lim_{M\to\infty} \frac{1}{2M+1} \frac{1-|a|^{2(M+1)}}{1-|a|^2} = 0$$

The signal $x[n] = a^n u[n]$ is an energy signal for $|a| < 1$.

(b) $$E_x = \sum_{n=0}^{\infty} \left|a^{2n}\right| \to \infty,$$

$$P_x = \lim_{M\to\infty} \frac{1}{2M+1} \sum_{n=0}^{M} \left|a^{2n}\right| = \lim_{M\to\infty} \frac{M+1}{2M+1} = \frac{1}{2}$$

The signal $x[n] = a^n u[n]$ is an power signal for $|a| = 1$.

(c) $$E_x = \sum_{n=0}^{\infty} \left|a^{2n}\right| \to \infty,$$

$$P_x = \lim_{M\to\infty} \frac{1}{2M+1} \sum_{n=0}^{M} \left|a^{2n}\right| = \lim_{M\to\infty} \frac{1}{2M+1} \frac{|a|^{2(M+1)} - 1}{|a|^2 - 1} \to \infty$$

The signal $x[n] = a^n u[n]$ is neither an energy signal nor a power signal for $|a| > 1$.

Symmetry properties

Even and odd symmetry

- A real-valued signal is said to have even symmetry if it has the property: $x[-n] = x[n]$ for all values of n.
- A real-valued signal is said to have odd symmetry if it has the property: $x[-n] = -x[n]$ for all values of n.

Decomposition into even and odd components

- Every real-valued signal $x[n]$ has a unique representation of the form: $x[n] = x_e[n] + x_o[n]$; where the signals x_e and x_o are even and odd, respectively.
- In particular, the signals x_e and x_o are given by:

$$x_e[n] = ½(x[n] + x[-n]) \text{ and } x_o[n] = ½(x[n] - x[-n])$$

Symmetry properties for complex signals

- A complex-valued signal is said to have conjugate symmetric if it has the property: $x[-n] = x^*[n]$ for all values of n.
- A complex-valued signal is said to have conjugate antisymmetric if it has the property: $x[-n] = -x^*[n]$ for all values of n.

Decomposition of complex signals

- Every complex-valued signal $x[n]$ has a unique representation of the form: $x[n] = x_E[n] + x_O[n]$; where the signals x_E and x_O are conjugate symmetric and conjugate antisymmetric, respectively.
- In particular, the signals x_E and x_O are given by:

$$x_E[n] = ½(x[n] + x^*[-n]) \text{ and } x_O[n] = ½(x[n] - x^*[-n])$$