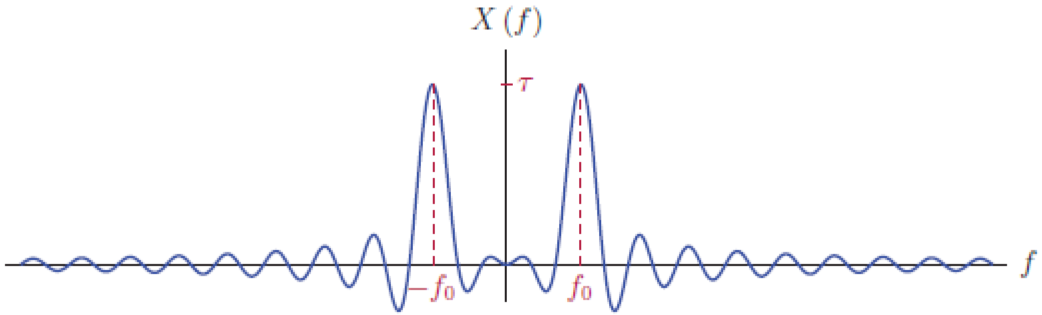
CEDC403: Signals and Systems

Lecture Notes 3: Analyzing Continuous Time Systems in the Time Domain



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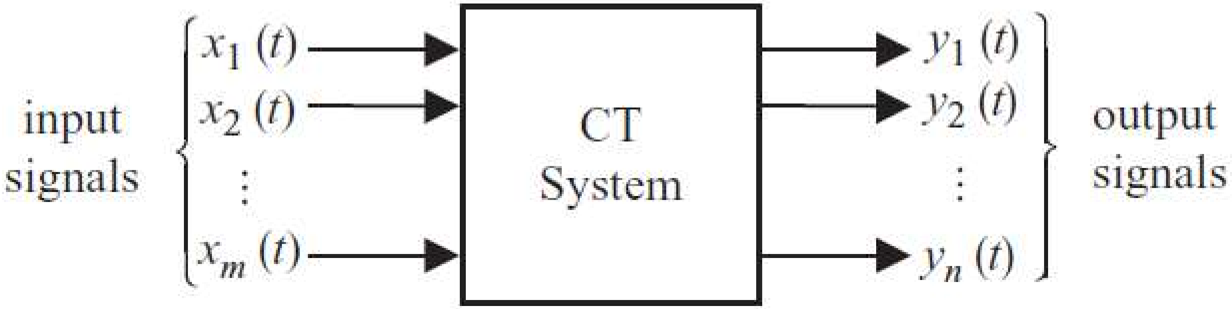
Chapter 2

Analyzing Continuous Time Systems in the Time Domain

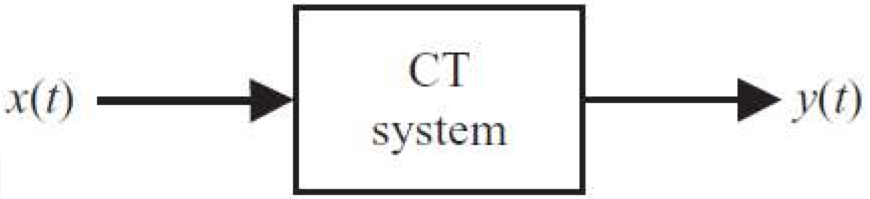
1. Linearity and Time Invariance
2. Differential Equations for Continuous-Time Systems
3. Constant-Coefficient Ordinary Differential Equations
4. Impulse Response and Convolution
5. Causality and Stability in Continuous-Time Systems
6. Block Diagram Representation of Continuous-Time Systems

Introduction

- A system is any physical entity that takes in a set of one or more physical signals and, in response, produces a new set of one or more physical signals.
- One representation of a general system is by a block diagram.

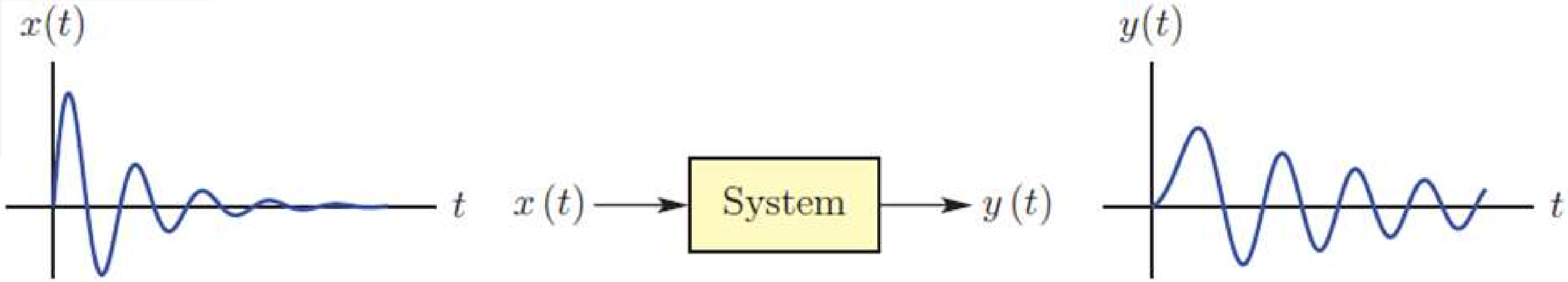


Multiple-input, multiple-output (MIMO) CT system



Single-input, single-output CT system

- If we focus our attention on single-input/single-output systems, the interplay between the system and its input and output signals can be graphically illustrated as:



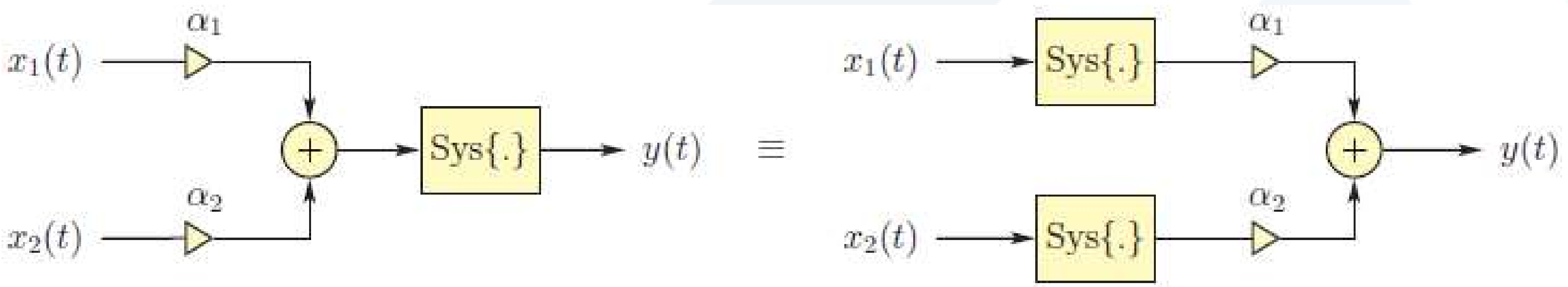
- The input signal is $x(t)$, and the output signal is $y(t)$. The system may be denoted by the equation $y(t) = T\{x(t)\} = \text{Sys}\{x(t)\}$, where $T\{.\} = \text{Sys}\{.\}$ indicates a transformation that defines the system in the time domain.
- A very simple example is a system that simply multiplies its input signal by a constant gain factor K to yield an output signal $y(t) = Kx(t)$,
- Or one that delays its input signal by a constant time delay τ $y(t) = x(t - \tau)$,
- Or one that produces an output signal proportional to the square of the input signal $y(t) = K[x(t)]^2$.

1. Linearity and Time Invariance

Linearity in continuous-time systems

- A system T is linear, if for all functions x_1 and x_2 and all constants α_1 and α_2, the following condition holds:

$$T\{\alpha_1 x_1(t) + \alpha_2 x_2(t)\} = \alpha_1 T\{x_1(t)\} + \alpha_2 T\{x_2(t)\}.$$



- A direct consequence of the linearity property is that, for linear systems, an input which is zero for all time results in an output which is zero for all time.

$$0 = T\{0x_1(t) + 0x_2(t)\} = 0\,T\{x_1(t)\} + 0\,T\{x_2(t)\} = 0 \text{ (zero-in/zero-out property)}$$

- Note: Linear systems are much easier to design and analyze than nonlinear systems.
- Example 1: Testing linearity of continuous-time systems

a. $y(t) = 5x(t)$ √

b. $y(t) = 5x(t) + 3$ ✗

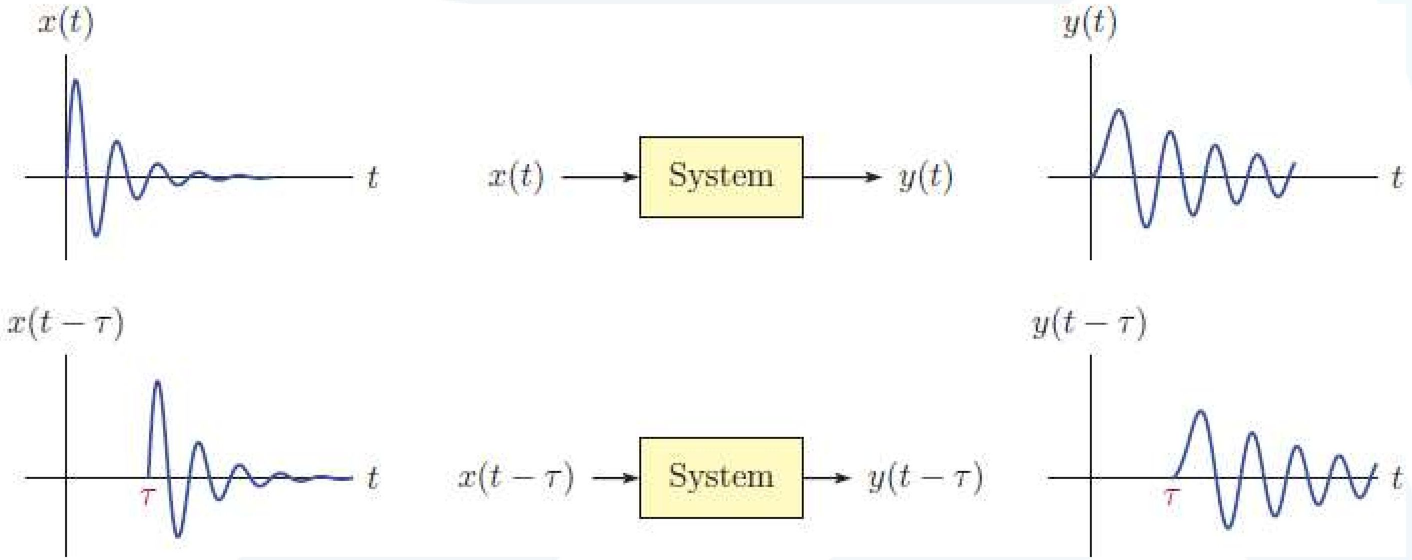
c. $y(t) = 3[x(t)]^2$ ✗

d. $y(t) = \cos(x(t))$ ✗

- The property defining a linear system can be separated into two statements as follows:
  - If $T\{\alpha x(t)\} = \alpha T\{x(t)\}$ for all signals $x(t)$ and any constant α, then the system is homogeneous.
  - If $\mathrm{T}\{x(t) + y(t)\} = T\{x(t)\} + T\{y(t)\}$ for all signals $x(t)$ and $y(t)$, then the system is additive.

Time Invariance in continuous-time systems

- A system T is said to be time invariant (TI) if, for every function x and every real constant τ, we have: $T\{x(t)\} = y(t) \Rightarrow T\{x(t-\tau)\} = y(t-\tau)$.

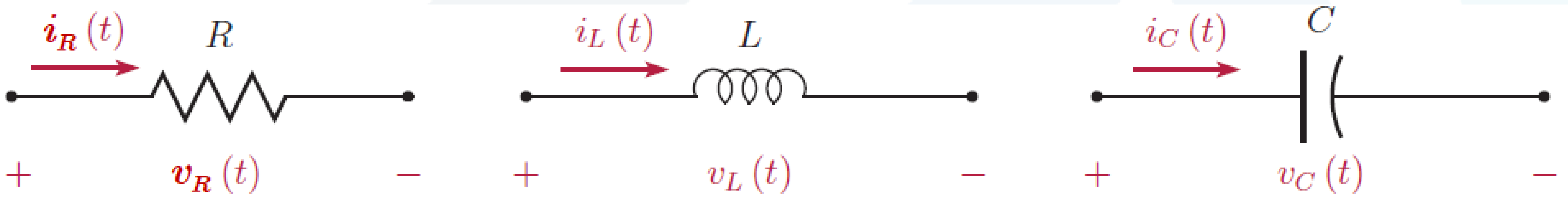


- Example 2: Testing time invariance of continuous-time systems

a. $y(t) = 5x(t)$ √ b. $y(t) = 3\cos(x(t))$ √ c. $y(t) = 3\cos(t)x(t)$ ✗

2. Differential Equations for Continuous-Time Systems

- One method of representing the relationship established by a system between its input and output signals is a differential equation (DE).
- model for an ideal resistor is: $v_R(t) = Ri_R(t)$
- model for an ideal inductor is: $v_L(t) = L\dfrac{di_L(t)}{dt}$
- model for an ideal capacitor is: $i_C(t) = C\dfrac{dv_C(t)}{dt}$

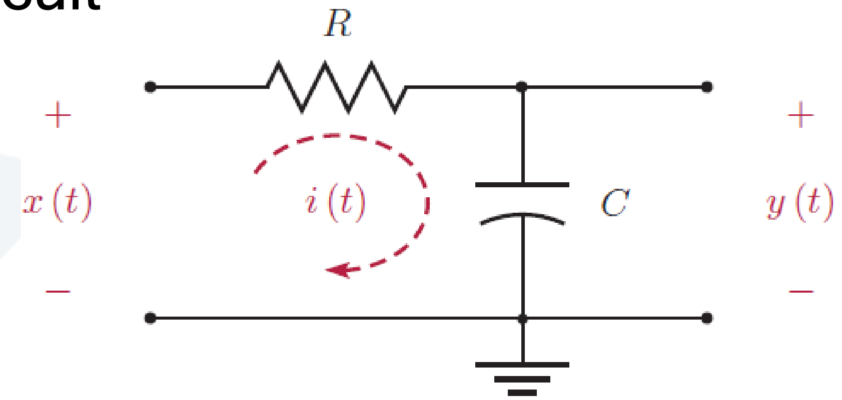


- Example 3: Differential equation for simple RC circuit

$$v_R(t) = Ri(t), \qquad i(t) = C\frac{dy(t)}{dt}$$

$$RC\frac{dy(t)}{dt} + y(t) = x(t)$$

$$\frac{dy(t)}{dt} + \frac{1}{RC}y(t) = \frac{1}{RC}x(t)$$

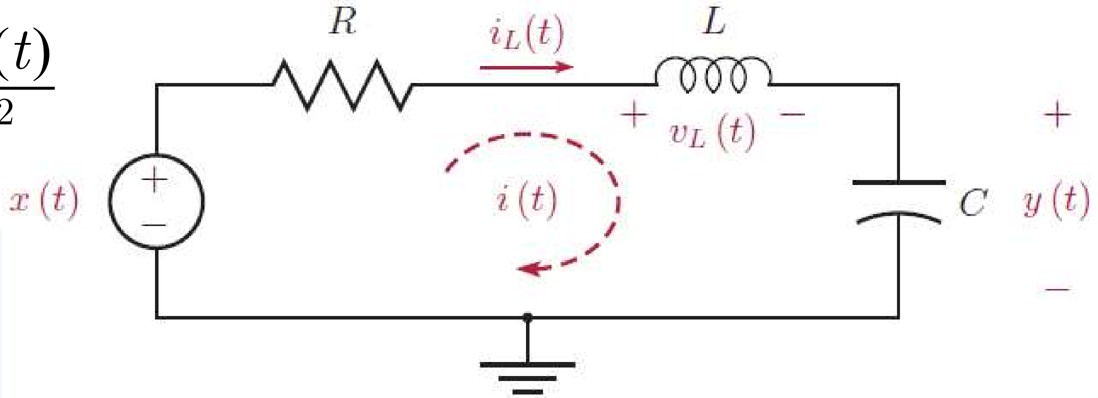


- Example 4: DE for RLC circuit

$$v_L(t) = L\frac{di(t)}{dt}, \quad i(t) = C\frac{dy(t)}{dt}, \quad \frac{di(t)}{dt} = C\frac{d^2y(t)}{dt^2}$$

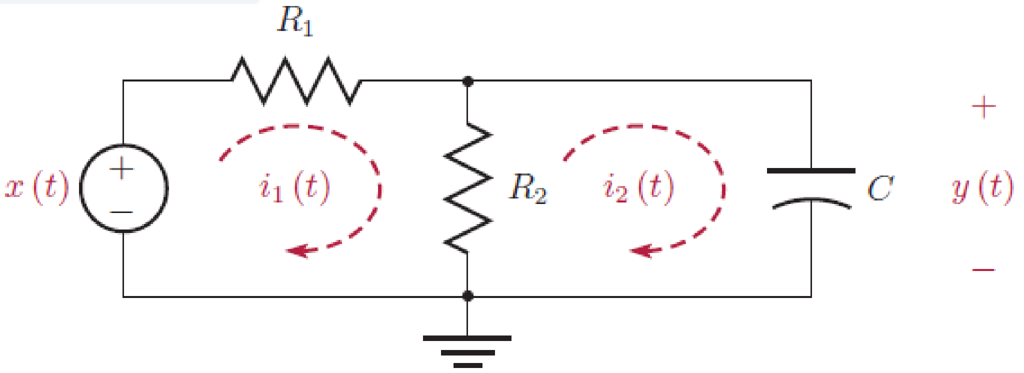
$$-x(t) + Ri(t) + v_L(t) + y(t) = 0$$

$$-x(t) + RC\frac{dy(t)}{dt} + LC\frac{d^2y(t)}{dt^2} + y(t) = 0$$



$$\frac{d^2y(t)}{dt^2} + \frac{R}{L}\frac{dy(t)}{dt} + \frac{1}{LC}y(t) = \frac{1}{LC}x(t)$$

- **Example 5:** Another RC circuit



$$-x(t) + R_1 i_1(t) + R_2[i_1(t) - i_2(t)] = 0$$

$$R_2[i_2(t) - i_1(t)] + y(t) = 0$$

$$i_2(t) = C\frac{dy(t)}{dt} \Rightarrow i_1(t) = C\frac{dy(t)}{dt} + \frac{1}{R_2}y(t)$$

$$-x(t) + R_1 C\frac{dy(t)}{dt} - \frac{R_1 + R_2}{R_2}y(t) = 0$$

$$\frac{dy(t)}{dt} + \frac{R_1 + R_2}{R_1 R_2 C}y(t) = \frac{1}{R_1 C}x(t)$$

3. Constant-Coefficient Ordinary Differential Equations

- In general, CTLTI systems can be modeled with ordinary differential equations that have constant coefficients.

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

- In general, a constant-coefficient ODE has a family of solutions. In order to find a unique solution for $y(t)$, initial values of the output signal and its first $N-1$ derivatives (initial conditions ICs) need to be specified at a time instant $t = t_0$.

$$y(t_0), \quad \left.\frac{dy(t)}{dt}\right|_{t=t_0}, \cdots, \quad \left.\frac{d^{N-1}y(t)}{dt^{N-1}}\right|_{t=t_0}$$ to find the solution for $t > t_0$

- Initial conditions (ICs) also represent the memory of continuous-time systems.

- The ICs in a DE of an LTI system are directly related to the initial values of the energy storage devices in the system, such as initial voltages on capacitors and initial currents through inductors.
- A system with zero ICs is said to be at rest (initially relaxed).

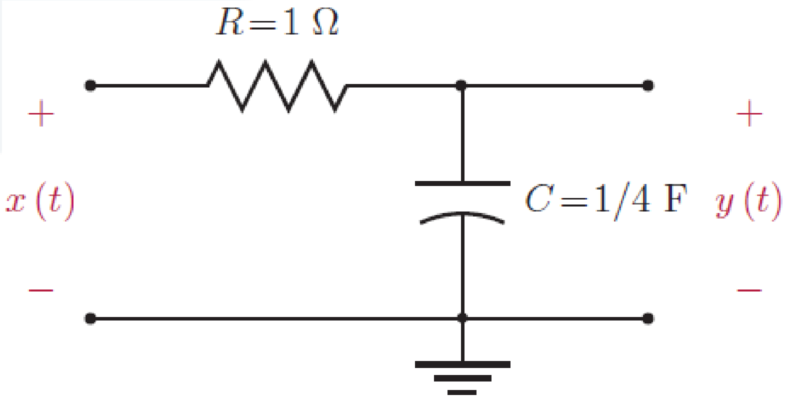
Solving Linear Differential Equations

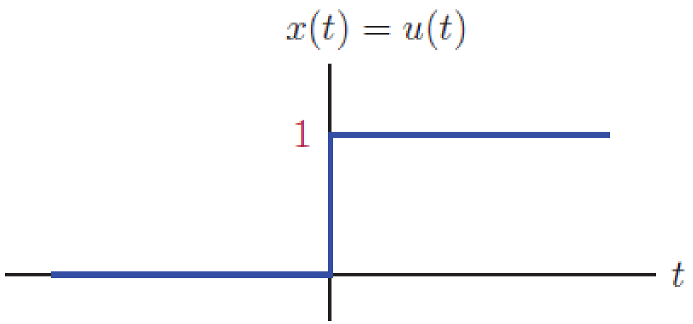
- The differential equation: $\frac{dy(t)}{dt} + \alpha y(t) = r(t), \quad y(t_0)$: specified is solved as:

$$y(t) = e^{-\alpha(t-t_0)}y(t_0) + \int_{t_0}^{t} e^{-\alpha(t-\tau)}r(\tau)d\tau$$

- Example 6: Unit-step response of the simple RC circuit ($y(0) = 0$)

$$\frac{dy(t)}{dt} + \frac{1}{RC}y(t) = \frac{1}{RC}u(t) \Rightarrow \frac{dy(t)}{dt} + 4y(t) = 4u(t)$$

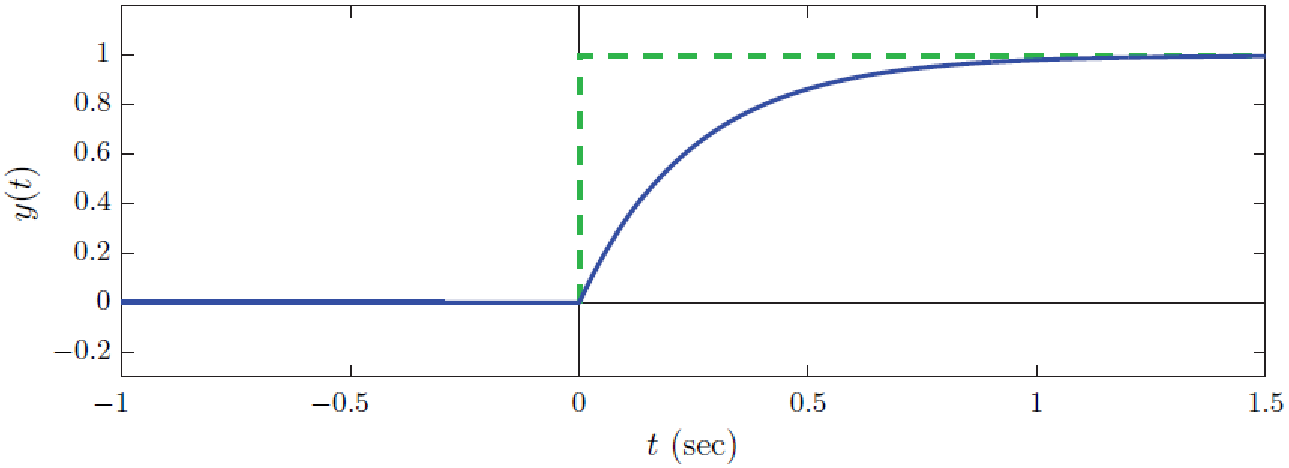




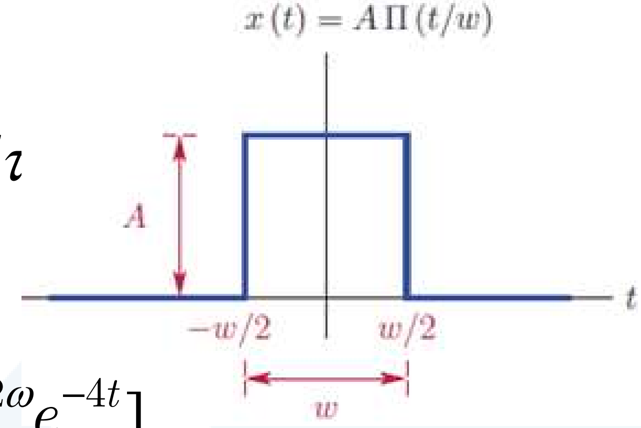
$$y(t) = \int_0^t e^{-(t-\tau)/RC}\,\frac{1}{RC}\,u(\tau)d\tau = \frac{e^{-t/RC}}{RC}\int_0^t e^{\tau/RC}d\tau = 1 - e^{-t/RC}, \quad t \geq 0$$

$$y(t) = (1 - e^{-t/RC})u(t)$$

$$y(t) = (1 - e^{-4t})u(t)$$



- Example 7: Pulse response of the simple RC circuit



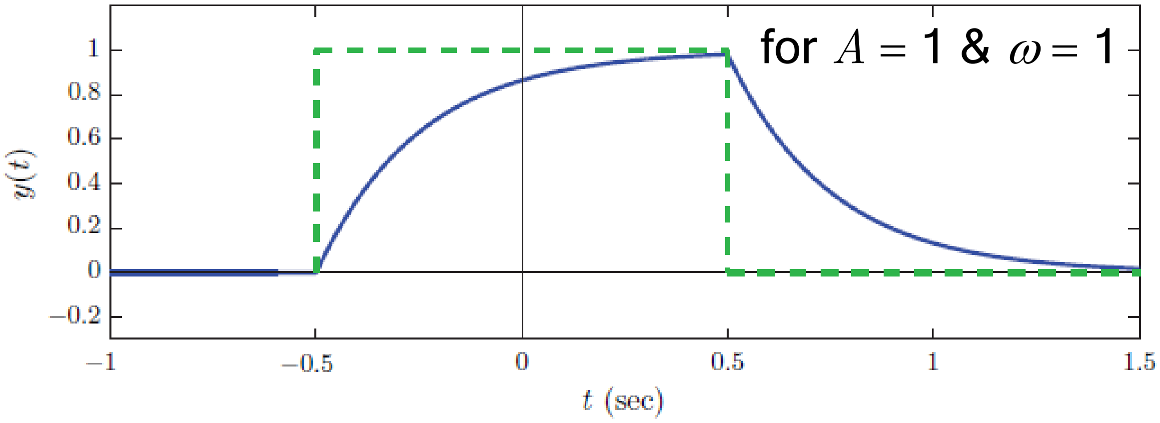
$$\frac{dy(t)}{dt} + 4y(t) = 4A\prod(t/\omega) \Rightarrow y(t) = \int_{-\omega/2}^{t} e^{-4(t-\tau)} 4A\prod(\tau/\omega)d\tau$$

Case 1: $t \le -\omega/2,\ y(t) = 0$

Case 2: $-\omega/2 < t \le \omega/2,\ y(t) = 4A\int_{-\omega/2}^{t} e^{-4(t-\tau)}d\tau = A[1 - e^{-2\omega}e^{-4t}]$

Case 3: $t > \omega/2,\ y(t) = 4A\int_{-\omega/2}^{\omega/2} e^{-4(t-\tau)}d\tau = Ae^{-4t}[e^{2\omega} - e^{-2\omega}]$

$$y(t) = \begin{cases} 0, & t < -\frac{\omega}{2} \\ A[1 - e^{-2\omega}e^{-4t}], & -\frac{\omega}{2} < t \le \frac{\omega}{2} \\ Ae^{-4t}[e^{2\omega} - e^{-2\omega}], & t > \frac{\omega}{2} \end{cases}$$



Solution of the general differential equation

- The complete solution of a linear constant coefficient differential equation can be decomposed into:

1. The point of view of Mathematics:

Homogenous solution $y_h(t)$ + Particular solution $y_p(t)$.

2. The point of view of Engineer:

Natural response $y_n(t)$ + Forced response $y_\phi(t)$.

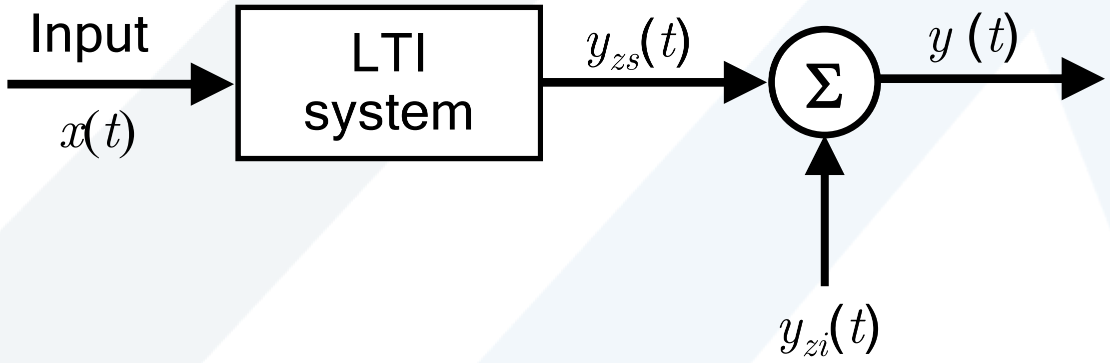
3. The point of view of control engineer:

Zero-input response $y_{zi}(t)$ + Zero-state response $y_{zs}(t)$.

Transient response $y_t(t)$ + Steady state response $y_{ss}(t)$.

Zero-input response and Zero-states response

- The zero-input response is the system output when the input $x(t) = 0$. It is the result of internal system conditions (energy storages, initial conditions).
- In contrast, the zero-state response is the system output to the external input $x(t)$ when the system is in zero state (all initial conditions are zero).
- Zero-state response is linear with respect to the input. While zero-input response is linear with respect to the initial conditions.



Zero-input response $y_{zi}(t)$

- The zero-input response $y_{zi}(t)$ is the output of the system when the external input $x(t)$ is zero. It is produced by the system because of the initial conditions.

  Zero-input response $y_{zi}(t)$ satisfies: $\sum_{k=0}^{N} a_k \frac{d^k y_{zi}(t)}{dt^k} = 0.$

  Determine the characteristic equation $\sum_{k=0}^{N} a_k \alpha^k = 0$ of the system by replacing the k-th derivative of the output signal $y_{zi}(t)$ with α^k.

$$\frac{d^k y_{zi}(t)}{dt^k} \to \alpha^k$$

  Let the roots of the characteristic polynomial $\sum_{k=0}^{N} a_k \alpha^k$ be α_1, α_2, ..., α_N.

- If all α_k are of order 1, $y_{zi}(t) = \sum_{k=1}^{N} c_{zik} e^{\alpha_k t}$, c_{zik} determined by the ICs $\dfrac{d^k y_{zi}}{dt^k}(0^+)$

Practical Initial Conditions and the Meaning of 0^- and 0^+

- In practical problems, we must derive ICs from the physical situation. For example, in an RLC circuit, we may be given the conditions (initial capacitor voltages, initial inductor currents, etc.).
- In practice, the input is assumed to start at $t = 0$. The conditions immediately before $t = 0$ are the conditions at $t = 0^-$ (Pre-initial conditions), and those immediately after $t = 0$ are the conditions at $t = 0^+$ (Post-initial conditions).
- In practice, we are likely to know the ICs at $t = 0^-$ rather than at $t = 0^+$. The two sets of conditions are different, although in some cases they may be identical.

- The total response $y(t)$ consists of two components: the zero-input response $y_{zi}(t)$ and the zero-state response $y_{zs}(t)$.
- At $t = 0^-$, the total response $y(t)$ consists solely of $y_{zi}(t)$ because the input has not started yet. Hence the initial conditions on $y(t)$ are identical to those of $y_{zi}(t)$. Thus, $y^{(j)}(0^-) = y_{zi}^{(j)}(0^-),\ j = 0, 1,\ \ldots, n-1$.
- Moreover, $y_{zi}(t)$ does not depend on the input $x(t)$. Hence, application of $x(t)$ at $t = 0$ does not affect $y_{zi}(t)$. Thus,

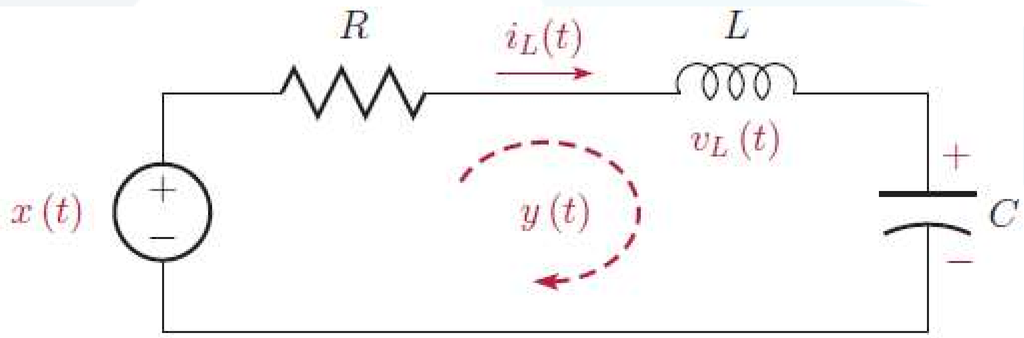
$$y_{zi}^{(j)}(0^+) = y_{zi}^{(j)}(0^-) = y^{(j)}(0^-), \quad j = 0, 1,\ \ldots, n-1$$

- Note: For $y_{zi}(t)$, there is no distinction between ICs at $t = 0^-$, 0, and 0^+. It is not the case for the total response, in general, .

$$y^{(j)}(0^-) \neq y^{(j)}(0^+), \quad j = 0, 1,\ \ldots, n-1$$

- **Example 8:** Zero-input response of RLC circuit

  A voltage $x(t) = 10e^{-3t}u(t)$ is applied at the input of the RLC circuit. Find the zero-input response for $t \geq 0$ if the initial inductor current is $i_L(0^-) = 0$, and the initial capacitor voltage $v_C(0^-) = 5$ V. Use $R = 3\ \Omega$, $L = 1$ H and $C = 1/2$ F.



$$v_L(t) + v_R(t) + v_C(t) = x(t)$$

$$\frac{dy(t)}{dt} + 3y(t) + 2\int_{-\infty}^{t} y(\tau)d\tau = x(t)$$

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = \frac{dx(t)}{dt}$$

$$y(0^-) = y_{zi}(0^-) = y_{zi}(0^+) = 0$$

$$\frac{dy(t)}{dt} + 3y(t) + v_C(t) = x(t) \Rightarrow \frac{dy(0^-)}{dt} + 3y(0^-) + v_C(0^-) = x(0^-) \Rightarrow \frac{dy(0^-)}{dt} = -5$$

$$\frac{dy_{zi}(0^+)}{dt} = \frac{dy_{zi}(0^-)}{dt} = \frac{dy(0^-)}{dt} = -5$$

$$\frac{d^2y_{zi}(t)}{dt^2} + 3\frac{dy_{zi}(t)}{dt} + 2y_{zi}(t) = 0 \Rightarrow y_{zi}(t) = c_{zi1}e^{-t} + c_{zi2}e^{-2t}$$

$$y_{zi}(0^+) = 0, \ \frac{dy_{zi}(0^+)}{dt} = -5 \Rightarrow c_{zi1} = -c_{zi2} = -5 \Rightarrow y_{zi}(t) = -5e^{-t} + 5e^{-2t}, \quad t \geq 0$$

Let us compare $y(0^-)$, $dy/dt(0^-)$ with $y(0^+)$, $dy/dt(0^+)$ for the total response.

$$\frac{dy(0^-)}{dt} + 3y(0^-) + v_C(0^-) = 0$$

$$\frac{dy(0^+)}{dt} + 3y(0^+) + v_C(0^+) = 10$$

$$\Rightarrow \frac{dy(0^-)}{dt} = -5 \neq \frac{dy(0^+)}{dt} = 5$$

The loop current $y(0^+) = y(0^-) = 0$ because it cannot change instantaneously in the absence of impulsive voltage. The same is true of the capacitor voltage. Hence, $v_C(0^+) = v_C(0^-) = 5$.

Zero-state response $y_{zs}(t)$

- The zero-state response $y_{zs}(t)$ is the output of the system to the external input signal $x(t)$. The initial conditions of the system are assumed to be zero.
- If all α_k are of order 1, $y_{zs}(t) = \sum_{k=1}^{N} c_{zsk} e^{\alpha_k t} + y_p(t)$

  c_{zsk} could be determined by the initial conditions $\frac{d^k y_{zs}}{dt^k}(0^+)$

  $y^{(j)}(0^+) = y_{zi}^{(j)}(0^+) + y_{zs}^{(j)}(0^+),\ j = 0, 1, \ldots, n-1$
- Example 9: Zero-state response of RLC circuit

  Find the zero-state response for $t \geq 0$ for the circuit in example 8

  $$\frac{d^2 y_{zs}(t)}{dt^2} + 3\frac{dy_{zs}(t)}{dt} + 2y_{zs}(t) = 10e^{-3t}$$

  A particular solution: $y_p(t) = ke^{-3t} \Rightarrow k = -15$

$$y_{zs}(t) = c_{zs1}e^{-t} + c_{zs2}e^{-2t} - 15e^{-3t}, \qquad t \geq 0$$

$$y(0^+) = y_{zi}(0^+) + y_{zs}(0^+) \Rightarrow y_{zs}(0^+) = 0$$

$$\frac{dy(0^+)}{dt} = \frac{dy_{zi}(0^+)}{dt} + \frac{dy_{zs}(0^+)}{dt} \Rightarrow \frac{dy_{zs}(0^+)}{dt} = 5 - (-5) = 10$$

$$y_{zs}(0^+) = 0, \frac{dy_{zs}(0^+)}{dt} = 10 \Rightarrow c_{zs1} = -5, c_{zs2} = 20$$

$$y_{zi}(t) = -5e^{-t} + 20e^{-2t} - 15e^{-3t}, \qquad t \geq 0$$

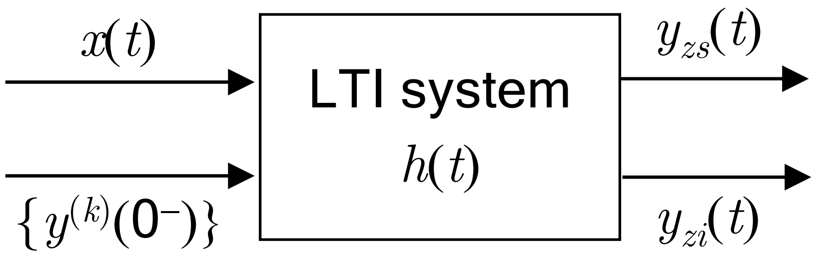
The complete solution = Zero-input response + Zero-state response

$$y(t) = y_{zi}(t) + y_{zs}(t) = \underbrace{-5e^{-t} + 5e^{-2t}}_{zero-input\ response} + \underbrace{-5e^{-t} + 20e^{-2t} - 15e^{-3t}}_{zero-state\ response}$$

$$y(t) = -10e^{-t} + 25e^{-2t} - 15e^{-3t}$$

Linearity properties of zero-input and zero-state response

- Zero-state response is linear with respect to the input.
- Zero-input response is linear with respect to the initial state.



$$y(t) = \underbrace{\sum_{k=1}^{N} c_{zik} e^{\alpha_k t}}_{zero-input\ response} + \underbrace{\sum_{k=1}^{N} c_{zsk} e^{\alpha_k t} + y_p(t)}_{zero-state\ response}$$

- Notes:

1. For LTI systems, the excitation and initial states can be thought of as two separate inputs.
2. When the ICs are not zero, there is no linear relationship between the complete response of the system and the external excitation.

Homogeneous solution (natural response) & particular solution (forced response)

Homogeneous solution $y_h(t)$, also called Natural response satisfies $\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = 0$

Determine the characteristic values $\sum_{k=0}^{N} a_k \alpha^k = 0$ as α_1, α_2, ..., α_N

a. If all α_i are of order 1, $y_h(t) = \sum_{i=1}^{N} c_i e^{\alpha_i t}$

b. If a root α_i is repeated k times (order k), $y_h(t) = \sum_{i=1}^{k} c_i t^{k-i} e^{\alpha_i t} + \sum_{j=k+1}^{N} c_j e^{\alpha_j t}$

- Note: the coefficients c_i or c_j should be determined by the initial conditions at $t = 0^+$ simultaneously with those in the particular solution.
- The particular solution $y_p(t)$ represents any solution of the DE for the given input. It is also called Forced response $y_\phi(t)$.

- A particular solution is usually obtained by assuming an output of the same general form as the input.

Input signal	Particular solution
t^n	$k_n t^n + k_{n-1} t^{n-1} + \dots k_1 t + k_0$ (Constant input is a special case with $n = 0$)
$e^{\alpha t}$	$ke^{\alpha t}$, α is not the characteristic value (c.v.) $k_1 te^{\alpha t} + k_0 e^{\alpha t}$, α is the characteristic value with order 1 $k_k t^k e^{\alpha t} + k_{k-1} t^{k-1} e^{\alpha t} + \dots k_1 te^{\alpha t} + k_0 e^{\alpha t}$, α is the c.v. with order k
$\cos(\omega t)$ or $\sin(\omega t)$	$k_1 \cos(\omega t) + k_2 \sin(\omega t)$

The complete solution $=$ homogeneous solution $+$ particular solution

$$y(t) = y_h(t) + y_p(t) = \underbrace{\sum_k c_k e^{\alpha_k t}}_{natural} + \underbrace{y_p(t)}_{forced}$$

- $y_h(t)$ depends on the structure and the initial state of the system.
- For a stable system, $y_h(t)$ tends to gradually disappear in time. Because of this, it is referred to as the transient response of the system.
- $y_p(t)$ depends on the input signal $x(t)$ and the internal structure of the system.
- $y_p(t)$ is the part of the response that remains active after the homogeneous solution disappears. It is referred to as the steady-state response of the system.
- Example 10: Output response of RLC circuit

  Find the complete solution, $y_h(t)$ and $y_p(t)$, for $t \geq 0$ for the circuit in example 8

$$\frac{d^2 y_{zs}(t)}{dt^2} + 3\frac{dy_{zs}(t)}{dt} + 2y_{zs}(t) = 10e^{-3t}$$

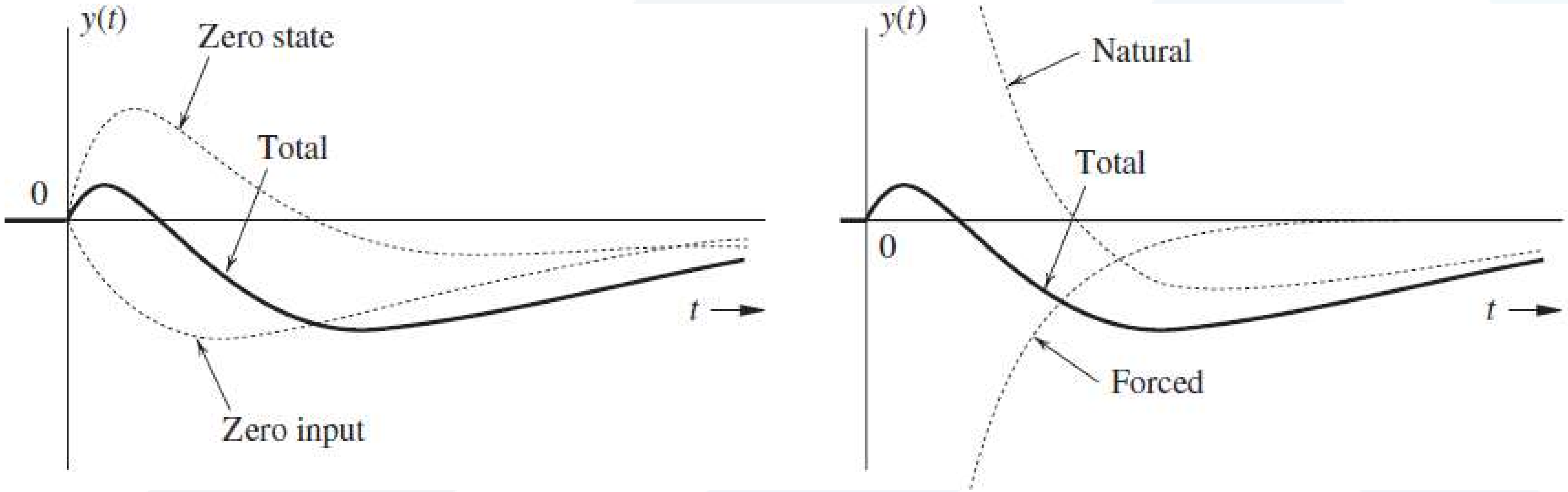
A particular solution: $y_p(t) = ke^{-3t} \Rightarrow k = -15$

The homogeneous solution: $y_h(t) = c_1e^{-t} + c_2e^{-2t}$

$$y(t) = c_1e^{-t} + c_2e^{-2t} - 15e^{-3t}, \quad t \geq 0$$

$$y(0^+) = 0, \frac{dy(0^+)}{dt} = 5 \Rightarrow c_1 = -10, c_2 = 25$$

$$y(t) = \underbrace{-10e^{-t} + 25e^{-2t}}_{y_h(t)} - \underbrace{15e^{-3t}}_{y_p(t)}, \; t \geq 0$$



4. Impulse Response and Convolution

Continuous-time impulse response

- The impulse response $h(t)$ of an CTLTI system is the zero-state output of the system when a unit impulse $\delta(t)$ is applied at the input.

$$\sum_{k=0}^{N} a_k \frac{d^k h(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k \delta(t)}{dt^k}$$

with post-initial conditions: $h^{(n-1)}(0^+) = 1/a_N,\ h^{(j)}(0^+) = 0, \quad j = 0, 1, \ldots, n-2$

- Example 11: Determine the impulse response of the CTLTI system given by:

(a) $\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y(t) = x(t)$ (b) $\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y(t) = \frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 3x(t)$

(a) $\frac{d^2h}{dt^2} + 5\frac{dh}{dt} + 6h(t) = \delta(t);\ \frac{dh}{dt}(0^+) = 1,\ h(0^+) = 0 \Rightarrow h(t) = (e^{-2t} - e^{-3t})u(t)$

(b) Suppose $h_1(t)$ satisfies:

$$\frac{d^2h_1}{dt^2} + 5\frac{dh_1}{dt} + 6h_1(t) = \delta(t)$$

Due to the differentiation property and linearity of the CTLTI system, the impulse response $h(t)$ satisfies: $h(t) = \frac{d^2h_1(t)}{dt^2} + 2\frac{dh_1(t)}{dt} + 3h_1(t)$

$$h_1(t) = (e^{-2t} - e^{-3t})u(t) \Rightarrow h(t) = \delta(t) + (3e^{-2t} - 6e^{-3t})u(t)$$

- Example 12: Determine the impulse response of the circuit in example 8

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = \frac{dx(t)}{dt}$$

$$h(t) = (-e^{-t} + 2e^{-2t})u(t)$$

Continuous-time step response

- The step response $s(t)$ of an CTLTI system is the zero-state output of the system when a unit step $u(t)$ is applied at the input.

$$\delta(t) = \frac{du(t)}{dt}, \quad u(t) = \int_{-\infty}^{t} \delta(\tau) d\tau$$

According to the differential (or integral) properties of the CTLTI system,

$$h(t) = \frac{ds(t)}{dt}, \quad s(t) = \int_{-\infty}^{t} h(\tau) d\tau$$

- Example 13: Impulse response of the simple RC circuit

Consider the RC circuit. Let the element values be $R = 1\ \Omega$ and $C = 1/4$ F. Assume $y(0) = 0$. Determine the impulse response of the system.

First method: using differential equation $y(t) = \int_{0}^{t} e^{-(t-\tau)/RC} \frac{1}{RC} x(\tau) d\tau$

Setting $x(t) = \delta(t)$ $h(t) = \int_0^t e^{-(t-\tau)/RC} \frac{1}{RC}\delta(\tau)d\tau = \frac{1}{RC}e^{-t/RC}u(t) = 4e^{-4t}u(t)$

Second method: unit-step response of the system

$$s(t) = (1 - e^{-t/RC})u(t) \Rightarrow h(t) = \frac{ds(t)}{dt} = \frac{1}{RC}e^{-t/RC}u(t) = 4e^{-4t}u(t)$$

The Convolution Integral

- The (CT) convolution of the functions x and h, denoted $x * h$, is defined as the function:

$$x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

Properties of Convolution

- Is commutative. For any two functions x and h, $x * h = h * x$.
- Is associative. For any functions x, h_1, and h_2, $(x * h_1) * h_2 = x * (h_1 * h_2)$.

- Is distributive with respect to addition. For any functions x, h_1, and h_2, $x * (h_1 + h_2) = x * h_1 + x * h_2$.
- For any function x, $x(t) * \delta(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau = x(t)$

  Moreover, δ is the convolutional identity. That is, for any function x, $x * \delta = x$.

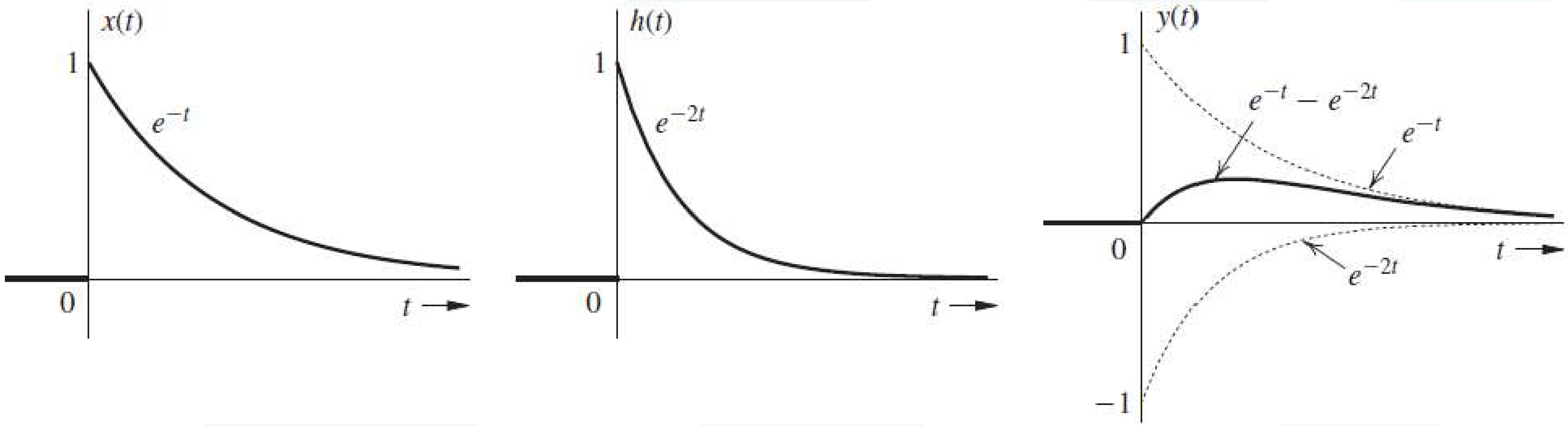
- According to the linear time-invariant property of CTLTI system, the zero-state response output can be obtained as follows:
  - When the input signal is $\delta(t)$, the zero-state response is $h(t)$: $\delta(t) \rightarrow h(t)$
  - According to time-invariant property: $\delta(t-\tau) \rightarrow h(t-\tau)$
  - According to homogeneous property: $x(\tau)\delta(t-\tau) \rightarrow x(\tau)h(t-\tau)$
  - According to additive property: $x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau \rightarrow \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$

- For any LTI system with input x, zero-state response y_{zs}, and impulse response h, the following relationship holds: $y_{zs}, = x * h$.

- Example 14: Computing the Zero-State Response

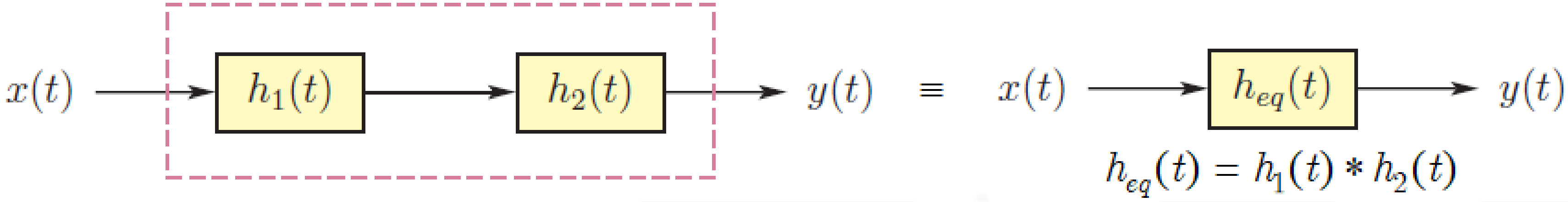
  CTLTI system with $h(t) = e^{-2t}u(t)$, determine the response $y_{zs}(t)$ for $x(t) = e^{-t}u(t)$.

$$y_{zs}(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{0}^{t} e^{-t}e^{-2(t-\tau)}d\tau = e^{-2t}\int_{0}^{t} e^{\tau}d\tau = (e^{-t} - e^{-2t})u(t)$$

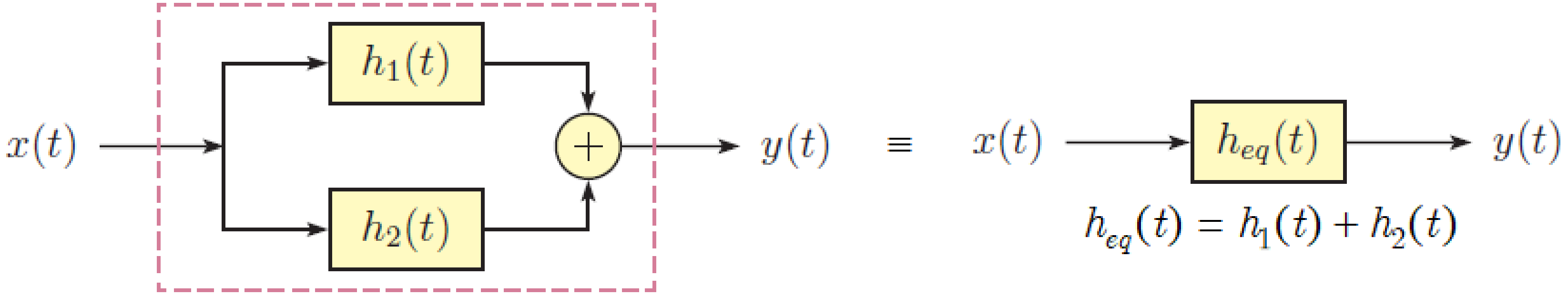


Interconnected Systems

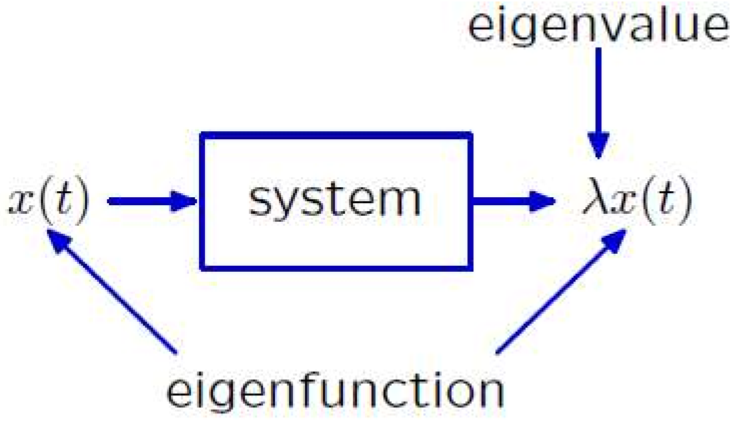
- Two CTLTI systems with impulse responses $h_1(t)$ and $h_2(t)$ are connected in cascade as shown in the figure below: $h_{eq}(t) = h_1(t) * h_2(t)$



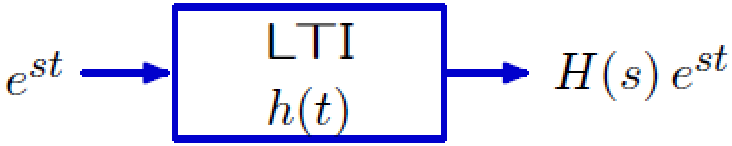
- Two CTLTI systems with impulse responses $h_1(t)$ and $h_2(t)$ are connected in parallel as shown in the figure below: $h_{eq}(t) = h_1(t) + h_2(t)$



Eigenfunctions of CTLTI system



- If the output signal is a scalar multiple of the input signal, we refer to the signal as an eigenfunction and the multiplier as the eigenvalue.
- Complex exponential are eigenfunctions of LTI systems.



$$y(t) = (h * x)(t) = \int_{-\infty}^{\infty} h(\tau) e^{s(t-\tau)} d\tau = e^{st} \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau = H(s) e^{st} \quad \text{where } s \in C$$

- Note: We are talking of an exponential, which starts at $t = -\infty$.
- We refer to H as the transfer function of the system.

5. Causality and Stability in Continuous-Time Systems

Causality in continuous-time systems

- A system T is said to be causal if, for every real constant t_0, $T\{x(t_0)\}$ does not depend on $x(t)$ for some $t > t_0$.
- A causal system is such that the value of its output at any given point in time can depend on the value of its input at only the same or earlier points in time.
- For CTLTI systems the causality property can be related to the impulse response of the system $h(t) = 0$ for all $t < 0$.

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau = \int_{0}^{\infty} h(\tau)x(t-\tau)d\tau$$

- Note: If the independent variable t represents time, a system must be causal in order to be physically realizable.

- Example 15: causal and non causal systems

  a. CT time-delay system $y(t) = x(t) + x(t - 0.01) + x(t - 0.02)$ √

  b. CT time-forward system $y(t) = x(t) + x(t + 0.1)$ X

Stability in continuous-time systems

- A system is said to be stable in the bounded-input bounded-output (BIBO) sense if any bounded input signal produce a bounded output signal. For stability of a continuous-time system: $x(t) < B_x < \infty \Rightarrow y(t) < B_y < \infty$.
- For a CTLTI system to be stable, its impulse response must be absolute integrable.

$$\int_{-\infty}^{\infty} |h(\tau)| d\tau < \infty$$

- Example 16: Stability of a first-order CTLTI system described by the DE:

$$\dot{y}(t) + ay(t) = x(t)$$

The step response of the system is when $x(t) = u(t)$

$$\frac{dy(t)}{dt} + ay(t) = u(t) \Rightarrow y(t) = ce^{-at} + \frac{1}{a}$$

$y(0) = 0$. (We take the initial value to be zero since the system is specified to be CTLTI. Non-zero initial conditions cannot be linear: Based on a zero input signal must produce a zero output signal).

$$y(0) = 0 \Rightarrow 0 = c + 1/a \Rightarrow c = -1/a$$

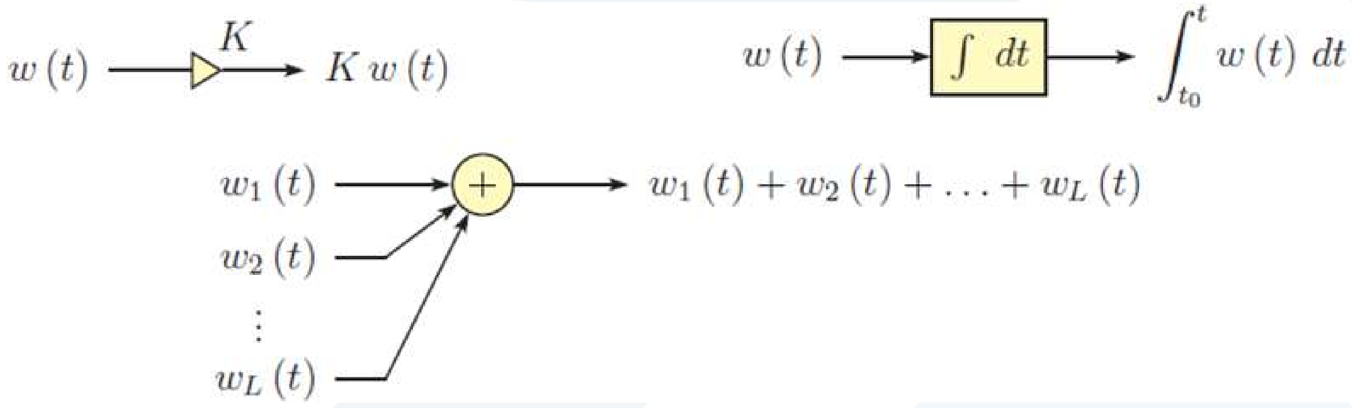
$$s(t) = \frac{1}{a}(1 - e^{-at})u(t) \qquad h(t) = \frac{ds(t)}{dt} = s(t) = e^{-at}u(t)$$

$$\int_{-\infty}^{\infty} |h(t)|\, dt = \int_{0}^{\infty} e^{-at}\, dt = \frac{1}{a}$$

Thus the system is stable if $a > 0$.

6. Block Diagram Representation of Continuous-Time Systems

- Block diagrams for CT systems are constructed using three types of components, namely constant-gain amplifiers, signal adders and integrators.

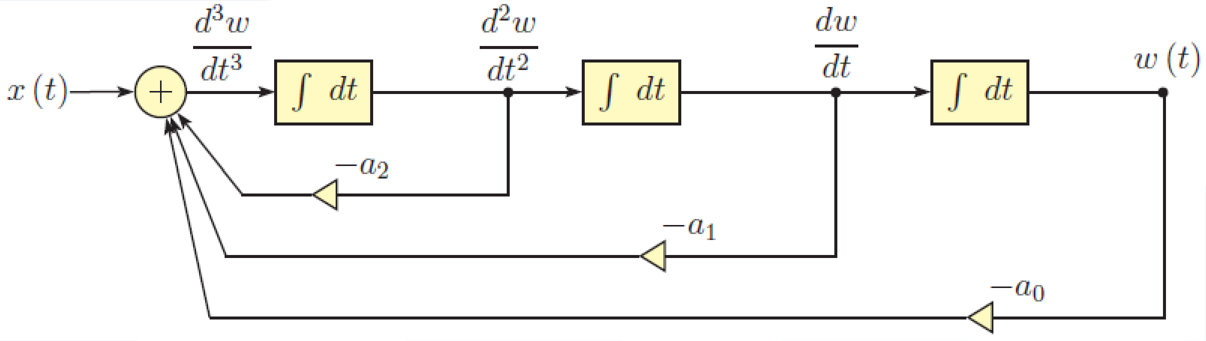


- Finding a block diagram from a DE is best explained with an example.

$$\frac{d^3y}{dt^3} + a_2\frac{d^2y}{dt^2} + a_1\frac{dy}{dt} + a_0 y = b_2\frac{d^2x}{dt^2} + b_1\frac{dx}{dt} + b_0 x$$

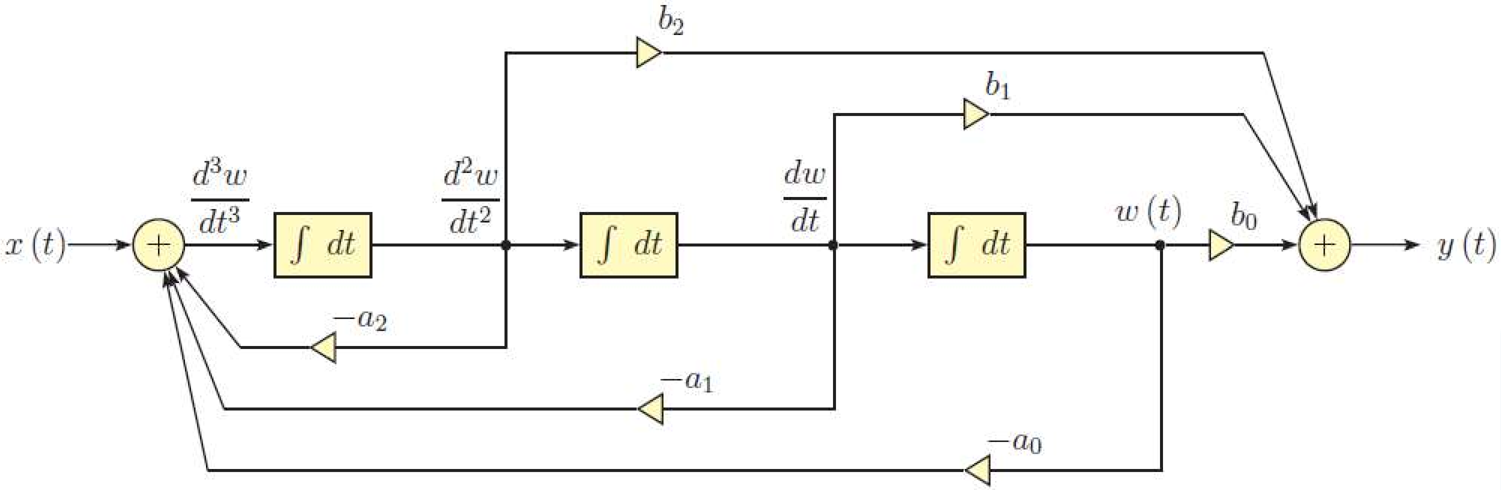
- We will introduce an intermediate variable $w(t)$

$$\frac{d^3w}{dt^3} + a_2\frac{d^2w}{dt^2} + a_1\frac{dw}{dt} + a_0w = x \Rightarrow \frac{d^3w}{dt^3} = x - a_2\frac{d^2w}{dt^2} - a_1\frac{dw}{dt} - a_0w$$



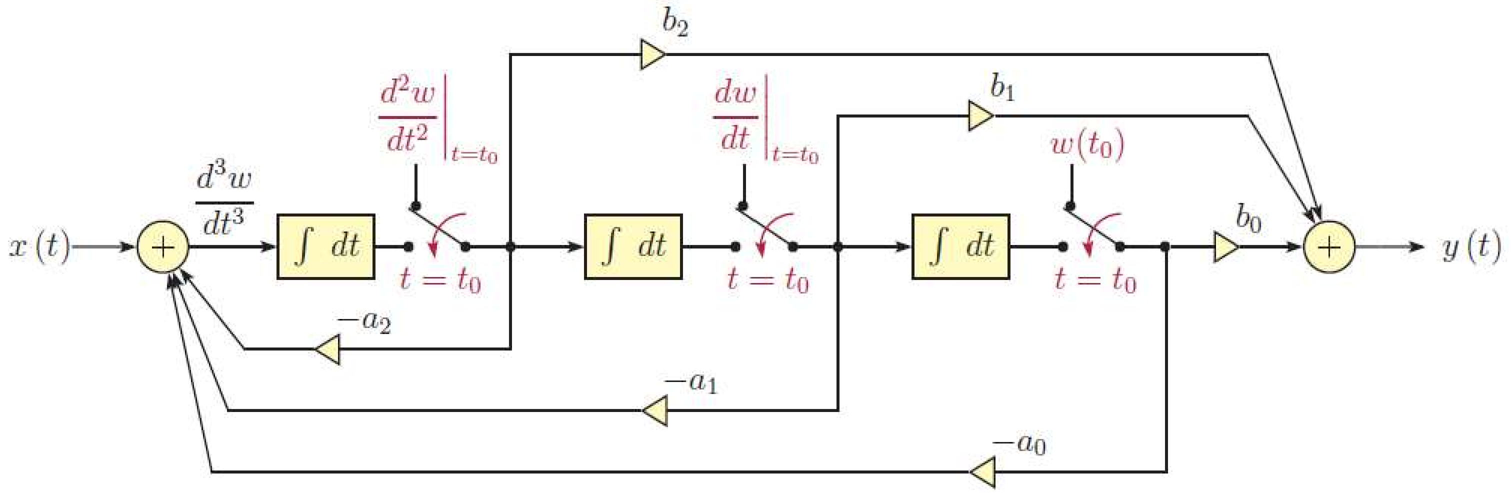
- The output signal $y(t)$ can be expressed in terms of $w(t)$ as:

$$y = b_2\frac{d^2w}{dt^2} + b_1\frac{dw}{dt} + b_0w$$



Imposing initial conditions

- Initial values of $y(t)$ and its first $N-1$ derivatives need to be converted to corresponding initial values of $w(t)$ and its first $N-1$ derivatives.



- Example 17: Block diagram for continuous-time system

$$\frac{d^3y}{dt^3}+5\frac{d^2y}{dt^2}+17\frac{dy}{dt}+13y=x+2\frac{dx}{dt} \qquad x(t)=\cos(20\pi t),\ y(0)=1,\ \left.\frac{dy}{dt}\right|_{t=0}=2,\ \left.\frac{d^2y}{dt^2}\right|_{t=0}=-4$$

$$\frac{d^3w}{dt^3} + 5\frac{d^2w}{dt^2} + 17\frac{dw}{dt} + 13w = x, \quad y = w + 2\frac{dw}{dt}$$

$$y(0) = 1 = w(0) + 2\left.\frac{dw}{dt}\right|_{t=0}, \quad \left.\frac{dy}{dt}\right|_{t=0} = 2 = \left.\frac{dw}{dt}\right|_{t=0} + 2\left.\frac{d^2w}{dt^2}\right|_{t=0},$$

$$\left.\frac{d^2y}{dt^2}\right|_{t=0} = -4 = \left.\frac{d^2w}{dt^2}\right|_{t=0} + 2\left.\frac{d^3w}{dt^3}\right|_{t=0}, \quad \left.\frac{d^3w}{dt^3}\right|_{t=0} = x(0) - 5\left.\frac{d^2w}{dt^2}\right|_{t=0} - 17\left.\frac{dw}{dt}\right|_{t=0} - 13w(0)$$

$x(0) = 1$. Solving Equations, the initial values of integrator outputs are:

$$w(0) = \frac{-71}{45}, \left.\frac{dw}{dt}\right|_{t=0} = \frac{58}{45},$$

$$\left.\frac{d^2w}{dt^2}\right|_{t=0} = \frac{16}{45}$$