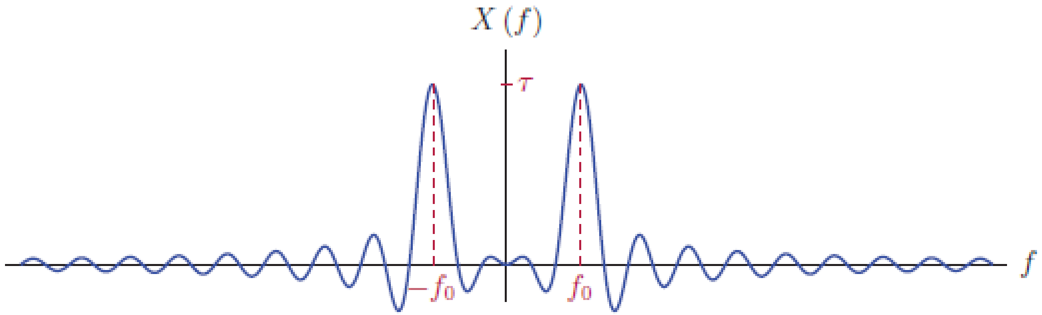
CEDC403: Signals and Systems

Lecture Notes 4: Analyzing Discrete Time Systems in the Time Domain



Ramez Koudsieh, Ph.D.

Faculty of Engineering

Department of Robotics and Intelligent Systems

Manara University

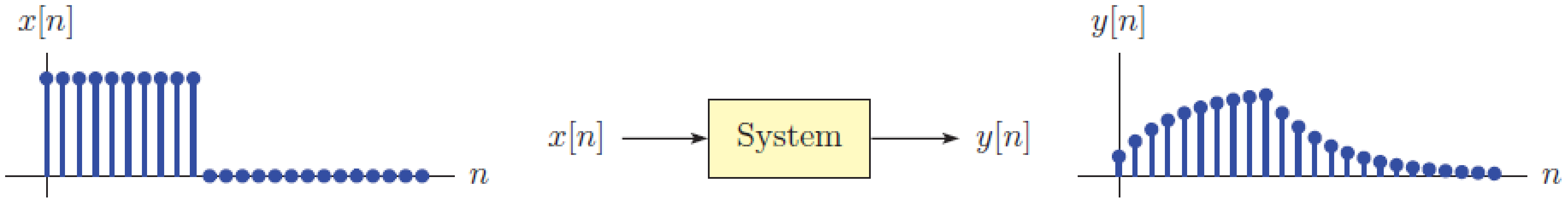
Chapter 3

Analyzing Discrete Time Systems in the Time Domain

1. Linearity and Time Invariance
2. Difference Equations for Discrete-Time Systems
3. Constant-Coefficient Linear Difference Equations
4. Impulse Response and Convolution
5. Causality and Stability in Discrete-Time Systems
6. Block Diagram Representation of Discrete-Time Systems

Introduction

- In general, a discrete-time (DT) system is a mathematical formula, method or algorithm that defines a cause-effect relationship between a set of discrete-time input signals and a set of discrete-time output signals.



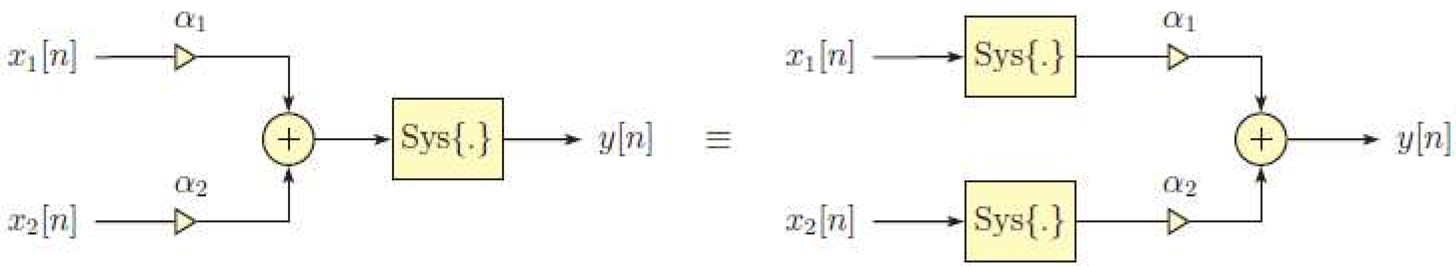
- The input signal is $x[n]$, and the output signal is $y[n]$. The system may be represented by $y[n] = T\{x[n]\}$, where $T\{.\} =$ Sys{.} denoted the transformation that defines the system in the time domain.
- A very simple example is a system that simply multiplies its input signal by a constant gain factor K to yield an output signal $y[n] = K\,x[n]$,

- Or one that delays its input signal by m samples $y[n] = x[n-m]$,
- Or one that produces an output signal proportional to the square of the input signal $y[n] = K[x[n]]^2$.

1. Linearity and Time Invariance

Linearity in discrete-time systems

- A system T is linear, if for all functions x_1 and x_2 and all constants α_1 and α_2, the following condition holds: $T\{\alpha_1 x_1[n] + \alpha_2 x_2[n]\} = \alpha_1 T\{x_1[n]\} + \alpha_2 T\{x_2[n]\}$.



- Linear systems are much easier to design and analyze than nonlinear systems.

- Example 1: Testing linearity of discrete-time systems

  a. $y[n] = 3x[n] + 2x[n-1]$ ✓ b. $y[n] = 3x[n] + 2x[n-1]\,x[n+1]$ ✗

  c. $y[n] = a^{-n}x[n]$ ✓

- A direct consequence of the linearity property is that, for linear systems, $T\{0\} = 0$ (zero-in/zero-out property).

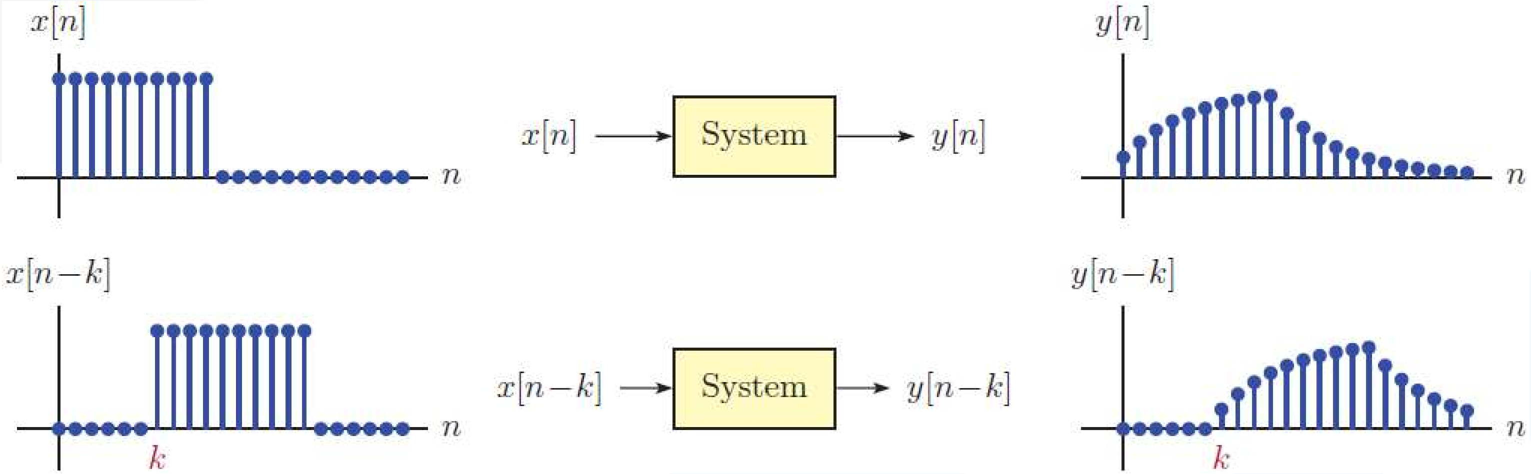
Time Invariance in discrete-time systems

- A system T is said to be time invariant if, for every function x and every integer constant k, the following condition holds:

$$T\{x[n]\} = y[n] \Rightarrow T\{x[n-k]\} = y[n-k]$$

- Example 2: Testing time invariance of discrete-time systems

  a. $y[n] = y[n-1] + 3x[n]$ ✓ b. $y[n] = x[n]\,y[n-1]$ ✓ c. $y[n] = nx[n-1]$ ✗



2. Difference Equations for Discrete-Time Systems

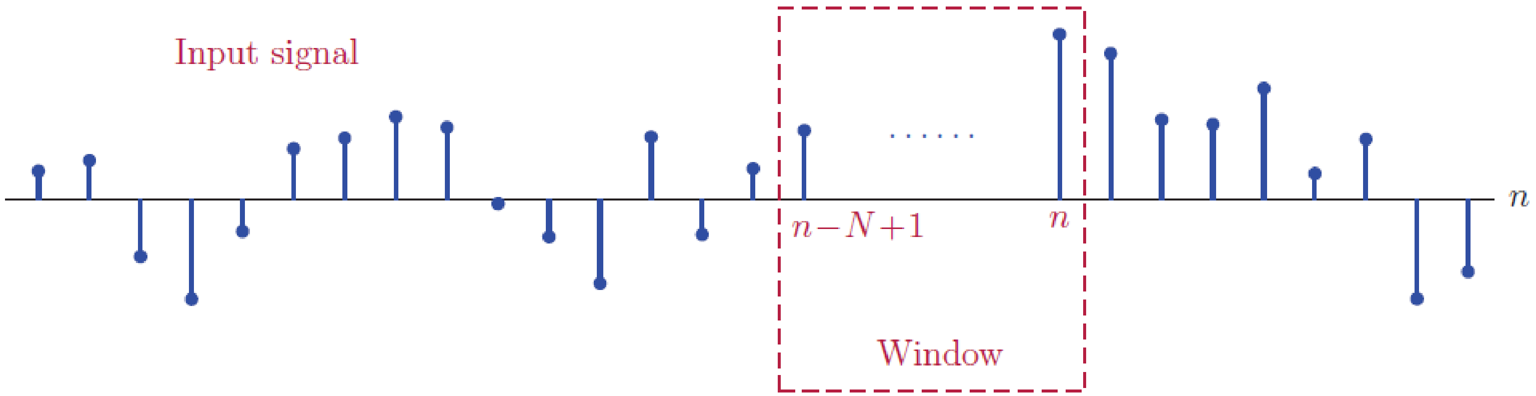
- One method of representing the relationship established by a system between its input and output signals is a difference equation (DE).
- A DT systems can be modeled with difference equations involving current, past, or future samples of input and output signals.

- Example 4: Moving-average filter

A length-N moving average filter is a simple system that produces an output equal to the arithmetic average of the most recent N samples of the input signal.

$$y[n] = \frac{x[n] + x[n-1] + \cdots + x[n-(N-1)]}{N} = \frac{1}{N}\sum_{k=0}^{N-1} x[n-k]$$

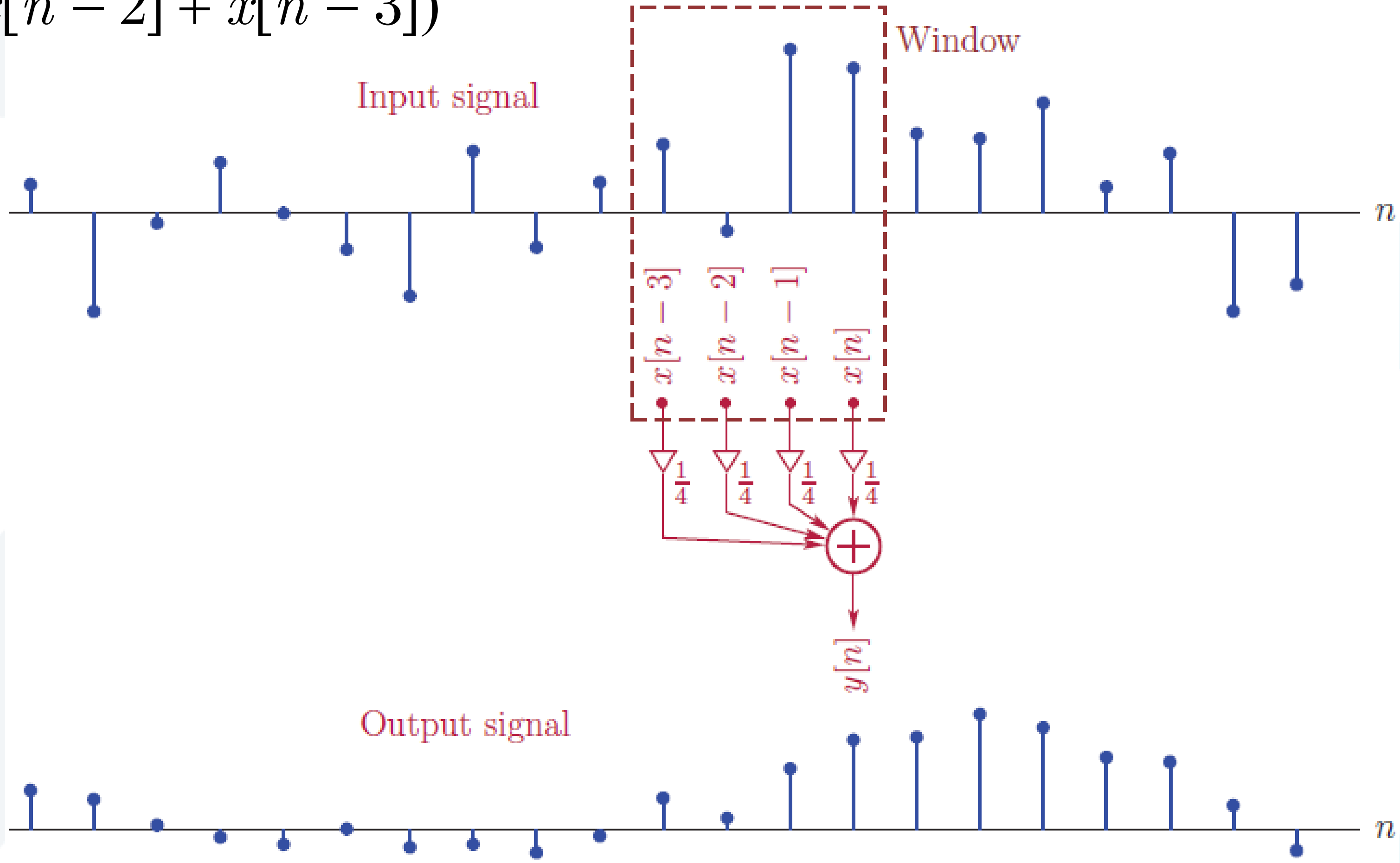
- Moving average filters are used in to smooth the variations in a signal.

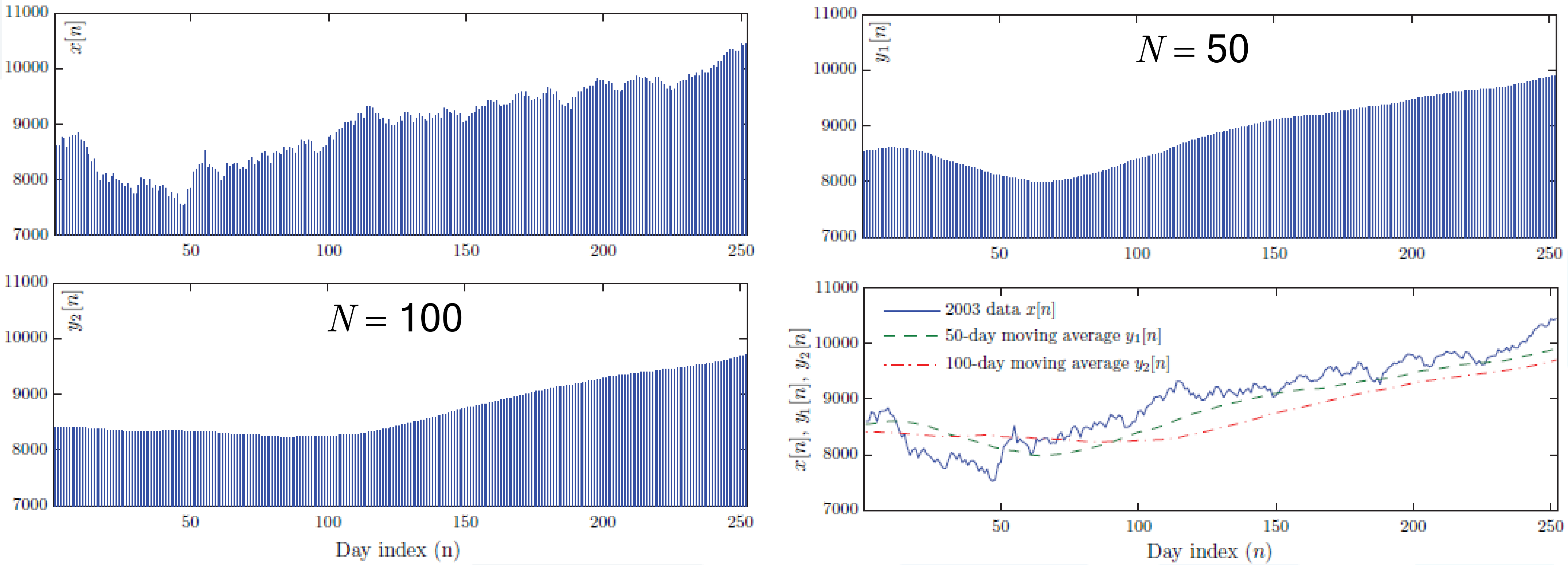


- The degree of smoothing is dependent on N, the size of the window.

- Example 5: Length-4 moving-average filter

$$y[n] = \frac{1}{4}(x[n] + x[n-1] + x[n-2] + x[n-3])$$





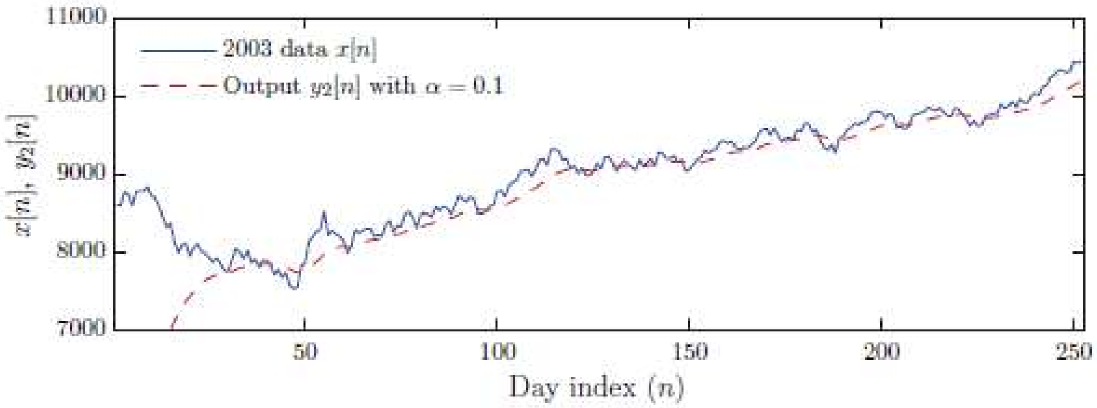
- Example 6: Exponential smoother

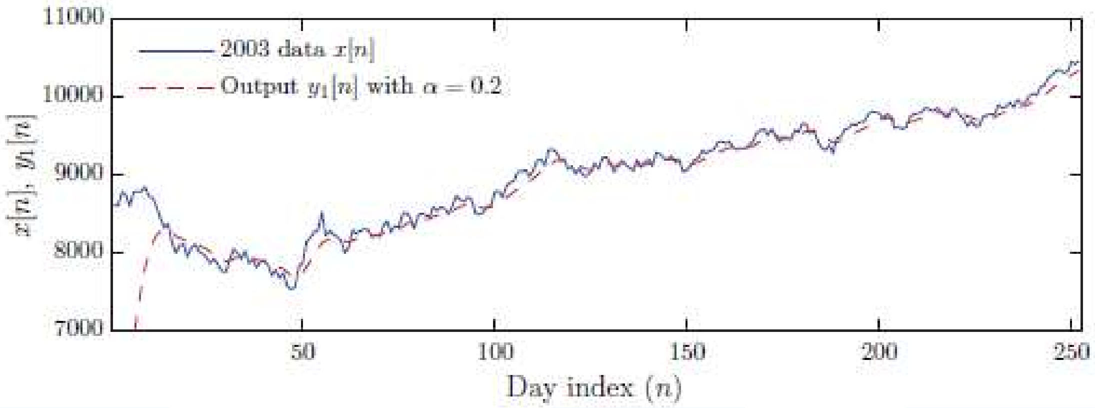
  An exponential smoother which employs a difference equation with feedback.

- The current output sample is computed as a mix of the current input sample and the previous output sample through the equation:

$$y[n] = (1 - \alpha)\, y[n-1] + \alpha x[n]$$

- The parameter $0 < \alpha < 1$ is a constant, it controls the degree of smoothing.
- Figure below illustrates the application of the linear exponential smoother to the 2003 Dow Jones Industrial Average data for $\alpha = 0.1$ and $\alpha = 0.2$.





3. Constant-Coefficient Linear Difference Equations

- In general, DTLTI systems can be modeled with linear difference equations that have constant coefficients in the form: $\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$.
- The order of the DE (= the order of the system it represents) = max(N, M).
- In general, a constant-coefficient linear DE has a family of solutions. To find a unique solution for $n \geq n_0$, the initial values $y[n_0 - 1], \ldots, y[n_0 - N]$ are needed.
- The linear difference equation $\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$

  represents a linear system provided that all initial conditions are equal to zero: $y[n_0 - k] = 0$ for $k = 1, \ldots, N$. And represents a time invariance system.

Solution of the general linear difference equation

Zero-input response and Zero-states response

- Considering the input signal $x[n]$ and the ICs two different inputs, using superposition we have that the complete response of the DE is composed of a zero-input response, due to the ICs when the input $x[n]$ is zero, and the zero-state response due to the input $x[n]$ with zero ICs.

- Zero-state response is linear with respect to the input. While zero-input response is linear with respect to the initial conditions.

Zero-input response $y_{zi}[n]$

- The zero-input response $y_{zi}[n]$ is the output of the system when the external input $x[n]$ is zero. It is produced by the system because of the initial conditions.

Zero-input response $y_{zi}[n]$ satisfies: $\sum_{k=0}^{N} a_k y_{zi}[n-k] = 0$

Determine the characteristic equation $\sum_{k=0}^{N} a_k z^{-k} = 0$ of the system by replacing delayed versions of the output signal $y_{zi}[n]$ the corresponding negative powers of the complex variable z: $y_{zi}[n-k] \to z^{-k}$.

Let the roots of the characteristic polynomial $\sum_{k=0}^{N} a_k z^{-k}$ be z_1, z_2, ..., z_N.

- If all z_k are of order 1, $y_{zi}[n] = \sum_{k=1}^{N} c_{zik} z_k^n$, c_{zik} could be determined by the ICs
- If a root z_1 is repeated k times, $y_{zi}[n] = c_{11} z_1^n + c_{12} n z_1^n + \cdots + c_{1k} n^{k-1} z_1^n$ + other terms
- If a root $z_{1,2}$ is complex-valued $z_1 = r_1 e^{j\Omega_1}$, $z_1 = r_1 e^{-j\Omega_1}$,

$$y_{zi}[n] = d_1 r_1^n \cos(\Omega_1 n) + d_2 r_1^n \sin(\Omega_1 n) + \text{other terms}$$

- Example 6: Zero-input response of second-order system

$y[n]-\frac{5}{6}y[n-1]+\frac{1}{6}y[n-2]=0$ $y[-1]=19$ and $y[-2]=53$

Determine the zero-input response of this system for $n \geq 0$.

$$z^2-\frac{5}{6}z+\frac{1}{6}=(z-\frac{1}{2})(z-\frac{1}{3})=0 \Rightarrow y_{zi}[n]=c_{zi1}\left(\frac{1}{2}\right)^n+c_{zi2}\left(\frac{1}{3}\right)^n, \text{ for } n \geq 0$$

$y[-1]=19$, and $y[-2]=53 \Rightarrow c_{zi1}=2,\ c_{zi2}=5$

$$y_{zi}[n]=2\left(\frac{1}{2}\right)^n u[n]+5\left(\frac{1}{3}\right)^n u[n]$$

- Example 7: Zero-input response of second-order system

$a.\ y[n]-1.4y[n-1]+0.85y[n-2]=0$ $y[-1]=5$ and $y[-2]=7$

$b.\ y[n]-1.6y[n-1]+0.64y[n-2]=0$ $y[-1]=2$ and $y[-2]=-3$

Determine the zero-input response of this system for $n \geq 0$.

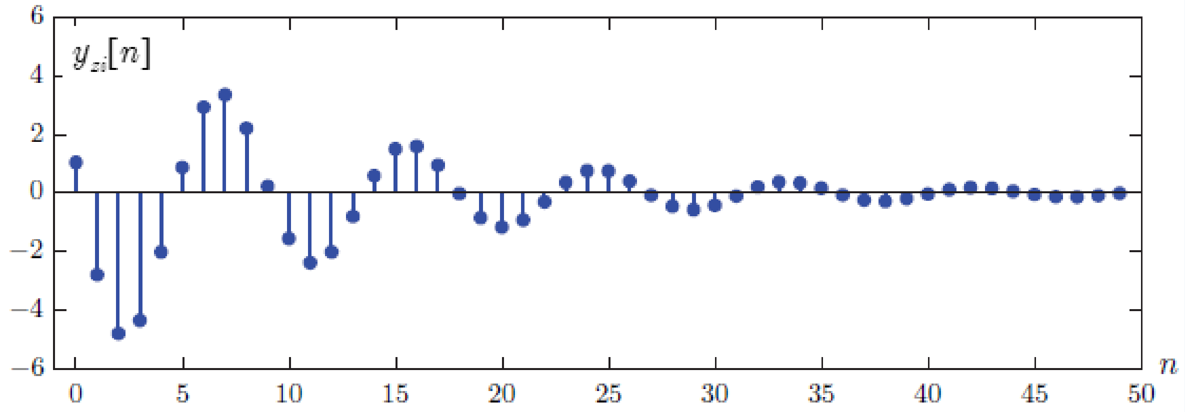
a. The characteristic equation is

$$z^2 - 1.4z + 0.85 = 0, \quad z_{1,2} = 0.7 \pm 0.6j = 0.922e^{\pm 0.7086j}$$

$$y_{zi}[n] = d_1 0.922^n \cos(0.7086n) + d_2 0.922^n \sin(0.7086n), \quad \text{for } n \geq 0$$

$y[-1] = 5$ and $y[-2] = 7 \Rightarrow d_1 = 1.05$ and $d_2 = -5.8583$

$$y_{zi}[n] = 1.05(0.922)^n \cos(0.7086n)u[n] - 5.8583(0.922)^n \sin(0.7086n)u[n]$$

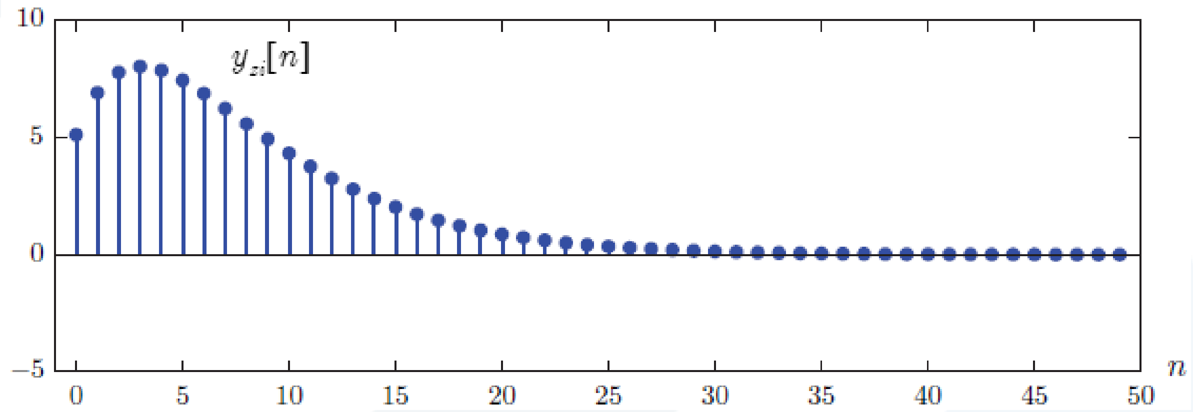


b. The characteristic equation is $z^2 - 1.6z + 0.64 = 0, \quad z_{1,2} = 0.8$

$$y_{zi}[n] = c_1(0.8)^n + c_2 n(0.8)^n, \quad \text{for } n \geq 0$$

$y[-1] = 2$ and $y[-2] = -3 \Rightarrow c_1 = 5.12$ and $c_2 = 3.52$

$$y_{zi}[n] = 5.12(0.8)^n u[n] + 3.52n(0.8)^n u[n]$$



Zero-state response $y_{zs}[n]$

- The zero-state response $y_{zs}[n]$ is the output of the system to the external input signal $x(t)$. The initial conditions of the system are assumed to be zero.
- If all z_k are of order 1, $y_{zs}[n] = \sum_{k=1}^{N} c_{zsk}\alpha_k^n + y_p[n]$, c_{zik} is determined by the ICs

- The complete solution = Zero-input response + Zero-state response

$$y[n] = y_{zi}[n] + y_{zs}[n] = \underbrace{\sum_k c_{zik}\alpha_k^n}_{zero-input\ response} + \underbrace{\sum_k c_{zsk}\alpha_k^n + y_p(t)}_{zero-state\ response}$$

- Example 9: Response of a second-order system

The DLTI system described by the difference equation:

$$y[n] + 3y[n-1] + 2y[n-2] = x[n]$$

has input $x[n] = 2^n u[n]$ and initial conditions $y[-1] = 0$ and $y[-2] = ½$. Determine the zero-input response and the zero-state response of this system.

$$z^2 + 3z + 2 = (z+1)(z+2) = 0 \Rightarrow y_{zi}[n] = c_{zi1}(-1)^n + c_{zi2}(-2)^n, \text{ for } n \geq 0$$

$y_{zi}[-1] = 0$, and $y_{zi}[-2] = ½ \Rightarrow c_{zi1} = 1$, $c_{zi2} = -2$

$$y_{zi}[n] = (-1)^n - 2(-2)^n, \text{ for } n \geq 0$$

$y_p[n] = k2^n \Rightarrow k = 1/3$ $y_{zs}[n] = c_{zs1}(-1)^n + c_{zs2}(-2)^n + \frac{1}{3}2^n$, for $n \geq 0$

$y_{zs}[-1] = y_{zs}[-2] = 0$

$y_{zs}[0] = -3y_{zs}[-1] - 2y_{zs}[-2] + x[0] = 1$

$y_{zs}[1] = -3y_{zs}[0] - 2y_{zs}[-1] + x[1] = -1$

$$\Rightarrow \begin{cases} c_{zs1} = -\frac{1}{3} \\ c_{zs2} = 1 \end{cases}$$

$y_{zs}[n] = -\frac{1}{3}(-1)^n + (-2)^n + \frac{1}{3}2^n$, for $n \geq 0$

The complete solution:

$$y[n] = y_{zi}[n] + y_{zs}[n] = \underbrace{(-1)^n - 2(-2)^n}_{zero-input\ response} \underbrace{-\frac{1}{3}(-1)^n + (-2)^n + \frac{1}{3}2^n}_{zero\text{-}state\ response}, \text{ for } n \geq 0$$

Linearity properties of zero-input and zero-state response

- Zero-state response is linear with respect to the input.
- Zero-input response is linear with respect to the initial state.

Homogeneous solution (natural response) & particular solution (forced response)

$y[n] = y_h[n] + y_p[n]$.

Homogeneous solution $y_h(t)$ satisfies $\sum_{k=0}^{N} a_k y[n-k] = 0$

- $y_h[n]$ depends on the structure of the system as well as the initial state of the system.
- For a stable system, $y_h[n]$ tends to gradually disappear in time.
- $y_p[n]$ is due to the input signal $x[n]$ being applied to the system. It is referred to as the particular solution of the difference equation.
- $y_p[n]$ depends on the input signal $x[n]$ and the internal structure of the system, but it does not depend on the initial state of the system.

- The particular solution $y_p[n]$ represents any solution of the DE for the given input.
- A particular solution is usually obtained by assuming an output of the same form as the input.

Input signal	Particular solution
n^k	$k_n n^k + k_{n-1} n^{k-1} + \dots k_1 n + k_0$ (Constant input is a special case with $k = 0$)
α^{an}	$k\alpha^{an}$, α is not the characteristic value $k_1 n\alpha^{an} + k_0\alpha^{an}$, α is the characteristic value with order 1 $k_k n^k \alpha^{an} + k_{k-1} n^{k-1}\alpha^{an} + \dots k_1 n\alpha^{an} + k_0\alpha^{an}$, α is the characteristic value with order k
$\cos(\Omega n)$ or $\sin(\Omega n)$	$k_1\cos(\Omega n) + k_2\sin(\Omega n)$

- Example 8: Find the total response of the exp. smoother $y[n] = (1-\alpha)y[n-1] + \alpha x[n]$ when the input is $x[n] = 20\cos(0.2\pi n)$. Use $\alpha = 0.1$ and $y[-1] = 2.5$.

The homogeneous solution is in the form: $y_h[n] = c(1-\alpha)^n$

The form of the particular solution is: $y_p[n] = k_1\cos(0.2\pi n) + k_2\sin(0.2\pi n)$

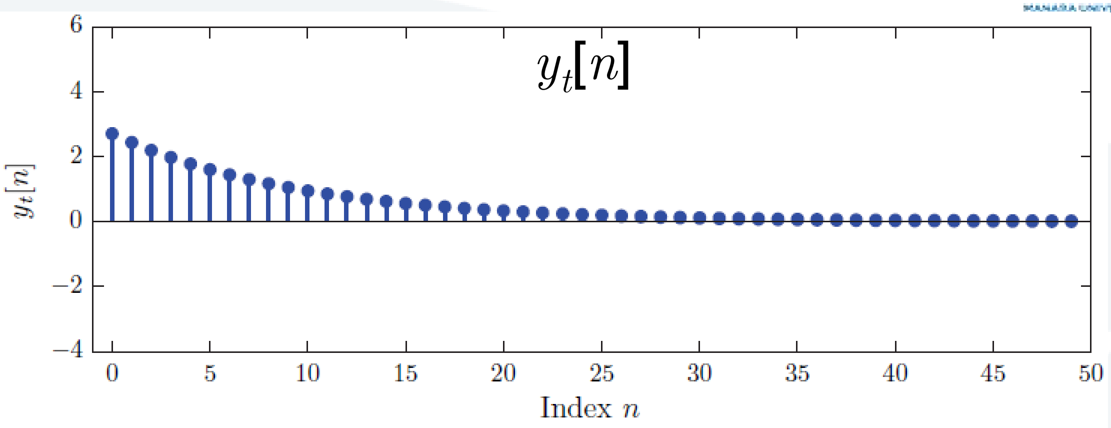
$$k_1 = \frac{\alpha A[1-(1-\alpha)\cos(0.2\pi)]}{1-2(1-\alpha)\cos(0.2\pi)+(1-\alpha)^2}, \quad k_2 = \frac{\alpha A(1-\alpha)\sin(0.2\pi)}{1-2(1-\alpha)\cos(0.2\pi)+(1-\alpha)^2}$$

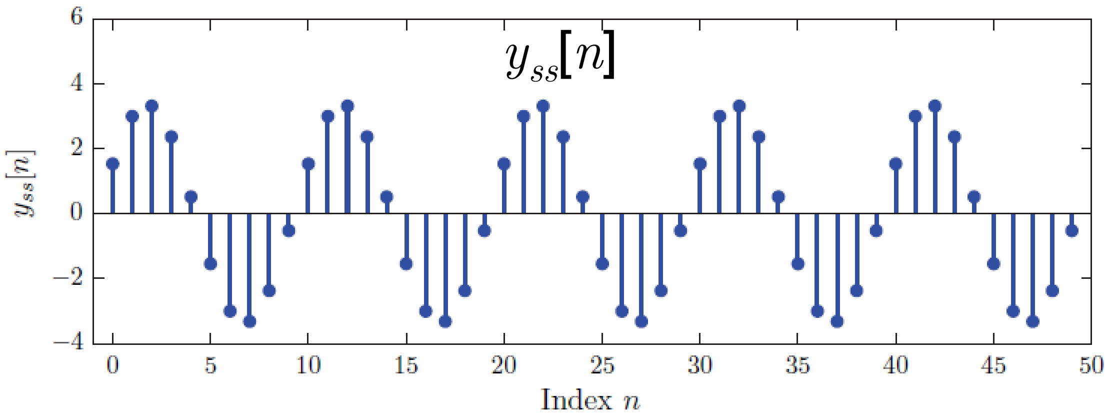
Using the specified parameter values: $k_1 = 1.537$ and $k_2 = 2.991$

$y[n] = c(0.9)^n + 1.537\cos(0.2\pi n) + 2.991\sin(0.2\pi n)$, $y[-1] = 2.5 \Rightarrow c = 2.713$

$y[n] = 2.7129(0.9)^n + 1.5371\cos(0.2\pi n) + 2.9907\sin(0.2\pi n)$, for $n \geq 0$

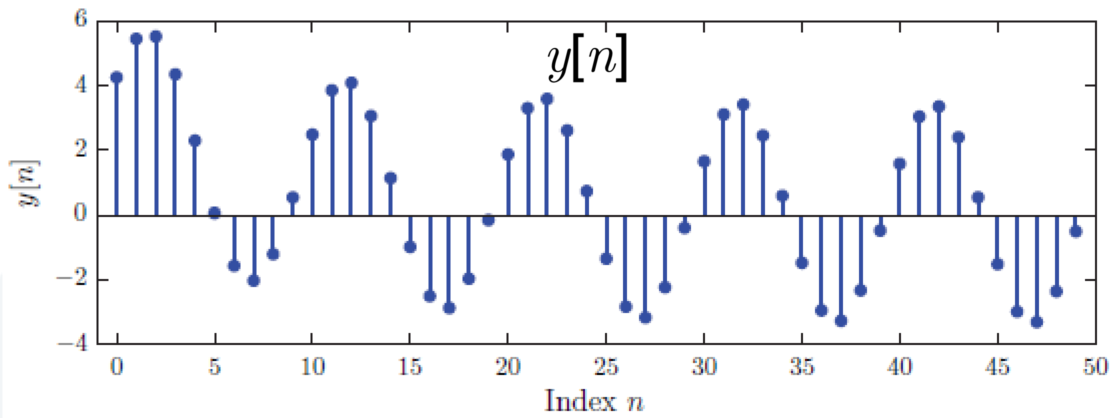
$y[n]$ consists of two components. The first term is the transient response: $y_t[n] = 2.7129(0.9)^n$, which is due to the initial state of the system.





The remaining terms represent the steady-state response of the system:

$$y_{ss}[n] = 1.5371\cos(0.2\pi n) + 2.9907\sin(0.2\pi n)$$



4. Impulse Response and Convolution

Discrete-time impulse response

- The impulse response $h[n]$ of an DTLTI system is the zero-state response of the system when a unit impulse $\delta[n]$ is applied at the input.

$$\sum_{k=0}^{N} a_k h[n-k] = \sum_{k=0}^{M} b_k \delta[n-k]$$

subject to initial conditions $h[-1] = h[-2] = \cdots = h[-N] = 0$

4. Impulse Response and Convolution

Discrete-time impulse response

- The impulse response $h[n]$ of an DTLTI system is the zero-state response of the system when a unit impulse $\delta[n]$ is applied at the input.

$$\sum_{k=0}^{N} a_k h[n-k] = \sum_{k=0}^{M} b_k \delta[n-k]$$

Convolution operation for DTLTI systems

- The (DT) convolution of x and h, denoted $x * h$, is defined as the function:

$$x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

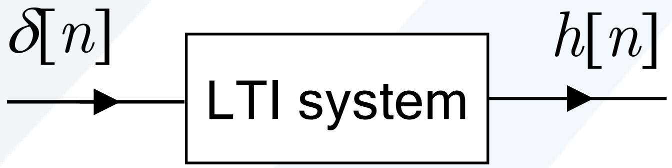
Properties of Convolution

- Commutative. That is, for any two functions x and h, $x * h = h * x$.
- Associative. That is, for any functions x, h_1, and h_2, $(x * h_1) * h_2 = x * (h_1 * h_2)$.

- Distributive. That is, for any functions x, h_1, and h_2, $x * (h_1 + h_2) = x * h_1 + x * h_2$.
- For any function x, $x[n] * \delta[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k] = x[n]$
- Moreover, δ is the convolutional identity. That is, for any function x, $x * \delta = x$.

Finding impulse response of a DTLTI system

- The response h of a system T to the input δ is called the impulse response of the system.
- For any LTI system with input x, output y, and impulse response h: $y = x * h$.
- A LTI system is completely characterized by its impulse response.



Step Response of a DTLTI system

- The response s of a system T to the input u is called the step response of the system.

$$s[n] = \sum_{k=-\infty}^{\infty} u[k]\,h[n-k] = \sum_{k=0}^{\infty} u[k]\,h[n-k]$$

- The impulse response h and step response s of a LTI system are related as:

$$h[n] = s[n] - s[n-1]$$

- Example 10: Impulse response of moving average filters

Length-N moving average filter $y[n] = \dfrac{1}{N}\sum_{k=0}^{N-1} x[n-k]$

$$h_N[n] = T\{\delta[n]\} = \frac{1}{N}\sum_{k=0}^{N-1} \delta[n-k]$$

$$h_N[n] = \begin{cases} \frac{1}{N}, & n = 0, \cdots, N-1 \\ 0, & \text{otherwise} \end{cases}$$

$$h_N[n] = \frac{1}{N}(u[n] - u[n-N])$$

- Example 11: Impulse response of exponential smoother

  Find the impulse response of the exponential smoother with $y[-1] = 0$ using step response s of a LTI system.

$$y_h[n] = c(1-\alpha)^n \qquad y_p[n] = k \Rightarrow k = 1$$

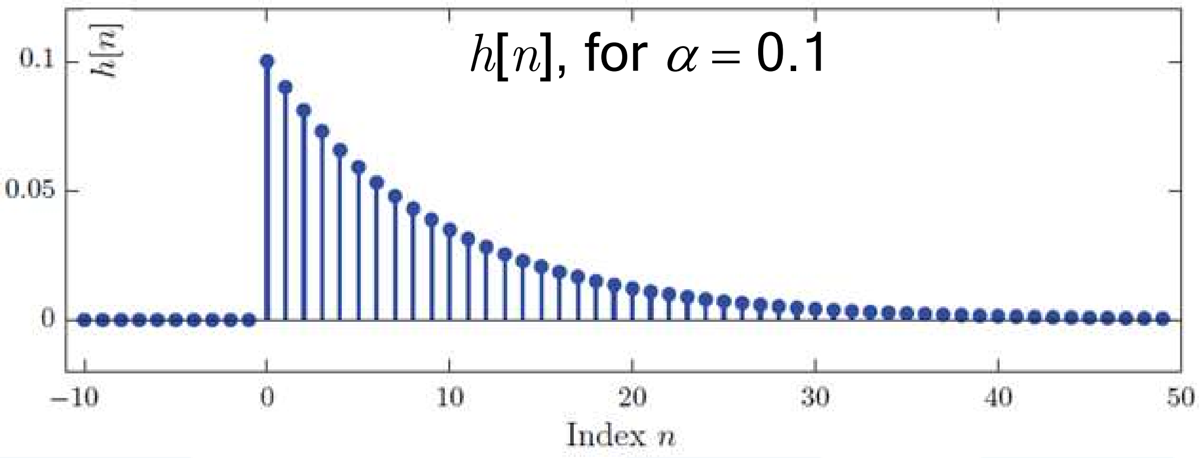
$$y[n] = y_h[n] + y_p[n] = c(1-\alpha)^n + 1$$

$$y[-1] = 0 \Rightarrow c = -(1-\alpha)$$

$$s[n] = 1 - (1-\alpha)^{n+1}, \text{ for } n \geq 0$$

$$s[n] = [1 - (1-\alpha)^{n+1}]u[n]$$

$$h[n] = s[n] - s[n-1] = \alpha(1-\alpha)^n u[n]$$



- Example 12: A simple discrete-time convolution example

A discrete-time system is described through the impulse response

$$h[n] = \{\underset{\uparrow}{4}, 3, 2, 1\}$$

Use the convolution operation to find the response of the system to the input signal $x[n] = \{\underset{\uparrow}{-3}, 7, 4\}$

$$x[k] = \{\underset{\uparrow}{-3}, 7, 4\} \qquad h[-k] = \{1, 2, 3, \underset{\substack{\uparrow \\ k=0}}{4}\} \qquad h[n-k] = \{1, 2, 3, \underset{\uparrow}{4}\}$$

$$y[n] = \sum_{k=\max(0,n-3)}^{\min(2,n)} x[k]\,h[n-k], \quad \text{for } n \geq 0$$

جامعة المنارة
MANARA UNIVERSITY

$$y[0]=\sum_{k=0}^{0}x[k]h[0-k]=x[0]h[0]=(-3)(4)=-12$$

$$y[1]=\sum_{k=0}^{1}x[k]h[1-k]=x[0]h[1]+x[1]h[0]=19$$

$$y[2]=\sum_{k=0}^{2}x[k]h[2-k]=x[0]h[2]+x[1]h[1]+x[2]h[0]=31$$

$$y[3]=\sum_{k=0}^{2}x[k]h[3-k]=x[0]h[3]+x[1]h[2]+x[2]h[1]=23$$

$$y[4]=\sum_{k=1}^{2}x[k]h[4-k]=x[1]h[3]+x[2]h[2]=15$$

$$y[5]=\sum_{k=2}^{2}x[k]h[5-k]=x[2]h[3]=4$$

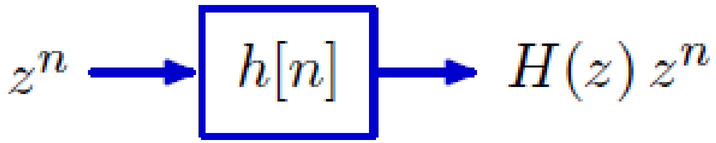
$$y[n]=\{-\underset{\uparrow}{12},19,31,23,15,4\}$$

Total Response of DTLTI system

$$y[n] = y_{zi}[n] + y_{zs}[n] = \underbrace{\sum_k c_{zik}\alpha_k^n}_{zero-input} + \underbrace{x[n] * h[n]}_{zero-state}$$

Eigenfunctions of DTLTI system

- Complex geometric sequences are eigenfunctions of DTLTI systems.



$$y[n] = (h * x)[n] = \sum_{k=-\infty}^{\infty} h[k]z^{n-k} = z^n \sum_{k=-\infty}^{\infty} h[k]z^{-k} = H(z)z^n$$

- We refer to H as the transfer function of the system.

Causality and Stability in Discrete-Time Systems

- For DTLTI systems the causality property can be related to the impulse response of the system $h[n] = 0$ for all $n < 0$:

$$y[n] = h[n] * x[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] = \sum_{k=0}^{\infty} h[k]x[n-k]$$

- For a DTLTI system to be stable, its impulse response must be absolute summable:

$$\sum_{k=-\infty}^{\infty} |h[k]| < \infty$$

- Example 13: Stability of a length-2 moving-average filter

  Comment on the stability of the length-2 moving-average filter described by the difference equation $y[n] = \frac{1}{2}x[n] + \frac{1}{2}x[n-1]$

$$|y[n]| = \left|\tfrac{1}{2}x[n] + \tfrac{1}{2}x[n-1]\right| \le \tfrac{1}{2}|x[n]| + \tfrac{1}{2}|x[n-1]|$$

Since we assume $|x[n]| < B_x$ for all n, $|y[n]| \le \frac{1}{2}B_x + \frac{1}{2}B_x = B_x$

- For a causal DTLTI system to be stable, the magnitudes of all roots of the characteristic polynomial must be less than unity.
- If a circle is drawn on the complex plane with its center at the origin and its radius equal to unity, all roots of the characteristic polynomial must lie inside the circle for the corresponding causal DTLTI system to be stable.

$$y[n] = \underbrace{\tfrac{2}{3}(-1)^n - (-2)^n}_{natural\ response} + \underbrace{\tfrac{1}{3}2^n}_{forced\ response}, \text{ for } n \geq 0$$

Causality in Discrete-Time Systems

- A system is said to be causal if the current value of the output signal depends only on current and past values of the input signal, but not on its future values.

- Example 3: causal and non causal systems

  $y[n] = y[n-1] + x[n] - 3x[n-1]$ is causal

  $y[n] = y[n-1] + x[n] - 3x[n+1]$ is non causal

- Causal systems can be implemented in real-time processing mode.

Stability in Discrete-Time Systems

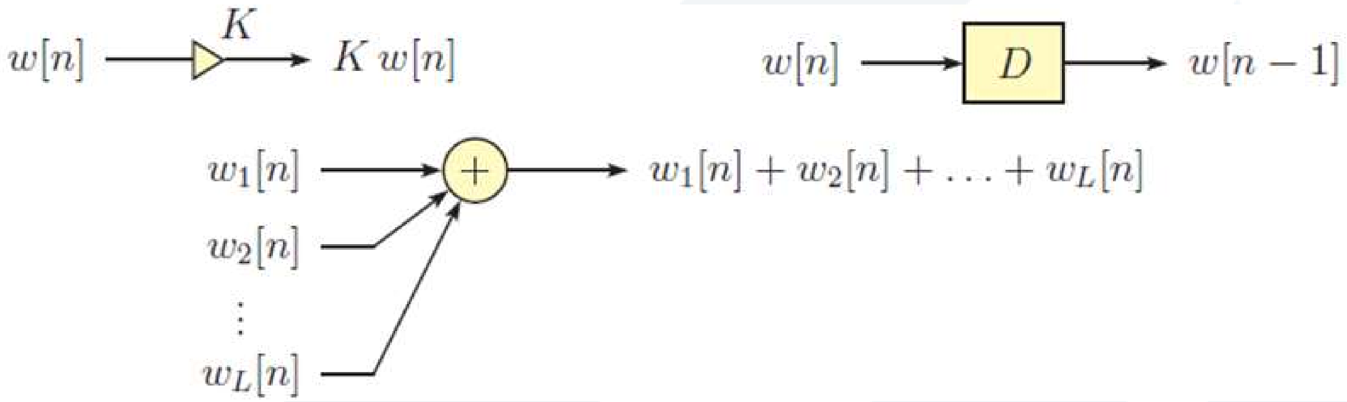
- A system is said to be stable in the bounded-input bounded-output (BIBO) sense if any bounded input signal is produce a bounded output signal.

- An input signal $x[n]$ is said to be bounded if an upper bound B_x exists such that $x[n] < B_x < \infty$ for all values of the integer index n.

- For stability of a discrete-time system: $x[n] < B_x < \infty \Rightarrow y[n] < B_y < \infty$.

5. Block Diagram Representation of Discrete-Time Systems

- Block diagrams for DT systems are constructed using three types of components, namely multiplication of a signal by a constant gain factor, addition of two signals, and time shift of a signal.

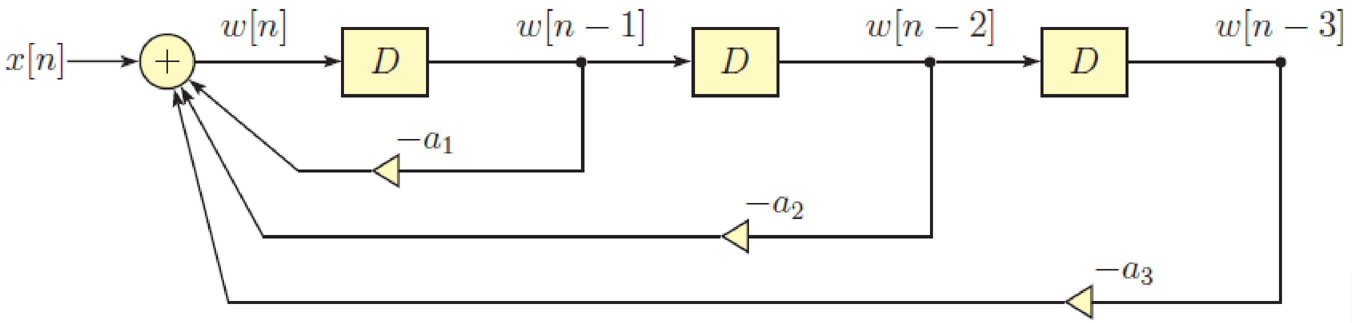


- Finding a block diagram from a DE is best explained with an example.

$$y[n] + a_1 y[n-1] + a_2 y[n-2] + a_3 y[n-3] = b_0 x[n] + b_1 x[n-1] + b_2 x[n-2]$$

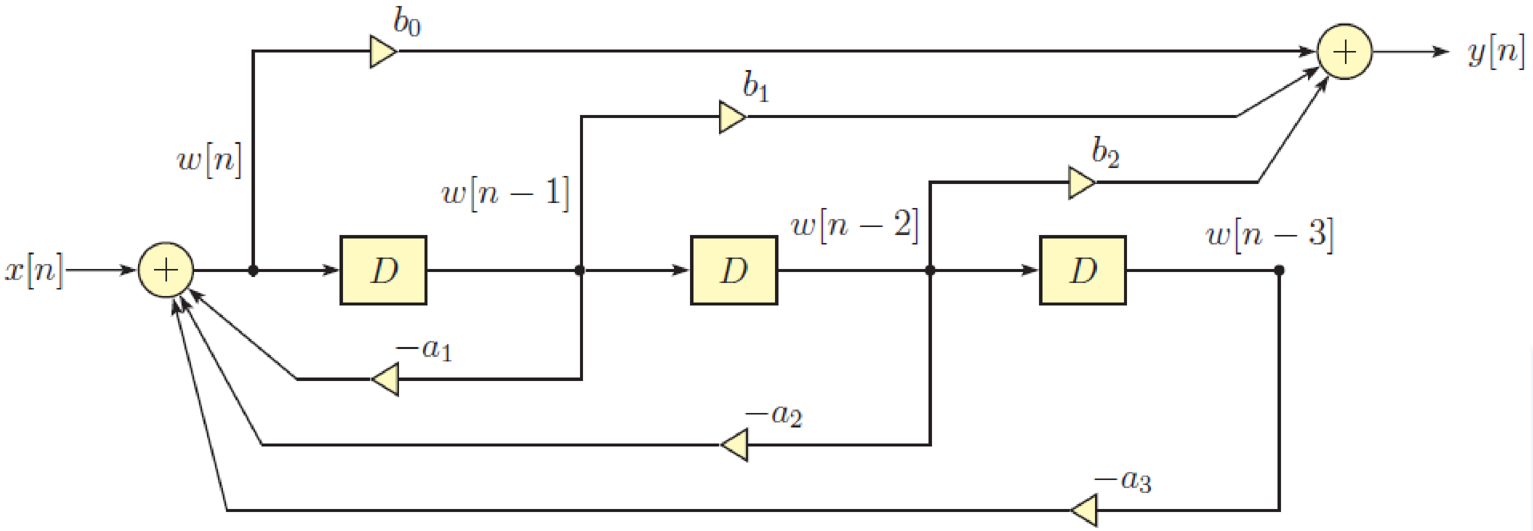
We will introduce an intermediate variable $w[n]$:

$$w[n] + a_1 w[n-1] + a_2 w[n-2] + a_3 w[n-3] = x[n]$$



- The output signal $y[n]$ can be expressed in terms of $w[n]$ as:

$$y[n] = b_0 w[n] + b_1 w[n-1] + b_2 w[n-2]$$



Imposing initial conditions

- Initial values of $y[-1]$, $y[-2]$, and $y[-3]$, need to be converted to corresponding initial values of $w[-1]$, $w[-2]$, and $w[-3]$ for the previous third-order DE. The outputs of the three delay elements should be set equal to these values.

6. Impulse Response and Convolution

Convolution operation for DTLTI systems

- The (DT) convolution of x and h, denoted $x * h$, is defined as the function:

$$x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Properties of Convolution

- Commutative. That is, for any two functions x and h, $x * h = h * x$.
- Associative. That is, for any functions x, h_1, and h_2, $(x * h_1) * h_2 = x * (h_1 * h_2)$.