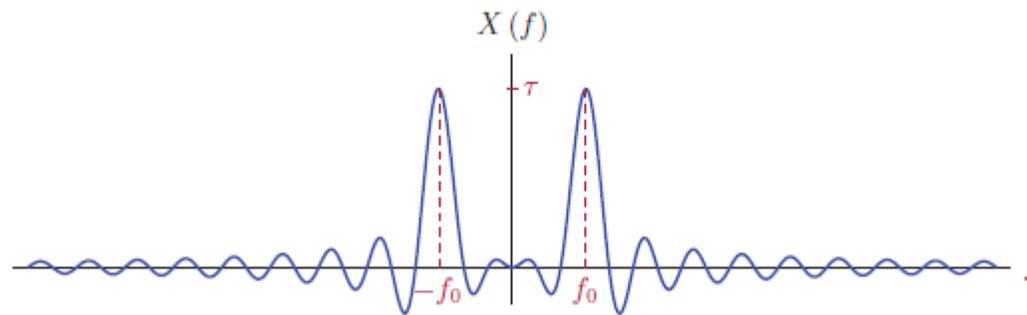


CEDC403: Signals and Systems

Lecture Notes 5 & 6: Fourier Analysis for Continuous Time Signals and Systems



Ramez Koudsieh, Ph.D.

Faculty of Engineering

Department of Robotics and Intelligent Systems

Manara University

Chapter 4

Fourier Analysis for Continuous Time Signals and Systems

- 1 Analysis of Periodic Continuous-Time Signals
- 2 Analysis of Non-Periodic Continuous-Time Signals
- 3 Energy and Power in the Frequency Domain
- 4 Transfer Function Concept
- 5 CTLTI Systems with Periodic Input Signals
- 6 CTLTI Systems with Non-Periodic Input Signals

Jean-Baptiste Joseph Fourier (1768-1830)



Born 21 March 1768 in Auxerre, Kingdom of France

Died 16 May 1830 (aged 62) in Paris, Kingdom of France

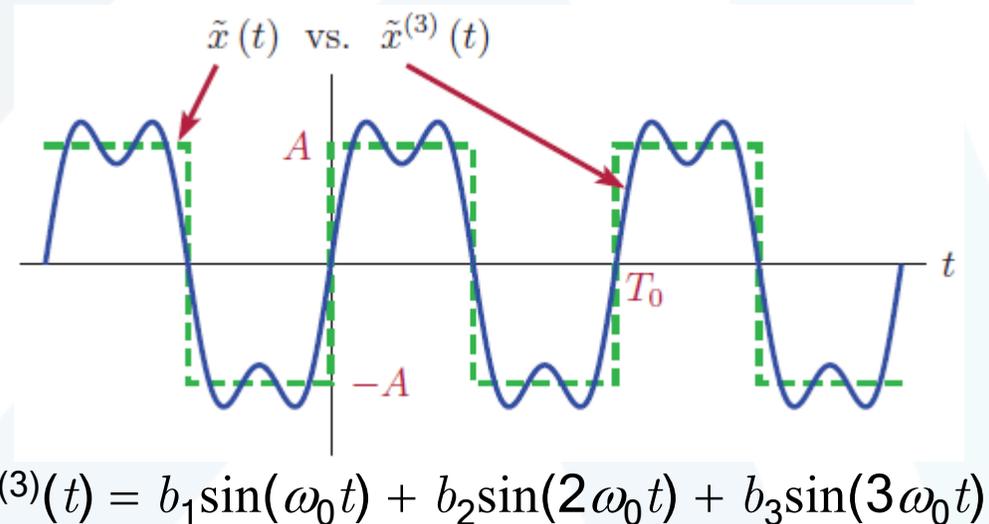
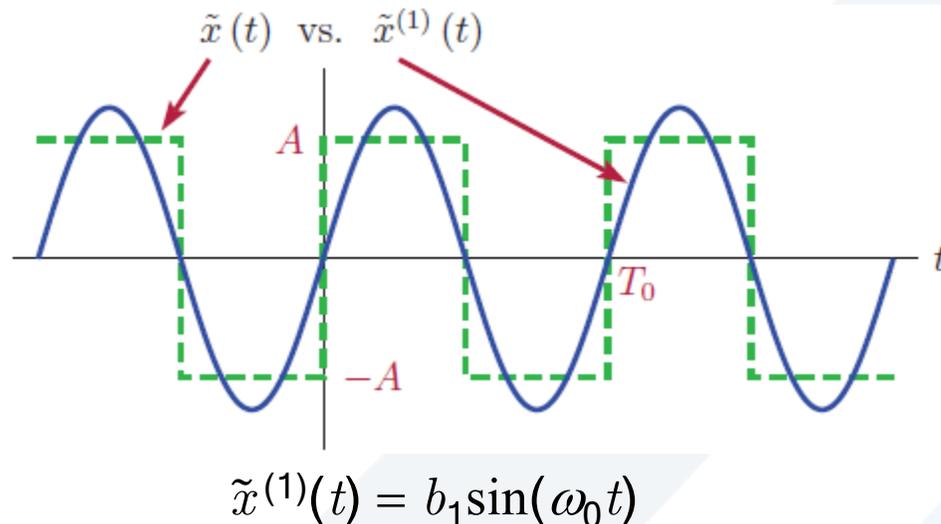
Introduction

- The frequency representation of **periodic** and **aperiodic** signals indicates how their **power** or **energy** are distributed to different frequency components. Such a distribution over frequency is called the **spectrum** of the signal.
- In the frequency domain, linear systems are described by linear **algebraic equations**, in contrast to the time-domain representation, where they are described by linear **differential equations**.
- A weighted summation of **Sines** and **Cosines** of different frequencies can be used to represent periodic (**Fourier Series**), or non-periodic (**Fourier Transform**) functions.
- The Fourier representation is also useful in finding the **frequency response** of LTI systems.

1. Analysis of Periodic Continuous-Time Signals

- We will study methods of expressing **periodic continuous-time signals** in two different but equivalent formats, namely the **trigonometric Fourier series (TFS)** and the **exponential Fourier series (EFS)**.

Approximating a periodic signal with trigonometric functions



- The **approximation error** $\tilde{\varepsilon}_1(t) = \tilde{x}(t) - \tilde{x}^{(1)}(t) = \tilde{x}(t) - b_1 \sin(\omega_0 t)$
- The best value for the coefficient b_1 would be to the value that makes the **normalized average power** of $\tilde{\varepsilon}_1(t)$ as small as possible.

$$P_\varepsilon = \frac{1}{T_0} \int_0^{T_0} [\tilde{\varepsilon}_1(t)]^2 dt = \frac{1}{T_0} \int_0^{T_0} [\tilde{x}(t) - b_1 \sin(\omega_0 t)]^2 dt, \quad \frac{dP_\varepsilon}{db_1} = 0 \Rightarrow b_1 = \frac{4A}{\pi}$$

$$\tilde{x}^{(1)}(t) = \frac{4A}{\pi} \sin(\omega_0 t), \quad \tilde{\varepsilon}_1(t) = \tilde{x}(t) - \frac{4A}{\pi} \sin(\omega_0 t)$$

$$\tilde{x}^{(2)}(t) = b_1 \sin(\omega_0 t) + b_2 \sin(2\omega_0 t)$$

$$\tilde{\varepsilon}_2(t) = \tilde{x}(t) - \tilde{x}^{(2)}(t)$$

$$P_\varepsilon = \frac{1}{T_0} \int_0^{T_0} [\tilde{\varepsilon}_2(t)]^2 dt, \quad \frac{dP_\varepsilon}{db_{1,2}} = 0 \Rightarrow \begin{cases} b_1 = \frac{4A}{\pi} \\ b_2 = 0 \end{cases} \Rightarrow \tilde{x}^{(2)}(t) = \tilde{x}^{(1)}(t)$$

$$\tilde{x}^{(3)}(t) = b_1 \sin(\omega_0 t) + b_2 \sin(2\omega_0 t) + b_3 \sin(3\omega_0 t)$$

$$P_\varepsilon = \frac{1}{T_0} \int_0^{T_0} [\tilde{\varepsilon}_3(t)]^2 dt, \quad \frac{dP_\varepsilon}{db_{1,2,3}} = 0 \Rightarrow \begin{cases} b_1 = \frac{4A}{\pi} \\ b_2 = 0 \\ b_3 = \frac{4A}{3\pi} \end{cases} \Rightarrow \tilde{\varepsilon}_3(t) = \tilde{x}(t) - \tilde{x}^{(3)}(t)$$

$$\tilde{x}^{(3)}(t) = \frac{4A}{\pi} \sin(\omega_0 t) + \frac{4A}{3\pi} \sin(3\omega_0 t)$$

Trigonometric Fourier series (TFS)

- The **Trigonometric Fourier Series** of the periodic signal $\tilde{x}(t)$ with **fundamental frequency** ω_0 :

$$\tilde{x}(t) = a_0 + a_1 \cos(\omega_0 t) + a_2 \cos(2\omega_0 t) + \dots + a_k \cos(k\omega_0 t) + \dots \\ + b_1 \sin(\omega_0 t) + b_2 \sin(2\omega_0 t) + \dots + b_k \sin(k\omega_0 t) + \dots$$

$$\tilde{x}(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(k\omega_0 t) + \sum_{k=1}^{\infty} b_k \sin(k\omega_0 t) \quad \omega_0 = 2\pi f_0 \text{ in rad/s}$$

- We call the frequencies that are integer multiples of the fundamental frequency the **harmonics**. The frequencies $2\omega_0$, $3\omega_0$, ..., $k\omega_0$ are the second, the third, and the k -th harmonics of the fundamental frequency ω_0 respectively.
- We need to determine the coefficients: a_0 , a_k , and b_k .

$$\int_{t_0}^{t_0+T_0} \cos(m\omega_0 t) \cos(k\omega_0 t) dt = \begin{cases} T_0/2, & m = k \\ 0, & m \neq k \end{cases}$$

The set $\{\cos(k\omega_0 t), k = 0, \dots, \infty\}$ is **orthogonal**

$$\int_{t_0}^{t_0+T_0} \sin(m\omega_0 t) \sin(k\omega_0 t) dt = \begin{cases} T_0/2, & m = k \\ 0, & m \neq k \end{cases}$$

The set $\{\sin(k\omega_0 t), k = 0, \dots, \infty\}$ is **orthogonal**

$$\int_{t_0}^{t_0+T_0} \sin(m\omega_0 t) \cos(k\omega_0 t) dt = 0$$

The two sets $\{\sin(k\omega_0 t)$ and $\cos(k\omega_0 t), k = 0, \dots, \infty\}$ are **orthogonal** to each other.

Trigonometric Fourier series (TFS)

1. Synthesis equation: $\tilde{x}(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(k\omega_0 t) + \sum_{k=1}^{\infty} b_k \sin(k\omega_0 t)$

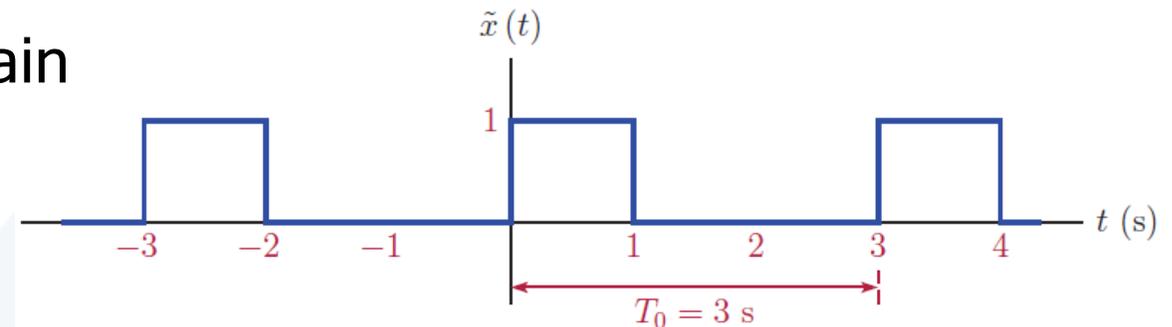
2. Analysis equation: $a_0 = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} \tilde{x}(t) dt$ (dc component)

$$a_k = \frac{2}{T_0} \int_{t_0}^{t_0+T_0} \tilde{x}(t) \cos(k\omega_0 t) dt, \quad \text{for } k = 1, 2, \dots, \infty$$

$$b_k = \frac{2}{T_0} \int_{t_0}^{t_0+T_0} \tilde{x}(t) \sin(k\omega_0 t) dt, \quad \text{for } k = 1, 2, \dots, \infty$$

- **Example 1:** TFS of a periodic pulse train

$$a_0 = \frac{1}{3} \int_0^1 (1) dt = \frac{1}{3}$$



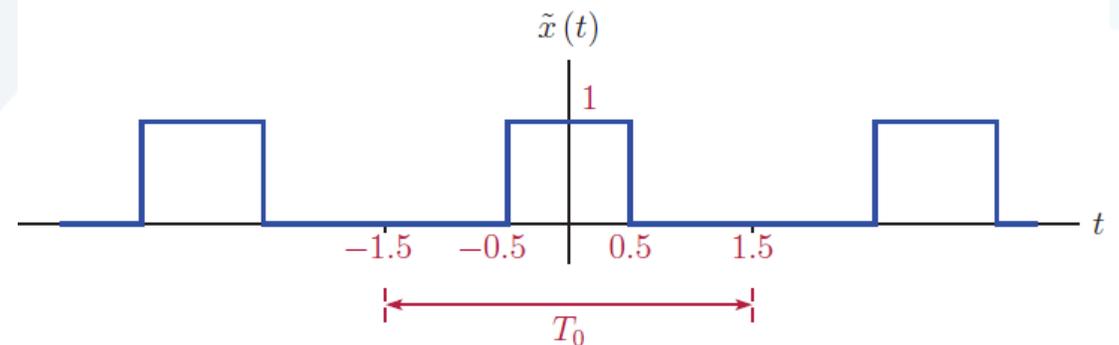
$$a_k = \frac{2}{3} \int_0^1 (1) \cos\left(\frac{2\pi kt}{3}\right) dt = \frac{\sin(2\pi k/3)}{\pi k}, \text{ for } k = 1, 2, \dots, \infty$$

$$b_k = \frac{2}{3} \int_0^1 (1) \sin\left(\frac{2\pi kt}{3}\right) dt = \frac{1 - \cos(2\pi k/3)}{\pi k}, \text{ for } k = 1, 2, \dots, \infty$$

$$\tilde{x}(t) = \frac{1}{3} + \sum_{k=1}^{\infty} \frac{\sin(2\pi k/3)}{\pi k} \cos\left(\frac{2\pi kt}{3}\right) + \sum_{k=1}^{\infty} \frac{1 - \cos(2\pi k/3)}{\pi k} \sin\left(\frac{2\pi kt}{3}\right)$$

- Example 2:** Periodic pulse train

$$\tilde{x}(t) = \frac{1}{3} + \sum_{k=1}^{\infty} \frac{2\sin(\pi k/3)}{\pi k} \cos\left(\frac{2\pi kt}{3}\right)$$



Exponential Fourier series (EFS)

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

Single-tone signals:

$$\tilde{x}(t) = A\cos(\omega_0 t + \theta) = \frac{A}{2} e^{j(\omega_0 t + \theta)} + \frac{A}{2} e^{-j(\omega_0 t + \theta)} = \frac{A}{2} e^{j\theta} e^{j\omega_0 t} + \frac{A}{2} e^{-j\theta} e^{-j\omega_0 t}$$

$$c_1 = \frac{A}{2} e^{j\theta}, \quad c_{-1} = \frac{A}{2} e^{-j\theta}, \quad \text{and} \quad c_k = 0 \text{ for all other } k$$

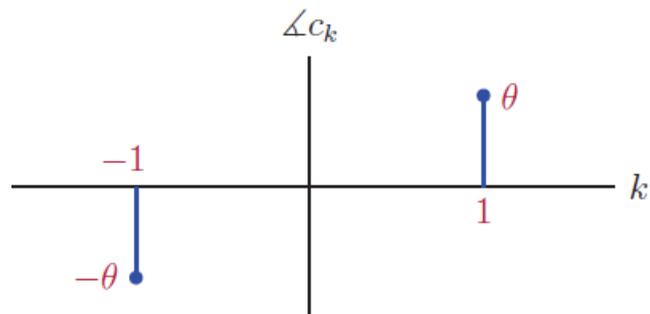
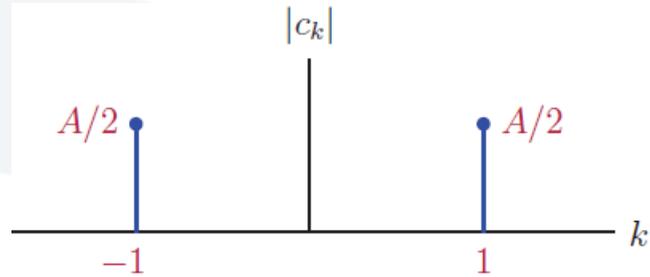
$$\tilde{x}(t) = A\sin(\omega_0 t + \theta) = \frac{A}{2} e^{j(\theta - \pi/2)} e^{j\omega_0 t} + \frac{A}{2} e^{-j(\theta - \pi/2)} e^{-j\omega_0 t}$$

$$c_1 = \frac{A}{2} e^{j(\theta - \pi/2)}, \quad c_{-1} = \frac{A}{2} e^{-j(\theta - \pi/2)}, \quad \text{and} \quad c_k = 0 \text{ for all other } k$$

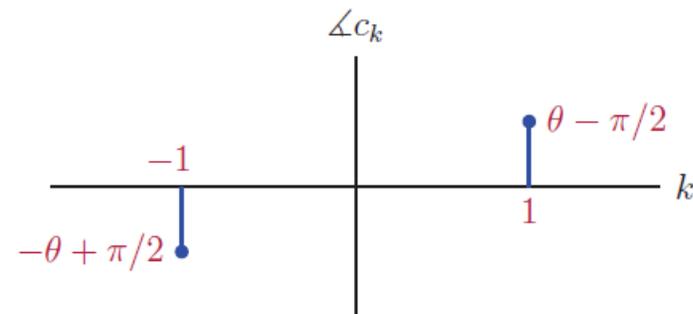
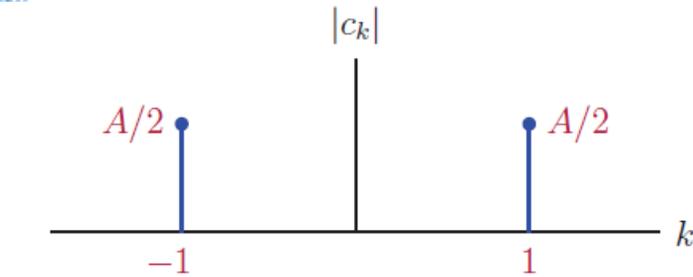
- The EFS representations of the two signals are shown graphically, in the form of a **line spectrum**.

The general case:

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} = a_0 + \sum_{k=1}^{\infty} a_k \cos(k\omega_0 t) + \sum_{k=1}^{\infty} b_k \sin(k\omega_0 t)$$



$$\tilde{x}(t) = A \cos(\omega_0 t + \theta)$$



$$\tilde{x}(t) = A \sin(\omega_0 t + \theta)$$

$$c_0 = a_0$$

$$c_k + c_{-k} = a_k \quad \text{and} \quad j(c_k - c_{-k}) = b_k, \quad \text{for } k = 1, \dots, \infty$$

$$c_k = \frac{1}{2}(a_k - jb_k) \quad \text{and} \quad c_{-k} = \frac{1}{2}(a_k + jb_k), \quad \text{for } k = 1, \dots, \infty$$

- To compute the EFS coefficients of a signal without first having to obtain the TFS coefficients:

$$\int_{t_0}^{t_0+T_0} e^{jm\omega_0 t} e^{-jk\omega_0 t} dt = \begin{cases} T_0, & m = k \\ 0, & m \neq k \end{cases}$$

The **exponential** basis functions form an **orthogonal** set

Exponential Fourier series (EFS):

1. Synthesis equation: $\tilde{x}(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$

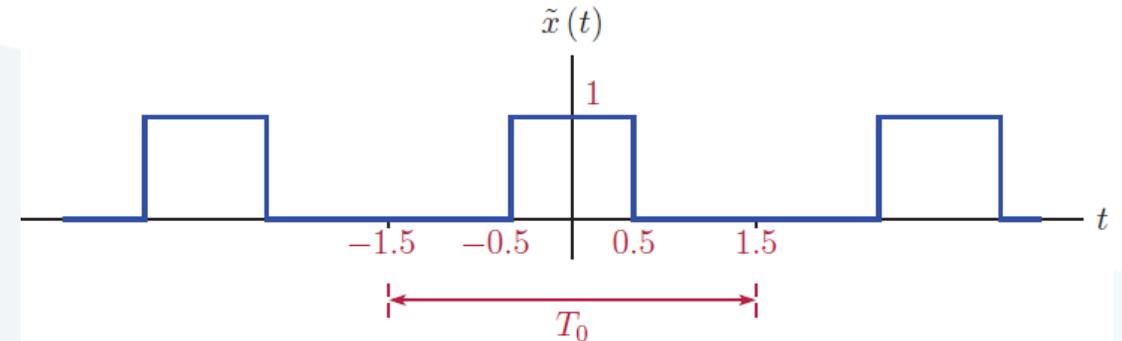
2. Analysis equation: $c_k = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} \tilde{x}(t) e^{-jk\omega_0 t} dt$

- In general, the coefficients of the EFS representation of a periodic signal $\tilde{x}(t)$ are **complex valued**. They can be graphed in the form of a **line spectrum** if they are expressed in polar form with magnitude and phase: $c_k = |c_k| e^{j\theta_k}$.

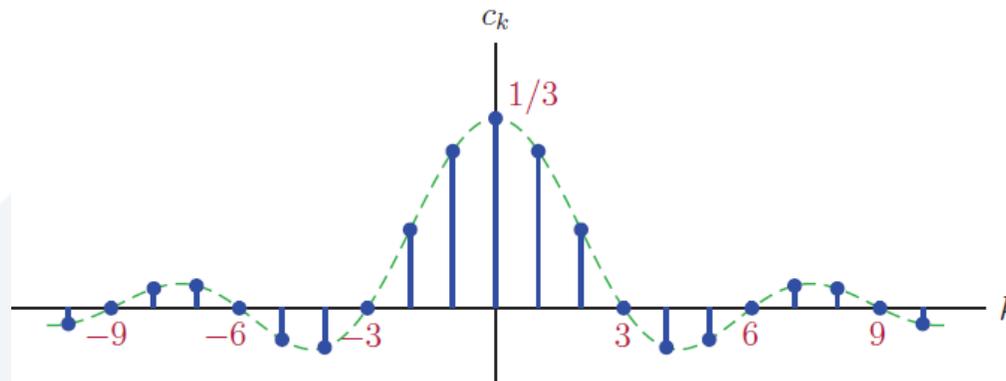
- **Example 3:** Exponential Fourier series for periodic pulse train

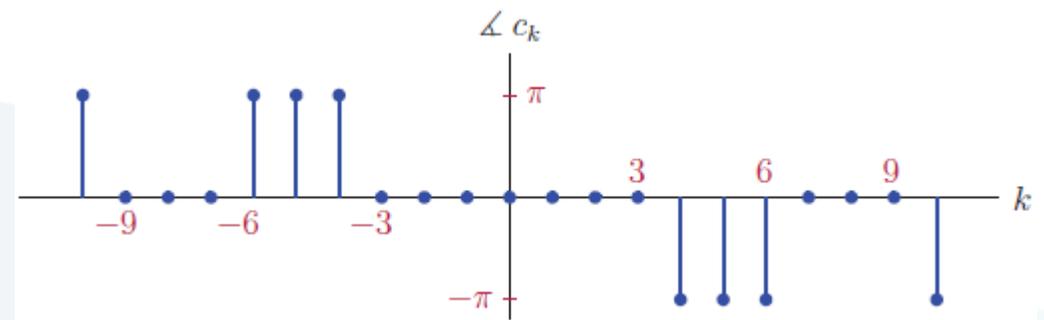
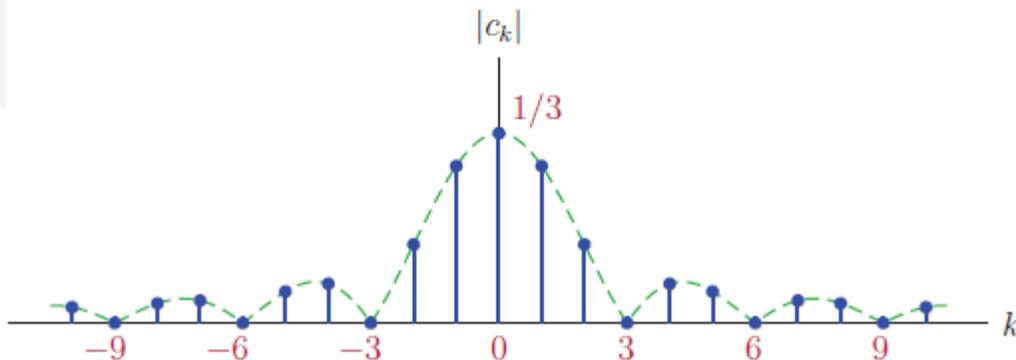
$$c_k = \frac{1}{3} \int_{-0.5}^{0.5} (1) e^{-j2\pi kt/3} dt = \frac{\sin(\pi k/3)}{\pi k}$$

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} \frac{\sin(\pi k/3)}{\pi k} e^{j2\pi kt/3}$$



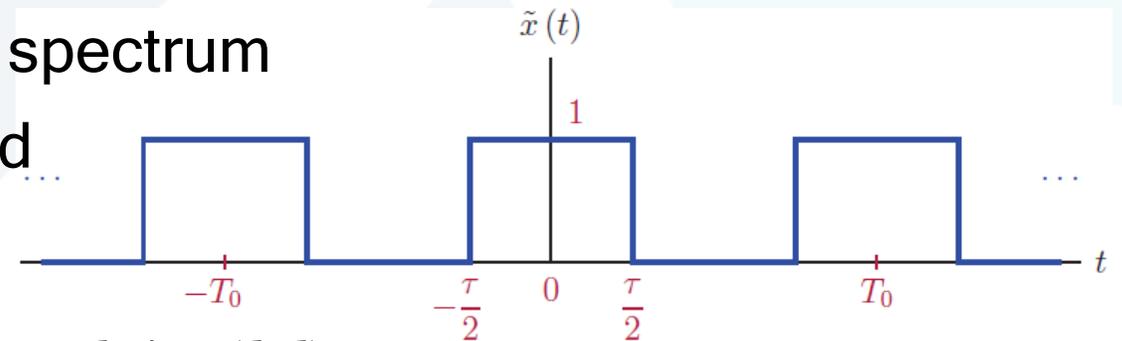
- A line graph of the set of coefficients c_k is useful for illustrating the make-up of the signal $\tilde{x}(t)$ in terms of its harmonics.





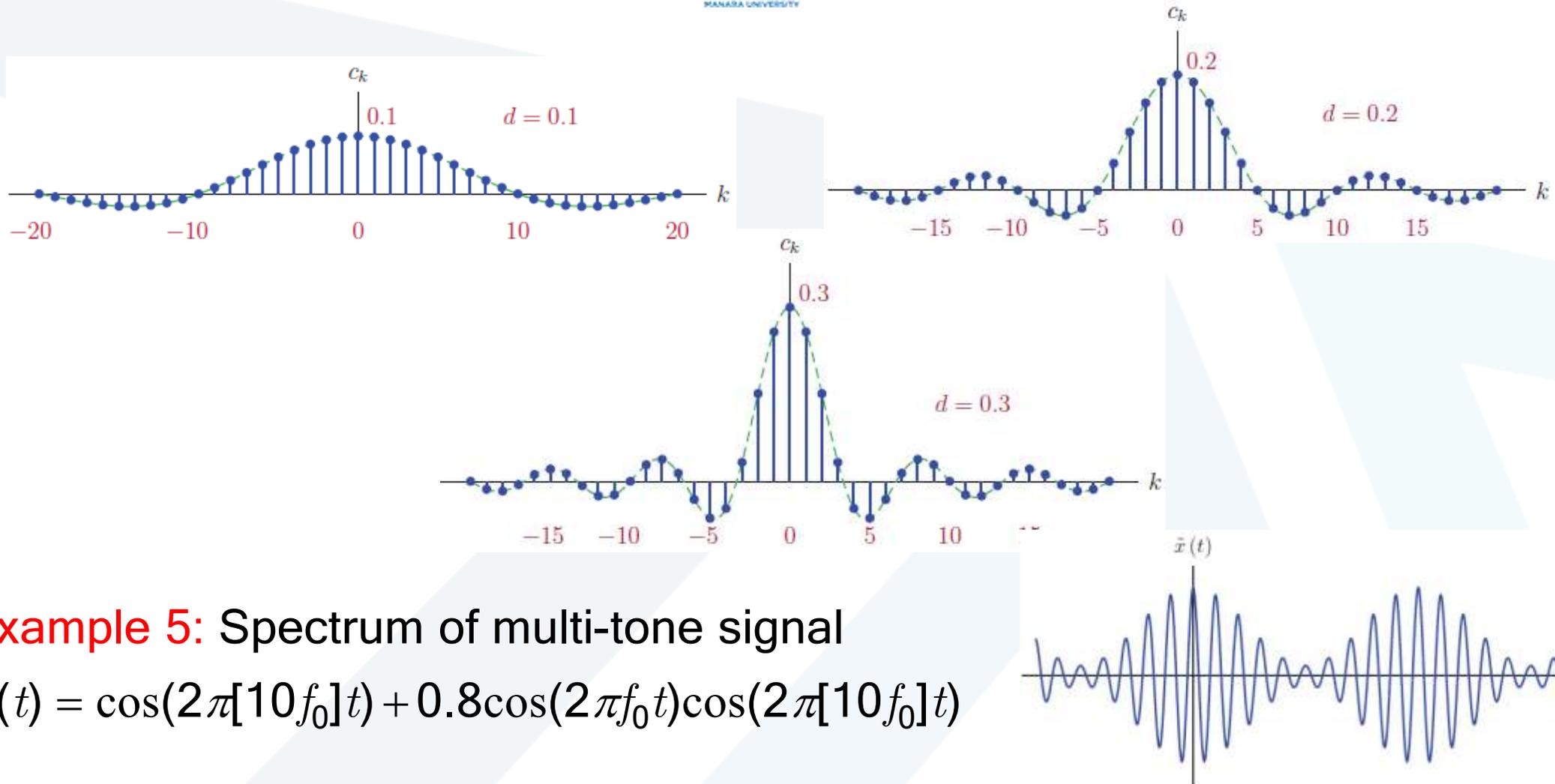
- **Example 4:** Effects of duty cycle on the spectrum

The **duty cycle** of a pulse train is defined as the ratio, $d = \tau/T_0$



$$c_k = \frac{1}{T_0} \int_{-\tau/2}^{\tau/2} (1) e^{-j2\pi kt/T_0} dt = \frac{\sin(\pi kd)}{\pi k} = d \operatorname{sinc}(kd)$$

- **Note:** Values of coefficients c_k depend only on the **duty cycle** and not on the period T_0 .



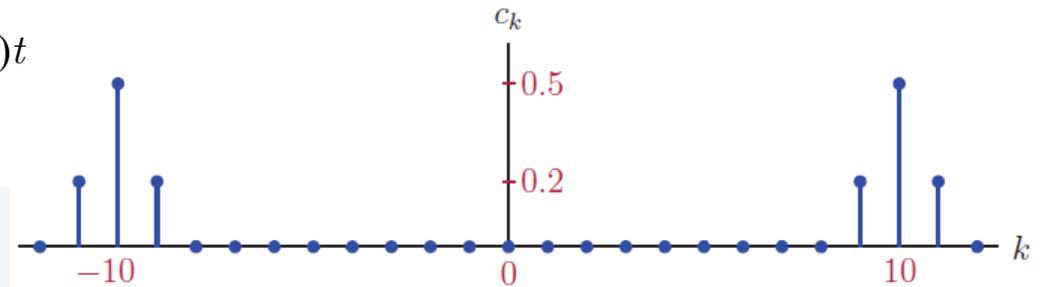
- **Example 5:** Spectrum of multi-tone signal

$$\tilde{x}(t) = \cos(2\pi[10f_0]t) + 0.8\cos(2\pi f_0 t)\cos(2\pi[10f_0]t)$$

$$\tilde{x}(t) = \frac{1}{2} e^{j2\pi(10f_0)t} + \frac{1}{2} e^{-j2\pi(10f_0)t} + \frac{1}{5} e^{j2\pi(11f_0)t} + \frac{1}{5} e^{-j2\pi(11f_0)t} + \frac{1}{5} e^{j2\pi(9f_0)t} + \frac{1}{5} e^{-j2\pi(9f_0)t}$$

The EFS coefficients for $\tilde{x}(t)$ are:

$$c_9 = c_{-9} = \frac{1}{5}, \quad c_{10} = c_{-10} = \frac{1}{2}, \quad c_{11} = c_{-11} = \frac{1}{5} \quad \text{all other coefficients} = 0$$



Existence of Fourier series

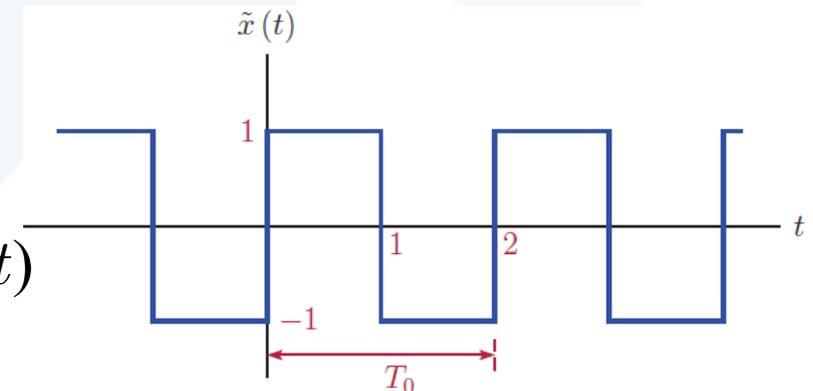
The **Dirichlet conditions** (for convergence of the FS) for the periodic function \tilde{x} :

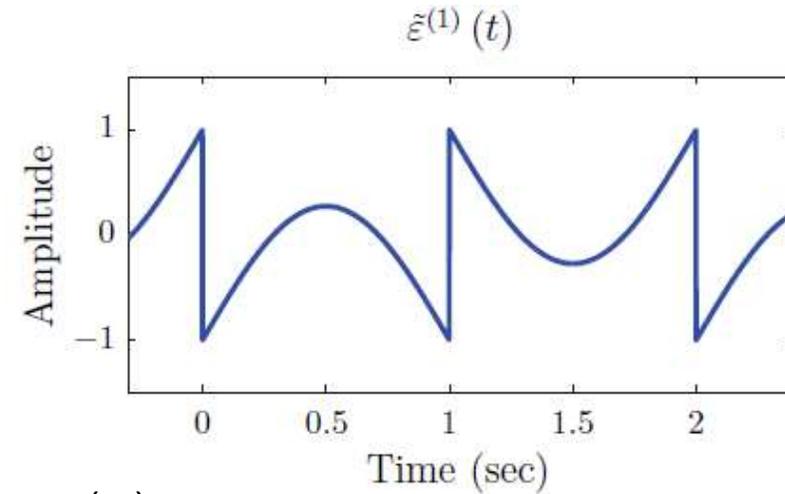
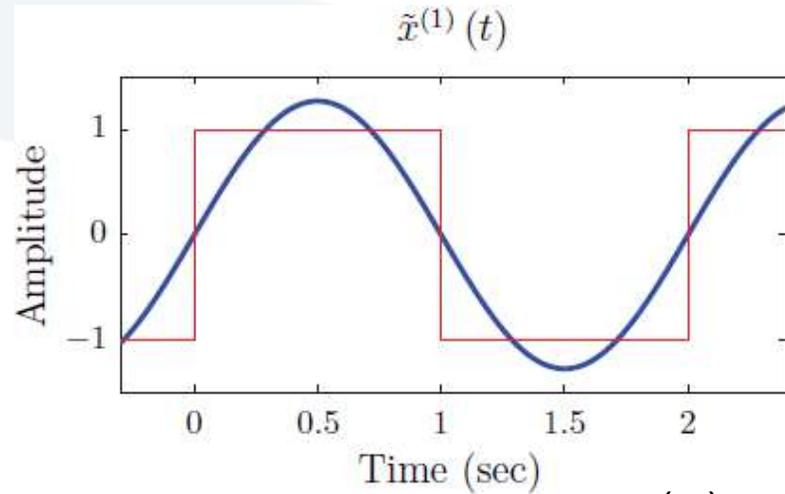
1. Over a single period, \tilde{x} is **absolutely integrable**
2. Over a single period, \tilde{x} has a finite number of maxima and minima (i.e., \tilde{x} is of **bounded variation**); and
3. Over any finite interval, \tilde{x} has a **finite number of discontinuities**, each of which is **finite**.

- If a periodic function \tilde{x} satisfies the Dirichlet conditions, then:
 1. The Fourier series converges everywhere to \tilde{x} , except at the points of discontinuity of \tilde{x} ; and
 2. At each point of discontinuity of $\tilde{x}(t)$, the Fourier series converges to $\frac{1}{2}[\tilde{x}(t^+) + \tilde{x}(t^-)]$, where $\tilde{x}(t^+)$ and $\tilde{x}(t^-)$ denote the values of the function \tilde{x} on the **left-** and **right-**hand sides of the discontinuity, respectively.
 3. If $\tilde{x}(t)$ is **continuous** everywhere, then the series converges **absolutely** and **uniformly**.

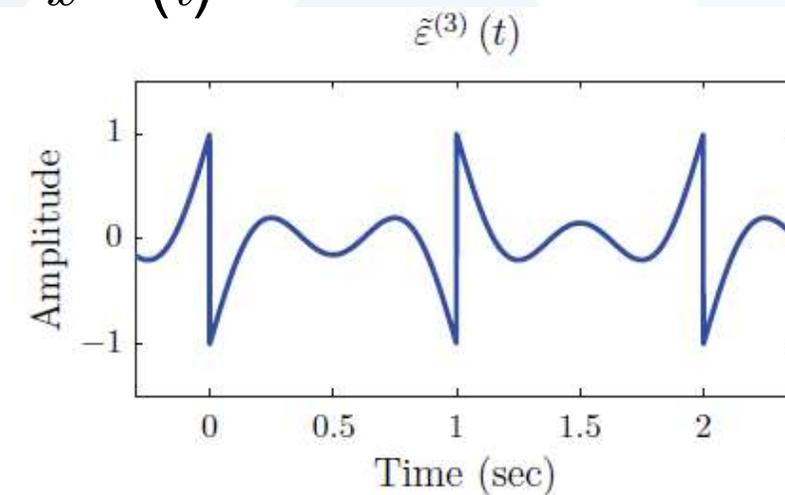
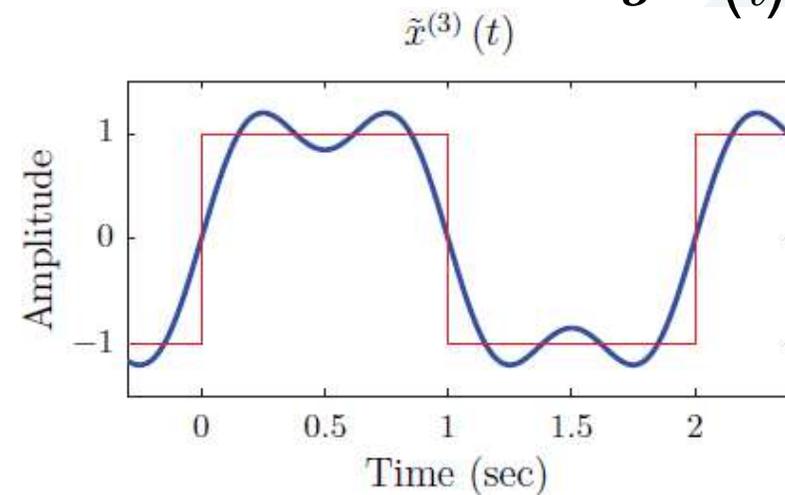
Gibbs phenomenon

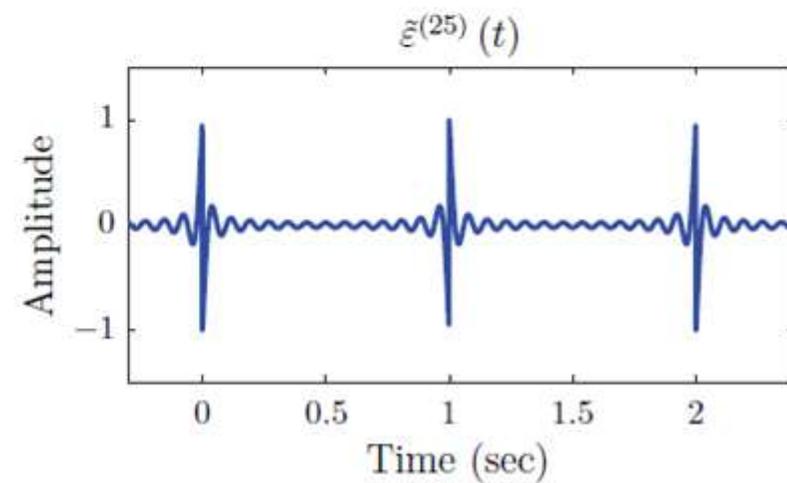
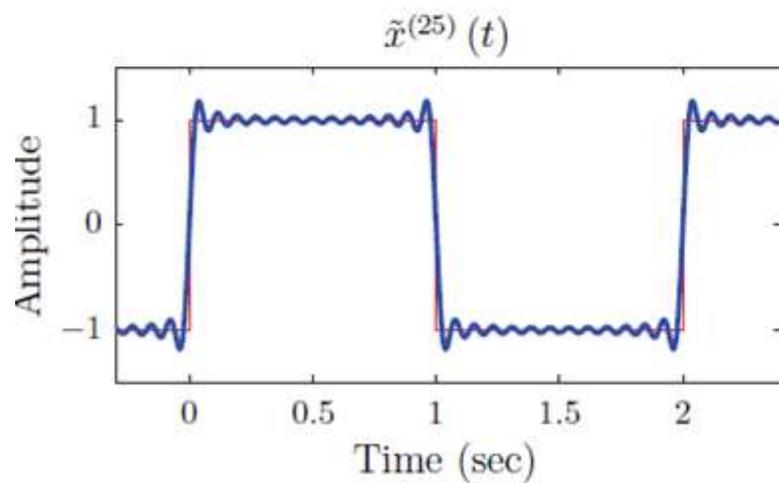
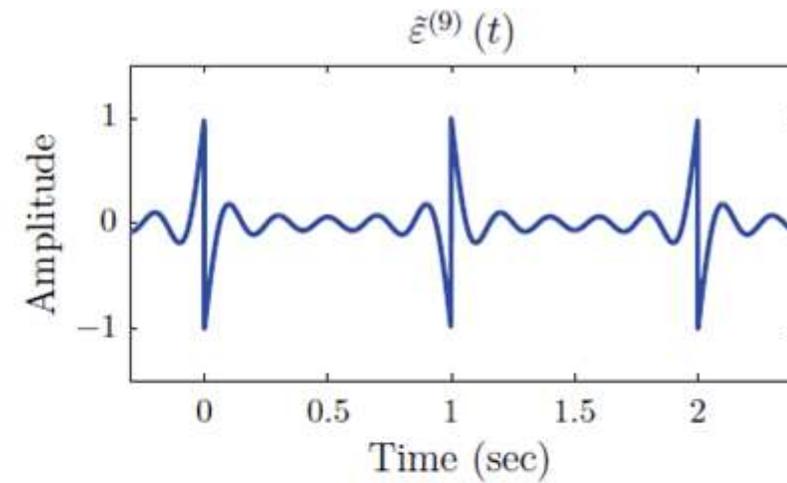
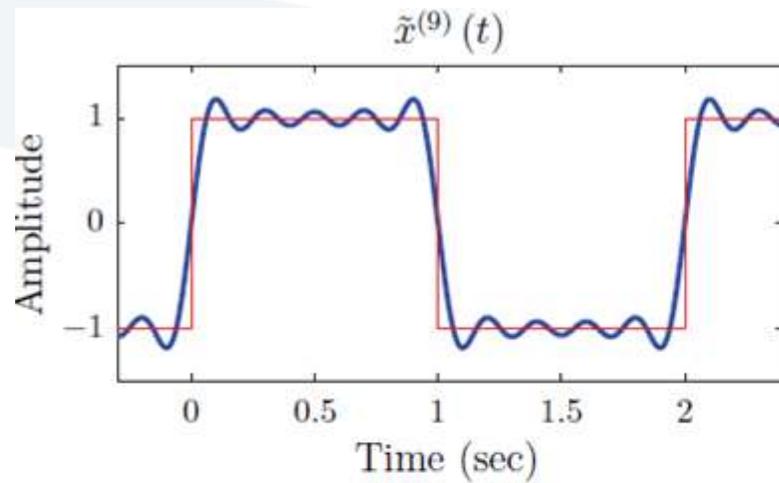
$$a_k = 0 \text{ and } b_k = \begin{cases} \frac{4}{\pi k}, & k \text{ odd} \\ 0, & k \text{ even} \end{cases} \quad \tilde{x}^{(m)}(t) = \sum_{\substack{k=1 \\ k \text{ odd}}}^m \frac{4}{\pi k} \sin(\pi kt)$$





$$\tilde{\varepsilon}^{(m)}(t) = \tilde{x}(t) - \tilde{x}^{(m)}(t)$$





- This behavior is known as **Gibbs phenomenon**.

- One way to explain the reason for the Gibbs phenomenon would be to link it to the **inability** of sinusoidal **basis functions** that are **continuous** at every point to approximate a **discontinuity** in the signal.

Properties of Fourier series

Linearity $\alpha_1 \tilde{x}(t) + \alpha_2 \tilde{y}(t) = \sum_{k=-\infty}^{\infty} [\alpha_1 c_k + \alpha_2 d_k] e^{jk\omega_0 t}$

Symmetry of Fourier series

$$\tilde{x}(t): \text{real}, \text{Im}\{\tilde{x}(t)\} = 0 \Rightarrow c_{-k} = c_k^*, \quad \tilde{x}(t): \text{imag}, \text{Re}\{\tilde{x}(t)\} = 0 \Rightarrow c_{-k} = -c_k^*$$

Fourier series for even and odd signals

- If the real-valued signal $\tilde{x}(t)$ is an even function of time, the resulting EFS spectrum c_k is real-valued for all k . $\tilde{x}(-t) = \tilde{x}(t)$, for all $t \Rightarrow \text{Im}\{c_k\} = 0$, for all k

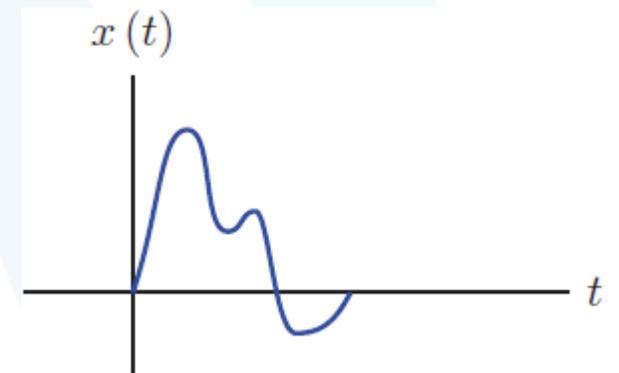
- If the real-valued signal $\tilde{x}(t)$ has odd-symmetry, the resulting EFS spectrum is purely imaginary. $\tilde{x}(-t) = -\tilde{x}(t)$, for all $t \Rightarrow \text{Re}\{c_k\} = 0$, for all k

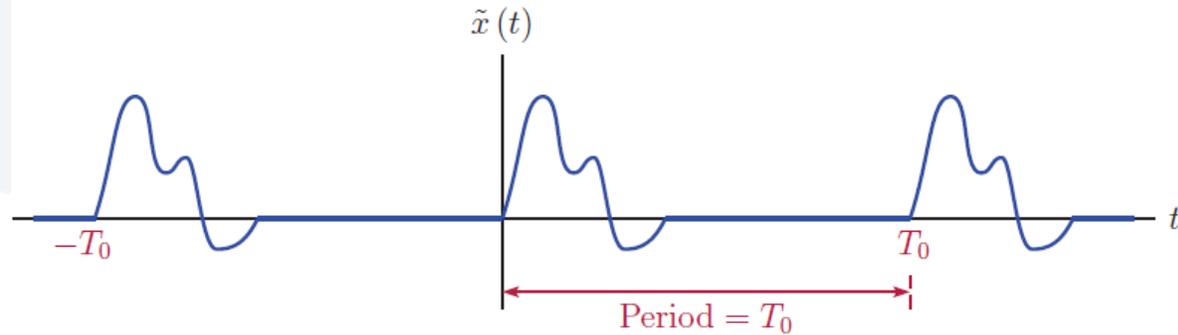
Time shifting $\tilde{x}(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \Rightarrow \tilde{x}(t - \tau) = \sum_{k=-\infty}^{\infty} [c_k e^{-jk\omega_0 \tau}] e^{jk\omega_0 t}$

3. Analysis of Non-Periodic Continuous-Time Signals

Fourier transform

- Consider the non-periodic signal $x(t)$
What frequencies are contained in this signal?
- Let us construct a periodic extension $\tilde{x}(t)$ of the signal $x(t)$ by repeating it at intervals of T_0 .





$$\tilde{x}(t) = \cdots + x(t + T_0) + x(t) + x(t - T_0) + x(t - 2T_0) + \cdots = \sum_{k=-\infty}^{\infty} x(t - kT_0)$$

- Since $\tilde{x}(t)$ is periodic, it can be analyzed in the frequency domain.

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}, \quad c_k = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} \tilde{x}(t) e^{-jk\omega_0 t} dt$$

- Realizing that $\tilde{x}(t) = x(t)$ within the span $-T_0/2 < t < T_0/2$

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} \left(\frac{1}{T_0} \int_{-T_0/2}^{T_0/2} \tilde{x}(\tau) e^{-jk(2\pi/T_0)\tau} d\tau \right) e^{jk(2\pi/T_0)t} \quad \lim_{T_0 \rightarrow \infty} [\tilde{x}(t)] = x(t)$$

- As $T_0 \rightarrow \infty$ implies that $\Delta\omega = 2\pi/T_0 \rightarrow 0$ (we switch to the notation $\Delta\omega$ instead of ω_0 to emphasize the **infinitesimal** nature of the fundamental frequency).

$$c_k = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} \tilde{x}(t) e^{-jk\Delta\omega t} dt$$

where c_k is the contribution of the complex exponential at the frequency $\omega = k\Delta\omega$. Each individual coefficient c_k is very small in magnitude, and in the limit we have $c_k \rightarrow 0$ when $T_0 \rightarrow \infty$. In addition, successive harmonics $k\Delta\omega$ are very close to each other due to infinitesimally small $\Delta\omega$.

$$c_k T_0 = \int_{-T_0/2}^{T_0/2} \tilde{x}(t) e^{-jk\Delta\omega t} dt$$

$$X(\omega) = \lim_{T_0 \rightarrow \infty} [c_k T_0] = \lim_{T_0 \rightarrow \infty} \int_{-T_0/2}^{T_0/2} x(t) e^{-jk\Delta\omega t} dt = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$X(\omega)$ is the **Fourier transform** of the non-periodic signal $x(t)$.

Fourier transform for continuous-time signals:

1. Synthesis equation: (Inverse transform)

$$x(t) = \mathcal{F}^{-1}\{X(\omega)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

$$x(t) = \mathcal{F}^{-1}\{X(f)\} = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df$$

(using f)

2. Analysis equation: (Forward transform)

$$X(\omega) = \mathcal{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$X(f) = \mathcal{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt$$

(using f)

Existence of Fourier transform

- The **Dirichlet conditions** (for convergence of the FT) for the function x :
 1. The function x is **absolutely integrable** $\int_{-\infty}^{\infty} |x(t)| dt < \infty$
 2. On any finite interval x has a finite number of maxima and minima (i.e. x is of **bounded variation**); and

3. On any finite interval, x has a **finite number of discontinuities**, and each discontinuity is itself **finite**.

- **Note:** All **energy signals** have Fourier transforms.

Fourier transforms of some signals

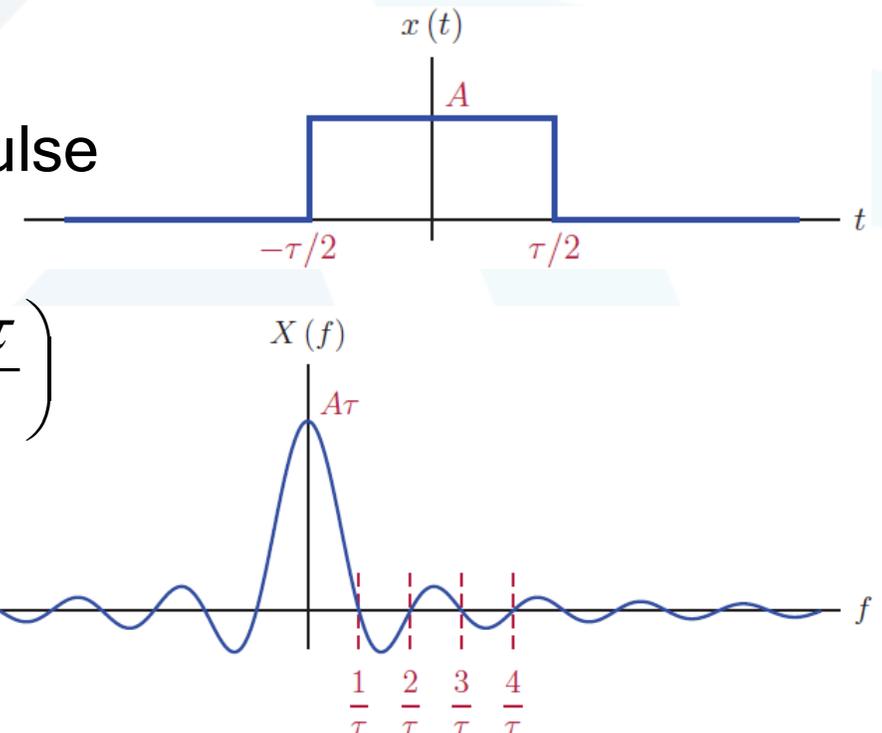
- **Example 6:** Fourier transform of a rectangular pulse

$$x(t) = A\Pi(t/\tau)$$

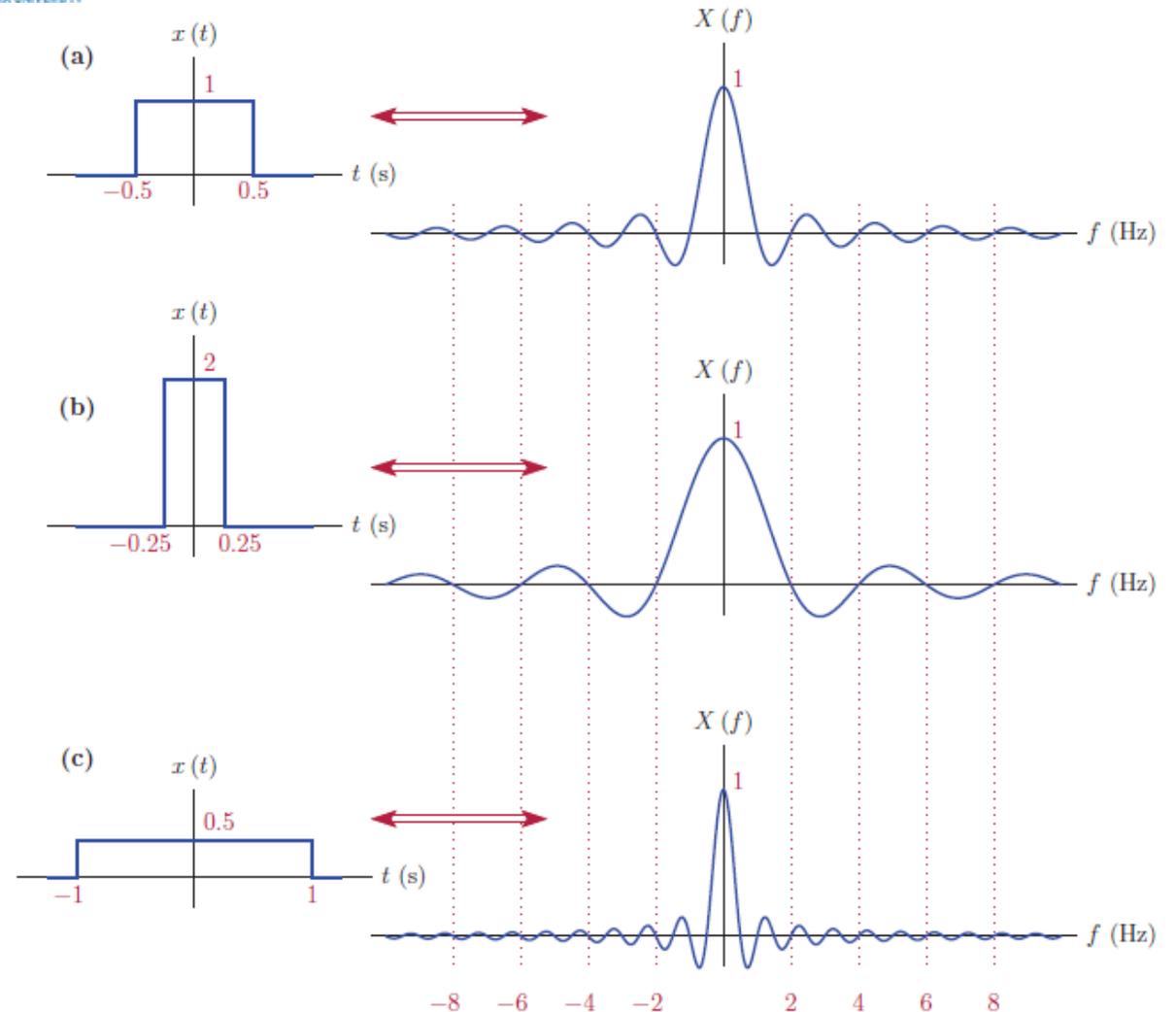
$$X(\omega) = \int_{-\tau/2}^{\tau/2} (A) e^{-j\omega t} dt = A \frac{e^{-j\omega t}}{-j\omega} \Big|_{-\tau/2}^{\tau/2} = \frac{2A}{\omega} \sin\left(\frac{\omega\tau}{2}\right)$$

$$X(\omega) = A\tau \frac{\sin(\omega\tau/2)}{(\omega\tau/2)} = A\tau \operatorname{sinc}\left(\frac{\omega\tau}{2\pi}\right)$$

$$X(f) = A\tau \operatorname{sinc}(f\tau)$$



- Effects of changing the pulse width on the frequency spectrum:



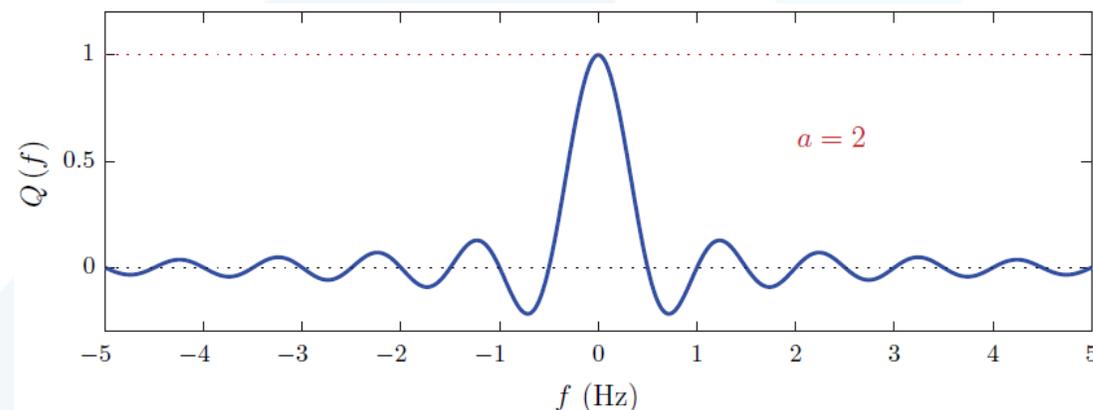
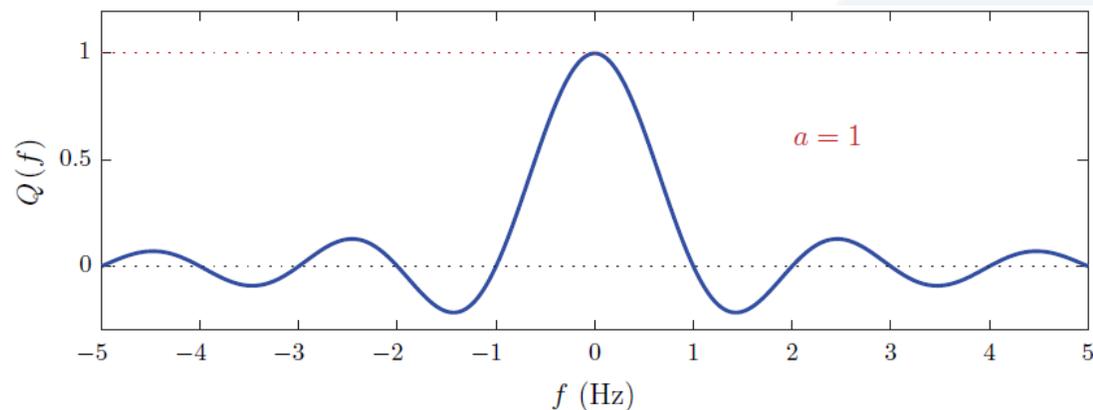
Example 7: Transform of the unit-impulse function

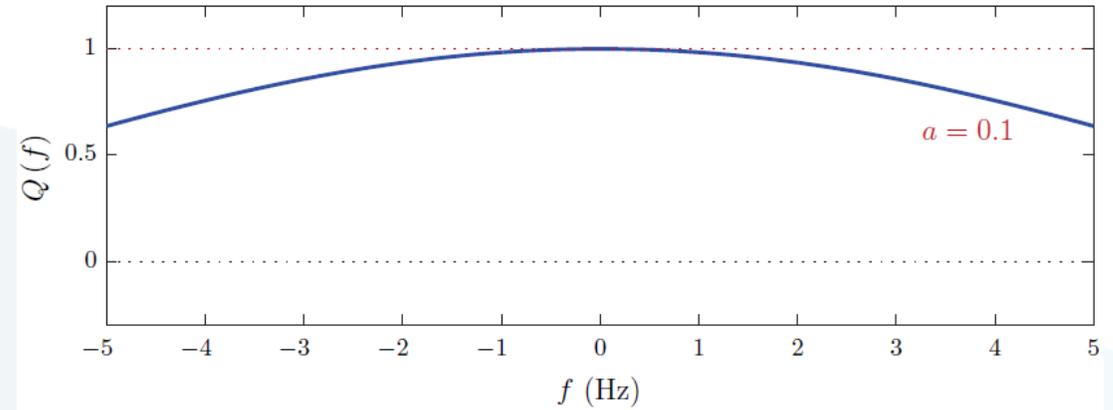
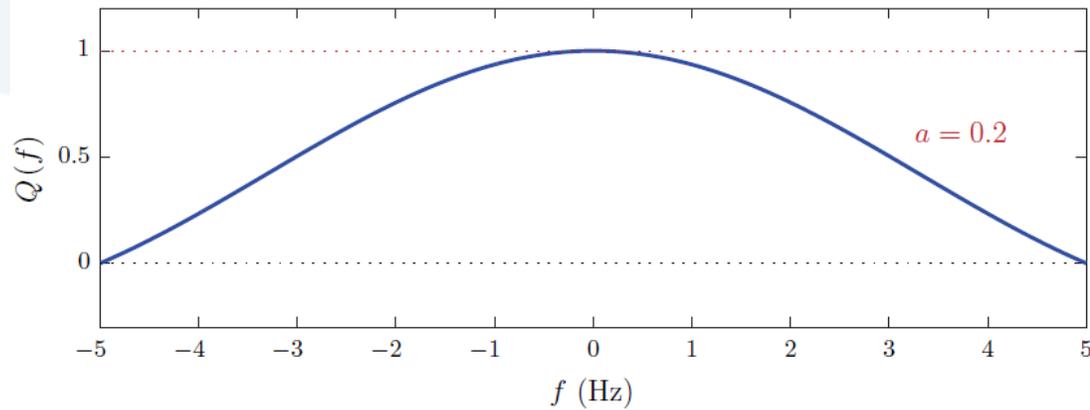
$$\mathcal{F}\{\delta(t)\} = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} \delta(t) dt = 1$$

$$q(t) = \frac{1}{a} \Pi\left(\frac{t}{a}\right) \Rightarrow \delta(t) = \lim_{a \rightarrow 0} q(t)$$

$$Q(f) = \mathcal{F}\{q(t)\} = \text{sinc}(fa)$$

$$\mathcal{F}\{\delta(t)\} = \lim_{a \rightarrow 0} \{Q(f)\} = \lim_{a \rightarrow 0} \{\text{sinc}(fa)\} = 1$$



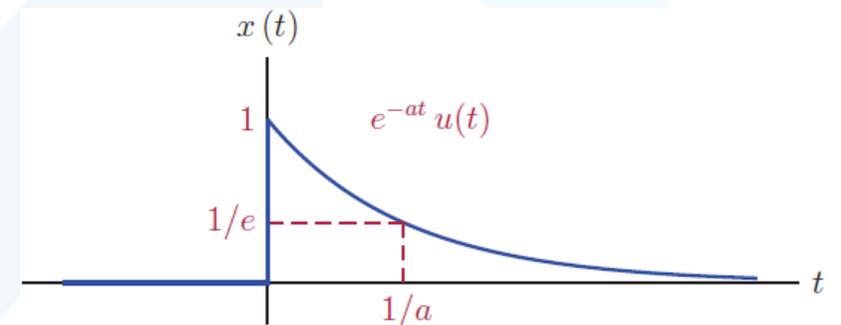


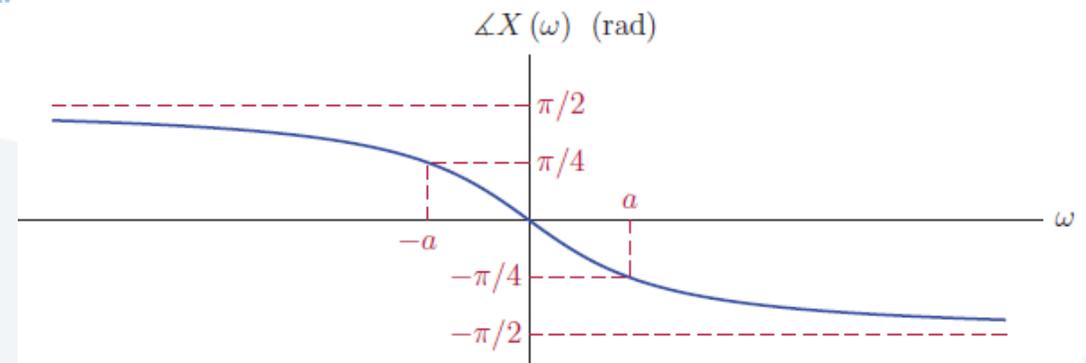
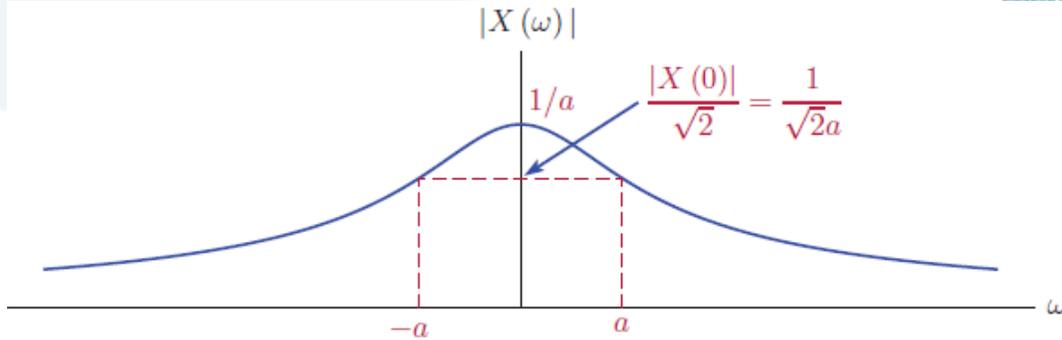
Example 8: Fourier transform of a right-sided exponential signal

$$x(t) = e^{-at}u(t), \quad a > 0$$

$$X(\omega) = \int_{-\infty}^{\infty} e^{-at} u(t) e^{-j\omega t} dt = \int_0^{\infty} e^{-at} e^{-j\omega t} dt = \frac{1}{a + j\omega}$$

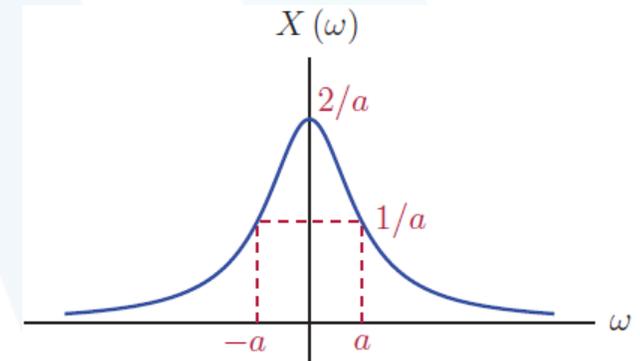
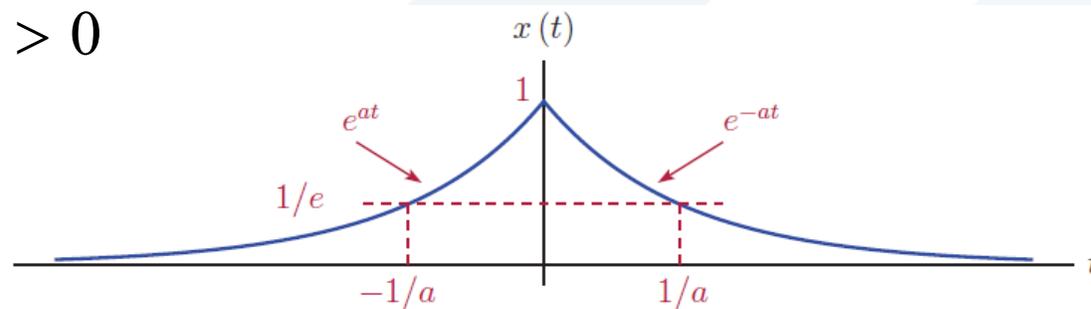
$$|X(\omega)| = \left| \frac{1}{a + j\omega} \right| = \frac{1}{\sqrt{a^2 + \omega^2}}, \quad \theta(\omega) = -\tan^{-1}(\omega/a)$$





- **Example 9:** Fourier transform of a two-sided exponential signal

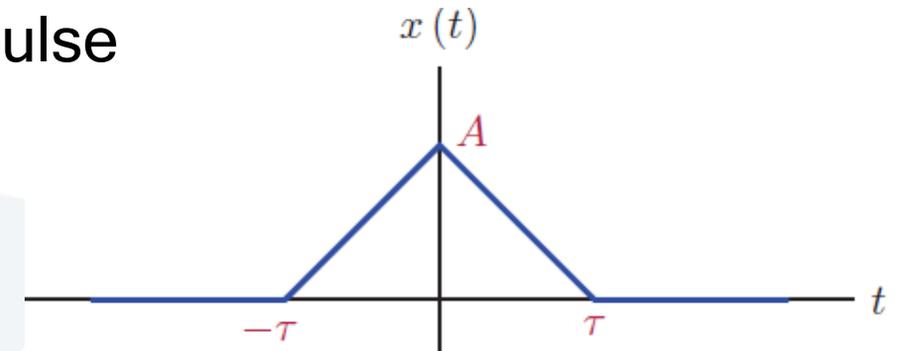
$$x(t) = e^{-a|t|}, \quad a > 0$$



$$X(\omega) = \int_{-\infty}^{\infty} e^{-a|t|} e^{-j\omega t} dt = \int_{-\infty}^0 e^{at} e^{-j\omega t} dt + \int_0^{\infty} e^{-at} e^{-j\omega t} dt = \frac{2a}{a^2 + \omega^2}$$

■ **Example 10:** Fourier transform of a triangular pulse

$$x(t) = A\Lambda\left(\frac{t}{\tau}\right) = \begin{cases} A + At/\tau, & -\tau < t < 0 \\ A - At/\tau, & 0 < t < \tau \\ 0, & |t| \geq \tau \end{cases}$$

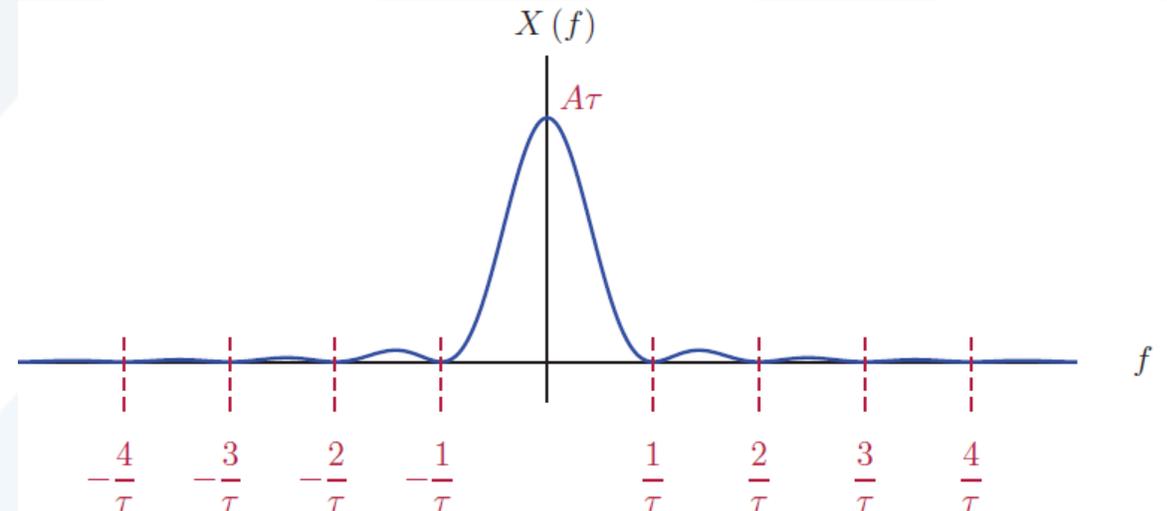


$$X(\omega) = \int_{-\tau}^0 (A + At/\tau) e^{-j\omega t} dt + \int_0^{\tau} (A - At/\tau) e^{-j\omega t} dt = \frac{2A}{\omega^2 \tau} [1 - \cos(\omega\tau)]$$

$$\text{sinc}\left(\frac{\omega\tau}{2\pi}\right) = \frac{\sin(\omega\tau/2)}{\omega\tau/2} = \frac{2}{\omega\tau} \sin\left(\frac{\omega\tau}{2}\right)$$

$$X(\omega) = A\tau \text{sinc}^2\left(\frac{\omega\tau}{2\pi}\right)$$

$$X(f) = A\tau \text{sinc}^2(f\tau)$$



The Generalized Fourier Transform

- There are some **important practical signals** that do not have Fourier transforms in the strict sense. Because of them, the Fourier transform has been “**generalized**” to include them.
- Example 11:** Fourier transform of constant-amplitude signal $x(t) = A$, all t

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} A e^{-j\omega t} dt = A \int_{-\infty}^{\infty} e^{-j\omega t} dt$$

The integral does not converge. Therefore, the Fourier transform does not exist. Let us define an intermediate signal $x_a(t) = A e^{-a|t|}$, $a > 0$.

$$X_a(\omega) = A \frac{2a}{a^2 + \omega^2} \quad \text{For } \omega \neq 0, \quad \lim_{a \rightarrow 0} \left[A \frac{2a}{a^2 + \omega^2} \right] = 0 \quad \int_{-\infty}^{\infty} A \frac{2a}{a^2 + \omega^2} d\omega = 2\pi A$$

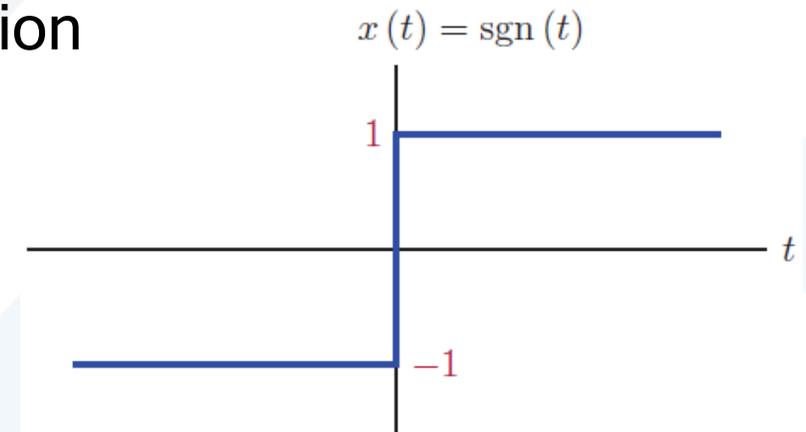
The area under the function is $2\pi A$ and is independent of the value of a .

Therefore the Fourier transform of the constant A is a function that is zero for $\omega \neq 0$ and has an area of $2\pi A$. Therefore $\mathcal{F}(A) = 2\pi A\delta(\omega)$

- **Example 12:** Fourier transform of the signum function

$$x(t) = \text{sgn}(t) = \begin{cases} -1, & t < 0 \\ 1, & t > 0 \end{cases}$$

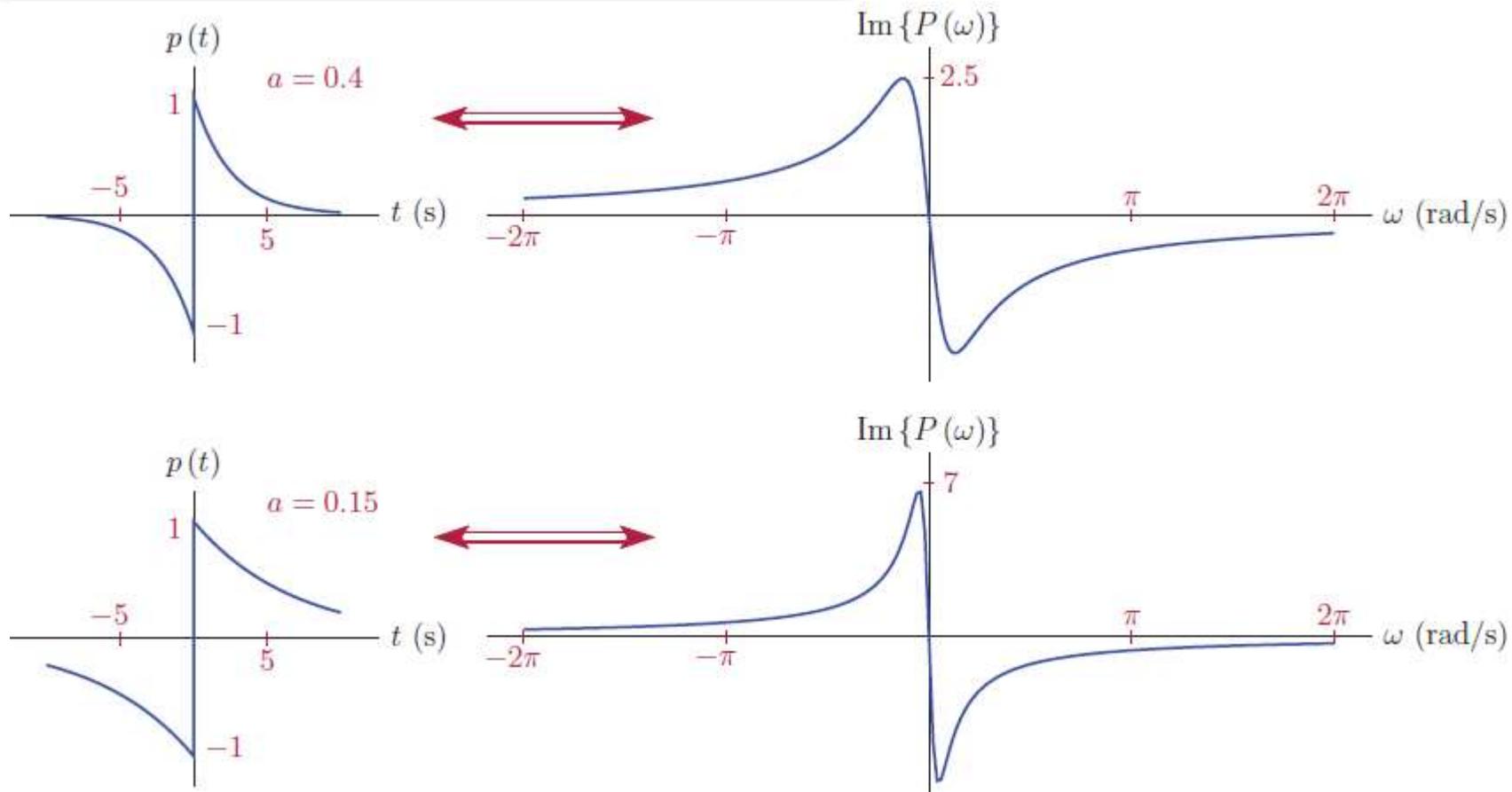
$$X(\omega) = \int_{-\infty}^0 (-1)e^{-j\omega t} dt + \int_0^{\infty} (1)e^{-j\omega t} dt$$

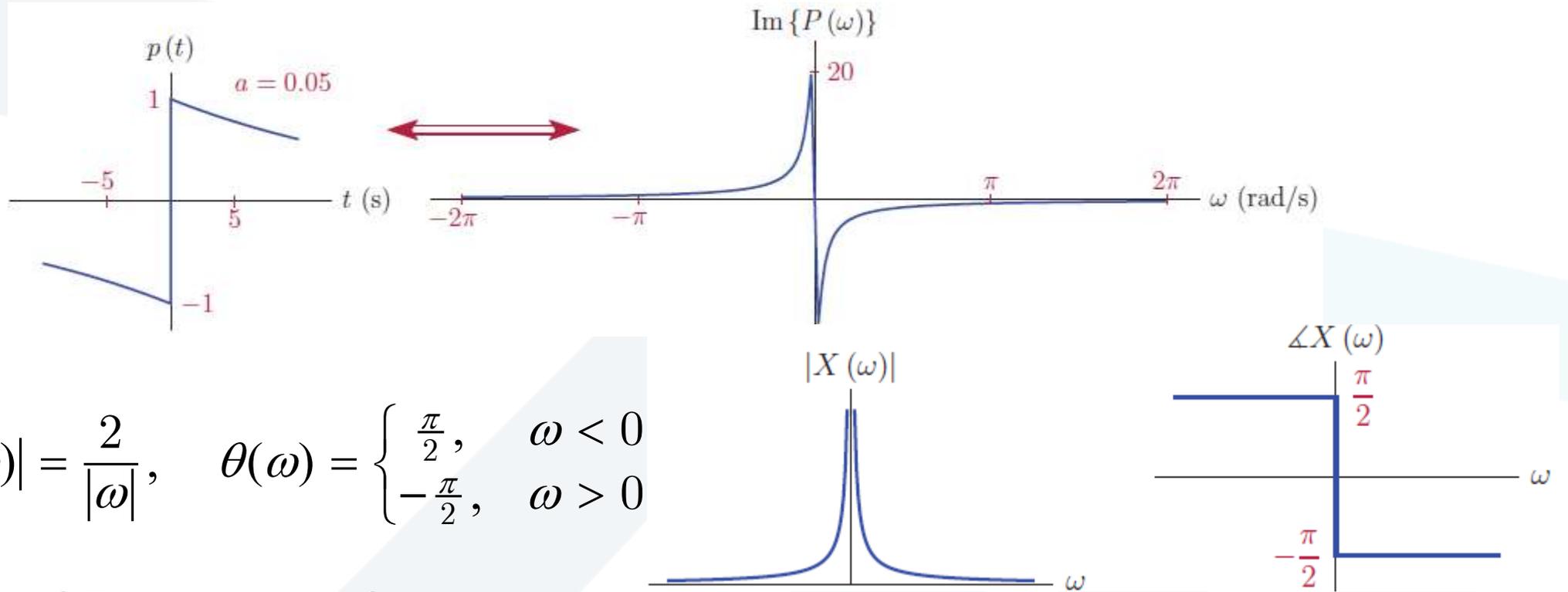


The two integrals cannot be evaluated. Instead, we will define an intermediate signal $p(t)$ as:

$$p(t) = \begin{cases} -e^{at}, & t < 0 \\ e^{-at}, & t > 0 \end{cases}, a \geq 0 \quad P(\omega) = \int_{-\infty}^0 (-e^{at})e^{-j\omega t} dt + \int_0^{\infty} (e^{-at})e^{-j\omega t} dt = \frac{-j2\omega}{a^2 + \omega^2}$$

$$X(\omega) = \mathcal{F}\{\text{sgn}(t)\} = \lim_{a \rightarrow 0} \frac{-j2\omega}{a^2 + \omega^2} = \frac{2}{j\omega}$$





$$|X(\omega)| = \frac{2}{|\omega|}, \quad \theta(\omega) = \begin{cases} \frac{\pi}{2}, & \omega < 0 \\ -\frac{\pi}{2}, & \omega > 0 \end{cases}$$

Properties of Fourier transform

Linearity of the Fourier transform: $\mathcal{F}\{\alpha_1 x(t) + \alpha_2 y(t)\} = \alpha_1 \mathcal{F}\{x(t)\} + \alpha_2 \mathcal{F}\{y(t)\}$

Duality property: $x(t) \xleftrightarrow{\mathcal{F}} X(\omega) \Rightarrow X(t) \xleftrightarrow{\mathcal{F}} 2\pi x(-\omega)$

Duality property (using f): $x(t) \xleftrightarrow{\mathcal{F}} X(f) \Rightarrow X(t) \xleftrightarrow{\mathcal{F}} x(-f)$

What is the relationship between $X(t)$ and $x(\omega)$?

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega, \quad X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

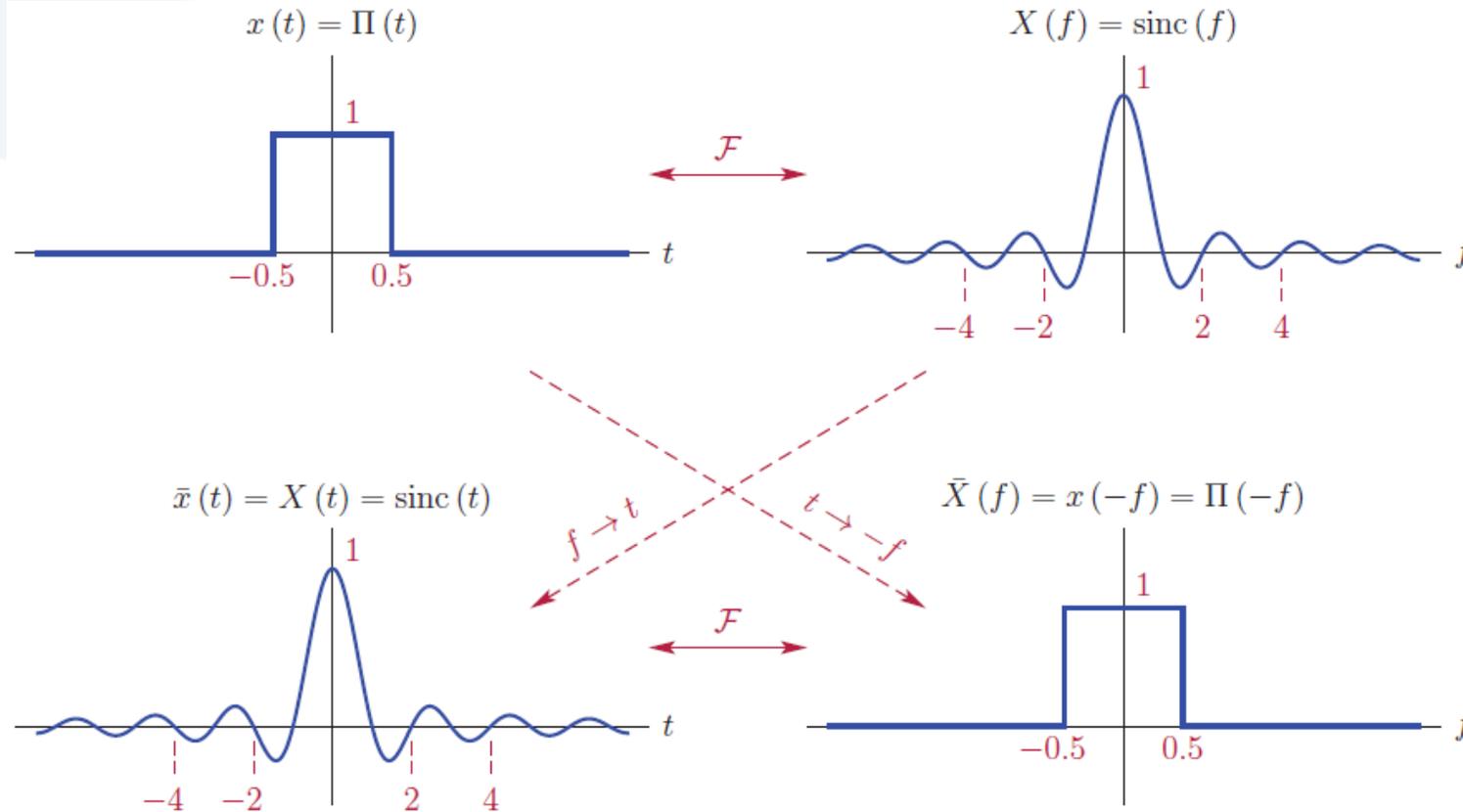
$$\omega \rightarrow \tau \Rightarrow x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\tau) e^{j\tau t} d\tau \quad t \rightarrow -\omega \Rightarrow x(-\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\tau) e^{-j\tau\omega} d\tau$$

$$\tau \rightarrow t \Rightarrow x(-\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(t) e^{-jt\omega} dt$$

$$2\pi x(-\omega) = \int_{-\infty}^{\infty} X(t) e^{-jt\omega} dt \Rightarrow X(t) \xleftrightarrow{\mathcal{F}} 2\pi x(-\omega)$$

- **Example 13:** Fourier transform of the sinc function

$$\mathcal{F} \left\{ \frac{1}{2\pi} \Pi \left(\frac{t}{2\pi} \right) \right\} = \text{sinc}(\omega) \Rightarrow \mathcal{F} \{ \text{sinc}(t) \} = \Pi \left(\frac{-\omega}{2\pi} \right) = \Pi \left(\frac{\omega}{2\pi} \right), \quad \mathcal{F} \{ \text{sinc}(t) \} = \Pi(f)$$



- Example 14:** Transform of a constant-amplitude signal $x(t) = 1$, all t
 $\mathcal{F}\{\delta(t)\} = 1$, all $\omega \Rightarrow \mathcal{F}\{1\} = 2\pi\delta(-\omega) = 2\pi\delta(\omega)$, $\mathcal{F}\{1\} = \delta(f)$ (duality)

- **Note 1:** The signal $x(t) = 1$ does not satisfy the existence conditions; it is neither absolute integrable nor square integrable. Its FT does not converge. We obtain a function $X(\omega)$ that has the characteristics of a Fourier transform, and that can be used in solving problems in the frequency domain.
- **Note 2:** the conversion from ω to f using: $X(\omega) = X(f)|_{f=\omega/2\pi}$, $X(f) = X(\omega)|_{\omega=2\pi f}$ is valid only when the transform does not contains a **singularity** function.
- **Example 15:** Fourier transform of the unit-step function

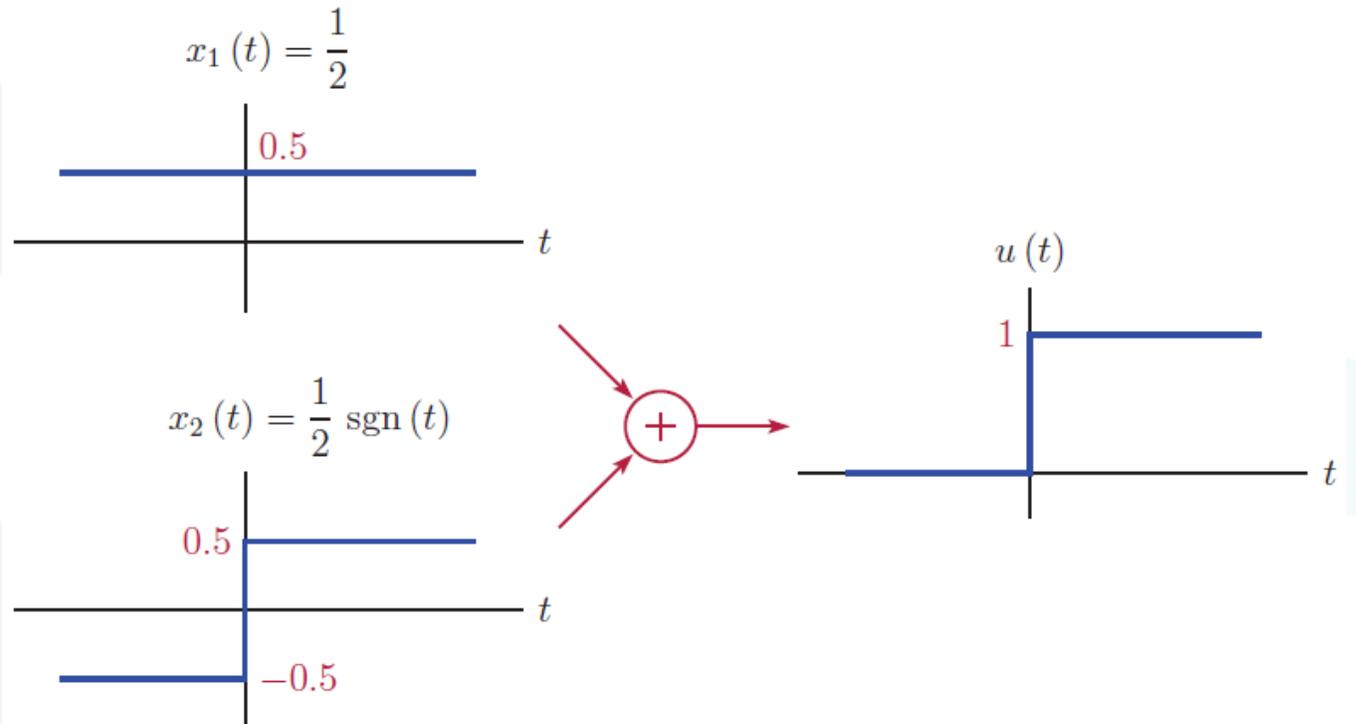
Determine the Fourier transform of the unit-step function $x(t) = u(t)$.

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt = \int_0^{\infty} e^{-j\omega t} dt \quad \text{could not be evaluated}$$

$$u(t) = \frac{1}{2} + \frac{1}{2} \text{sgn}(t) \Rightarrow \mathcal{F}\{u(t)\} = \mathcal{F}\{\frac{1}{2} + \frac{1}{2} \text{sgn}(t)\} = \frac{1}{2} \mathcal{F}\{1\} + \frac{1}{2} \mathcal{F}\{\text{sgn}(t)\}$$

$$\mathcal{F}\{u(t)\} = \pi \delta(\omega) + \frac{1}{j\omega}$$

$$\mathcal{F}\{u(t)\} = \frac{1}{2} \delta(f) + \frac{1}{j2\pi f}$$



Symmetry of the Fourier transform

$$x(t): \text{real}, \text{Im}\{x(t)\} = 0 \Rightarrow X^*(\omega) = X(-\omega)$$

$$x(t): \text{imag}, \text{Re}\{x(t)\} = 0 \Rightarrow X^*(\omega) = -X(-\omega)$$

Transforms of even and odd signals

- If the real-valued signal $x(t)$ is an even function of time, the resulting Fourier transform $X(\omega)$ is real-valued for all ω .

$$x(-t) = x(t), \text{ for all } t \Rightarrow \text{Im}\{X(\omega)\} = 0, \text{ for all } \omega$$

- If the real-valued signal $x(t)$ has odd-symmetry, the resulting Fourier transform $X(\omega)$ is purely imaginary.

$$x(-t) = -x(t), \text{ for all } t \Rightarrow \text{Re}\{X(\omega)\} = 0, \text{ for all } \omega$$

Modulation property

$$x(t) \xleftrightarrow{\mathcal{F}} X(\omega) \Rightarrow$$

$$x(t) \cos(\omega_0 t) \xleftrightarrow{\mathcal{F}} \frac{1}{2} [X(\omega - \omega_0) + X(\omega + \omega_0)]$$

$$x(t) \sin(\omega_0 t) \xleftrightarrow{\mathcal{F}} \frac{1}{2} [X(\omega - \omega_0) e^{-j\pi/2} + X(\omega + \omega_0) e^{j\pi/2}]$$

Time shifting

$$x(t) \xleftrightarrow{\mathcal{F}} X(\omega) \Rightarrow x(t - \tau) \xleftrightarrow{\mathcal{F}} X(\omega) e^{-j\omega\tau}$$

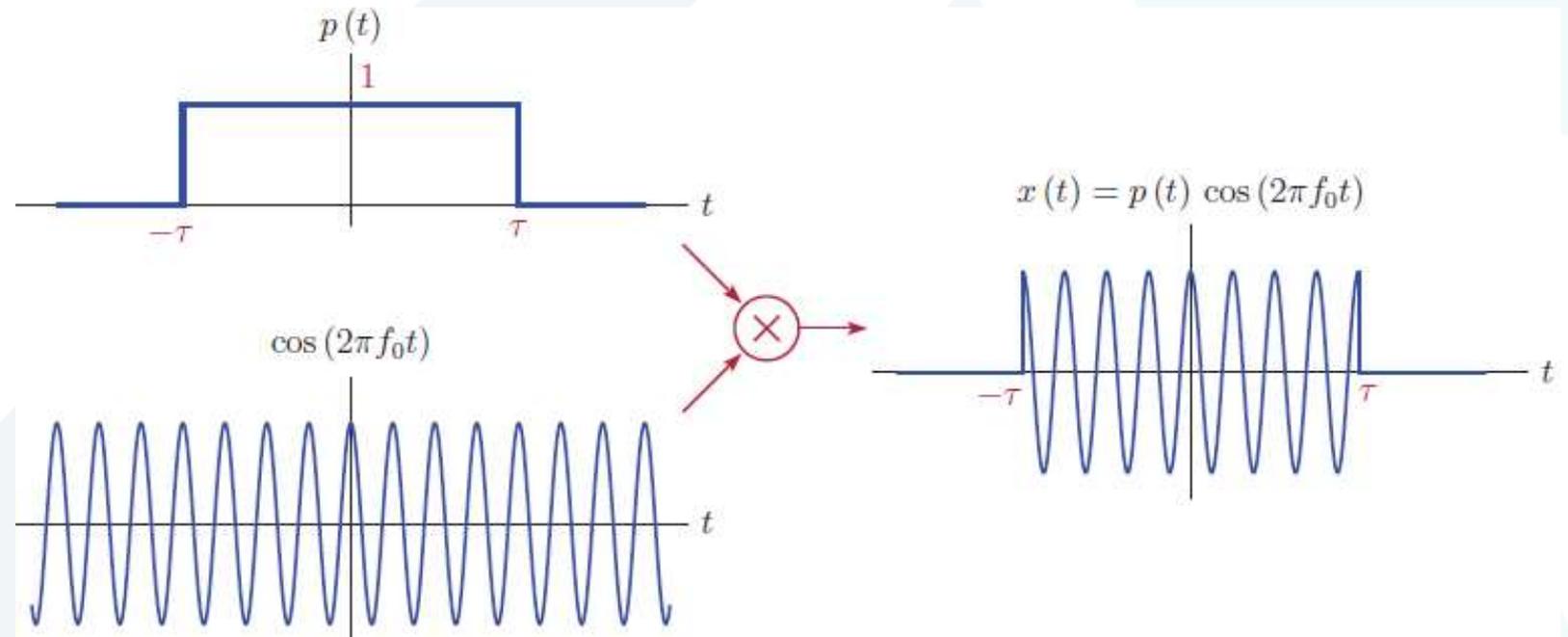


Frequency shifting $x(t) \xleftrightarrow{F} X(\omega) \Rightarrow x(t)e^{j\omega_0 t} \xleftrightarrow{F} X(\omega - \omega_0)$

- **Example 16:** Modulated pulse $x(t) = \begin{cases} \cos(2\pi f_0 t), & |t| < \tau \\ 0, & |t| > \tau \end{cases}$

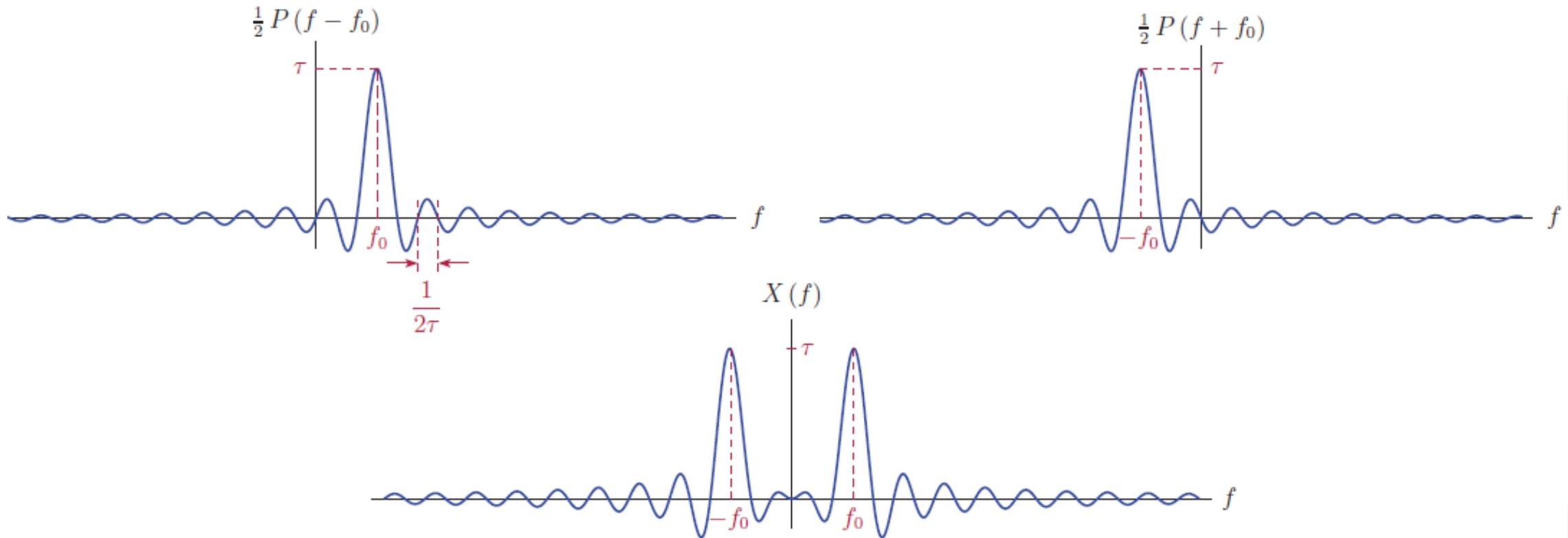
Using the rectangular pulse $p(t)$, $x(t)$ can be expressed as: $x(t) = p(t) \cos(2\pi f_0 t)$

where $p(t) = \Pi\left(\frac{t}{2\tau}\right)$



$$P(f) = 2\tau \operatorname{sinc}(2\tau f)$$

$$X(f) = \frac{1}{2} [P(f - f_0) + P(f + f_0)] = \tau \operatorname{sinc}(2\tau(f + f_0)) + \tau \operatorname{sinc}(2\tau(f - f_0))$$



Time and frequency scaling $x(t) \xleftrightarrow{\mathcal{F}} X(\omega) \Rightarrow x(at) \xleftrightarrow{\mathcal{F}} \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$

The parameter a is any non-zero real-valued constant.

Differentiation in the time domain

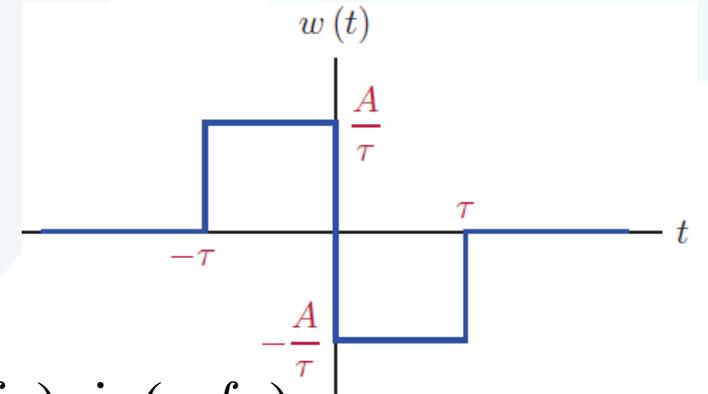
$$x(t) \xleftrightarrow{\mathcal{F}} X(\omega) \Rightarrow \frac{d^n}{dt^n} [x(t)] \xleftrightarrow{\mathcal{F}} (j\omega)^n X(\omega), \quad \frac{d^n}{dt^n} [x(t)] \xleftrightarrow{\mathcal{F}} (j2\pi f)^n X(f)$$

- Example 17:** Triangular pulse revisited $x(t) = A\Lambda(t/\tau)$

$$w(t) = \frac{dx(t)}{dt} = \frac{A}{\tau} \left[\Pi\left(\frac{t + \tau/2}{\tau}\right) - \Pi\left(\frac{t - \tau/2}{\tau}\right) \right]$$

$$W(f) = A \operatorname{sinc}(f\tau) e^{j2\pi f \frac{\tau}{2}} - A \operatorname{sinc}(f\tau) e^{-j2\pi f \frac{\tau}{2}} = 2jA \operatorname{sinc}(f\tau) \sin(\pi f\tau)$$

$$W(f) = (j2\pi f)X(f) \Rightarrow X(f) = \frac{W(f)}{j2\pi f} = \frac{2jA \operatorname{sinc}(f\tau) \sin(\pi f\tau)}{j2\pi f} = A\tau \operatorname{sinc}^2(f\tau)$$



Differentiation in the frequency domain

$$x(t) \xleftrightarrow{\mathcal{F}} X(\omega) \quad \Rightarrow \quad (-jt)^n x(t) \xleftrightarrow{\mathcal{F}} \frac{d^n}{d\omega^n} [X(\omega)]$$

Convolution property $x_1(t) \xleftrightarrow{\mathcal{F}} X_1(\omega)$ and $x_2(t) \xleftrightarrow{\mathcal{F}} X_2(\omega)$

$$\Rightarrow x_1(t) * x_2(t) \xleftrightarrow{\mathcal{F}} X_1(\omega) X_2(\omega)$$

Multiplication of two signals $x_1(t) \xleftrightarrow{\mathcal{F}} X_1(\omega)$ and $x_2(t) \xleftrightarrow{\mathcal{F}} X_2(\omega)$

$$\Rightarrow x_1(t)x_2(t) \xleftrightarrow{\mathcal{F}} \frac{1}{2\pi} X_1(\omega) * X_2(\omega), \quad x_1(t)x_2(t) \xleftrightarrow{\mathcal{F}} X_1(f) * X_2(f)$$

- **Example 18:** Transform of a truncated sinusoidal signal

$$x(t) = \begin{cases} \cos(2\pi f_0 t), & |t| < \tau \\ 0, & |t| > \tau \end{cases}$$

$$x(t) = x_1(t)x_2(t), \quad x_1(t) = \cos(2\pi f_0 t), \quad x_2(t) = \Pi\left(\frac{t}{2\tau}\right)$$

$$X_1(f) = \frac{1}{2} \delta(f + f_0) + \frac{1}{2} \delta(f - f_0), \quad X_2(f) = 2\tau \operatorname{sinc}(2\tau f)$$

$$X(f) = X_1(f) * X_2(f) = \tau \operatorname{sinc}(2\tau(f + f_0)) + \tau \operatorname{sinc}(2\tau(f - f_0))$$

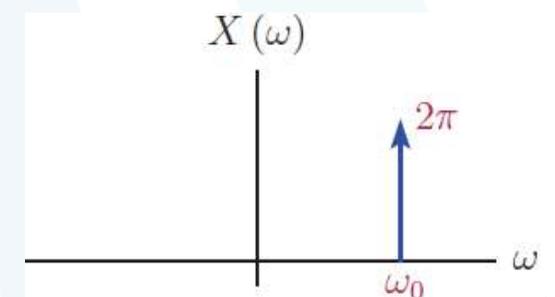
Integration $x(t) \xleftrightarrow{F} X(\omega) \Rightarrow \int_{-\infty}^t x(\tau) d\tau \xleftrightarrow{F} \frac{X(\omega)}{j\omega} + \pi X(0)\delta(\omega)$

Applying Fourier transform to periodic signals

- **Example 19:** Fourier transform of complex exponential signal

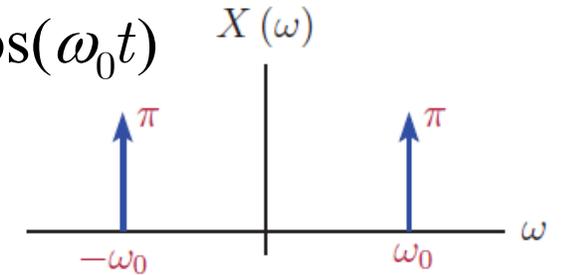
$$x(t) = e^{j\omega_0 t}$$

$$F\{1\} = 2\pi\delta(\omega) \Rightarrow F(e^{j\omega_0 t}) = 2\pi\delta(\omega - \omega_0)$$



- **Example 20:** Fourier transform of sinusoidal signal $x(t) = \cos(\omega_0 t)$

$$F\{1\} = 2\pi\delta(\omega) \Rightarrow F\{\cos(\omega_0 t)\} = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$

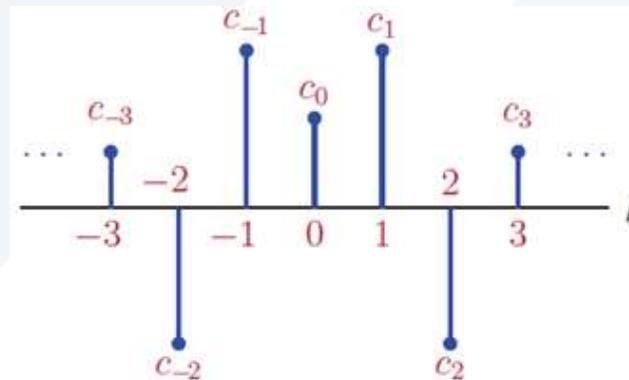


- In general for a periodic CT signal $\tilde{x}(t)$ that has an EFS representation:

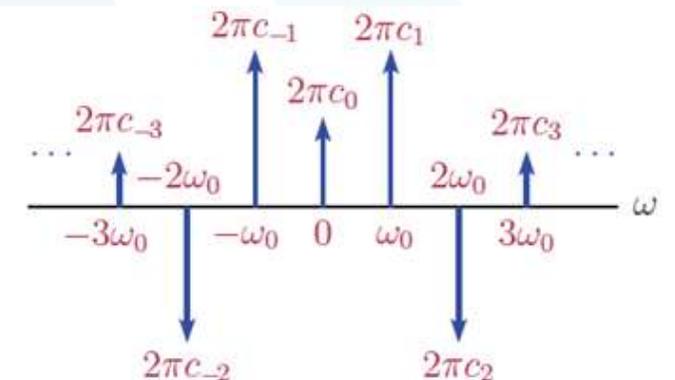
$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \Rightarrow X(\omega) = \int_{-\infty}^{\infty} \tilde{x}(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \right] e^{-j\omega t} dt$$

$$X(\omega) = \sum_{k=-\infty}^{\infty} c_k \left[\int_{-\infty}^{\infty} e^{jk\omega_0 t} e^{-j\omega t} dt \right]$$

$$= \sum_{k=-\infty}^{\infty} c_k [2\pi\delta(\omega - k\omega_0)]$$



EFS coefficients for a signal



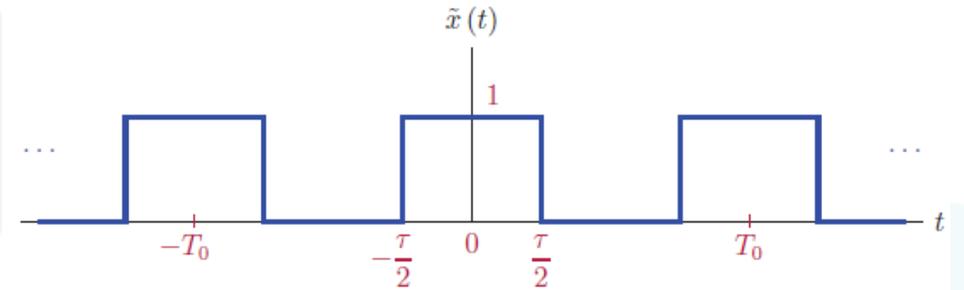
Fourier transform obtained

- Example 21:** Fourier transform of periodic pulse train

Determine the FT of the periodic pulse train with duty cycle $d = \tau/T_0$

$$c_k = d \operatorname{sinc}(kd)$$

$$X(\omega) = \sum_{k=-\infty}^{\infty} 2\pi d \operatorname{sinc}(kd) \delta(\omega - k\omega_0)$$

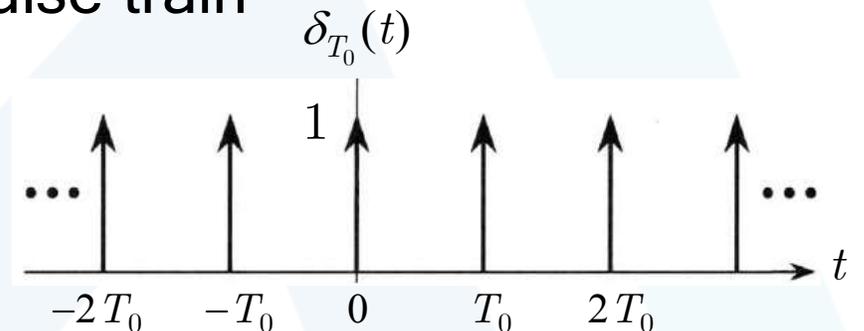


- Example 22:** Fourier transform of periodic impulse train

$$\delta_{T_0}(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_0)$$

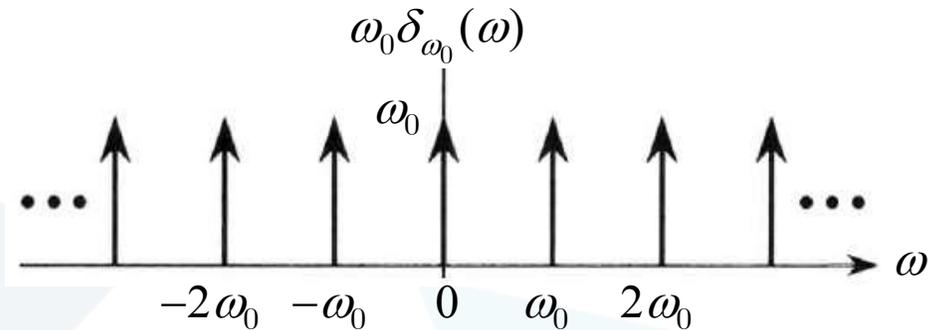
$$\delta_{T_0}(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_0) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

$$c_k = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} \delta(t) e^{-jk\omega_0 t} dt = \frac{1}{T_0}$$



$$\delta_{T_0}(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_0) = \frac{1}{T_0} \sum_{k=-\infty}^{\infty} e^{jk\omega_0 t}$$

$$\mathcal{F}\{\delta_{T_0}(t)\} = \frac{2\pi}{T_0} \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_0) = \omega_0 \delta_{\omega_0}(\omega)$$



4. Energy and Power in the Frequency Domain

Parseval's theorem

- For a periodic power signal $\tilde{x}(t)$ with period T_0 and EFS coefficients $\{c_k\}$:

$$P_x = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} |\tilde{x}(t)|^2 dt = \sum_{k=-\infty}^{\infty} |c_k|^2$$

- For a non-periodic energy signal $x(t)$ with a Fourier transform $X(f)$:

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df$$

Energy and power spectral density

$$S_x(f) = \sum_{k=-\infty}^{\infty} |c_k|^2 \delta(f - kf_0) \quad \text{power spectral density of the signal } x(t)$$

$$\int_{-\infty}^{\infty} S_x(f) df = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_x(\omega) d\omega = \sum_{k=-\infty}^{\infty} |c_k|^2 \quad P_x \text{ in } (-f_0, f_0) = \int_{-f_0}^{f_0} S_x(f) df$$

$$G_x(f) = |X(f)|^2 \quad \text{energy spectral density of the signal } \tilde{x}(t)$$

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} G_x(f) df = \frac{1}{2\pi} \int_{-\infty}^{\infty} G_x(\omega) d\omega \quad E_x \text{ in } (-f_0, f_0) = \int_{-f_0}^{f_0} G_x(f) df$$

- **Example 23:** Power spectral density of a periodic pulse train

Determine the power spectral density for the periodic pulse train $\tilde{x}(t)$. Also find the total power, the DC power, the power in the first three harmonics, and the power above 1 Hz.

$$c_k = \frac{1}{3} \text{sinc}(k/3)$$

$$S_x(f) = \sum_{k=-\infty}^{\infty} \left| \frac{1}{3} \text{sinc}(k/3) \right|^2 \delta(f - k/3)$$

The total power in the signal $x(t)$:

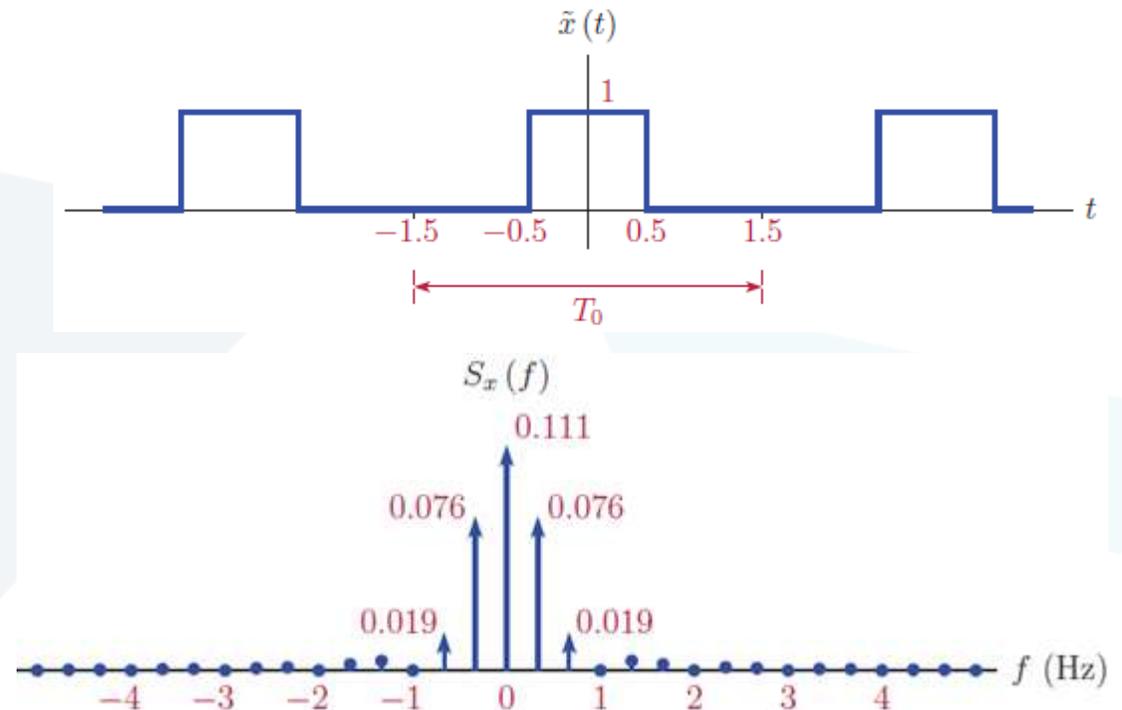
$$\frac{1}{T_0} \int_{t_0}^{t_0+T_0} |\tilde{x}(t)|^2 dt = \frac{1}{3} \int_{-0.5}^{0.5} (1)^2 dt = \frac{1}{3}$$

$$P_{dc} = |c_0|^2 = \left(\frac{1}{3} \right)^2 = \frac{1}{9} \approx 0.1111$$

$$P_1 = |c_{-1}|^2 + |c_1|^2 = \frac{3}{2\pi^2} \approx 0.1520, \quad P_2 = |c_{-2}|^2 + |c_2|^2 = \frac{3}{8\pi^2} \approx 0.0380, \quad P_3 = 0$$

The third harmonic is at frequency $f = 1$ Hz. Thus, the power above 1 Hz:

$$P_{hf} = P_x - P_{dc} - P_1 - P_2 - P_3 = 0.3333 - 0.1111 - 0.1520 - 0.0380 - 0 = 0.0322$$



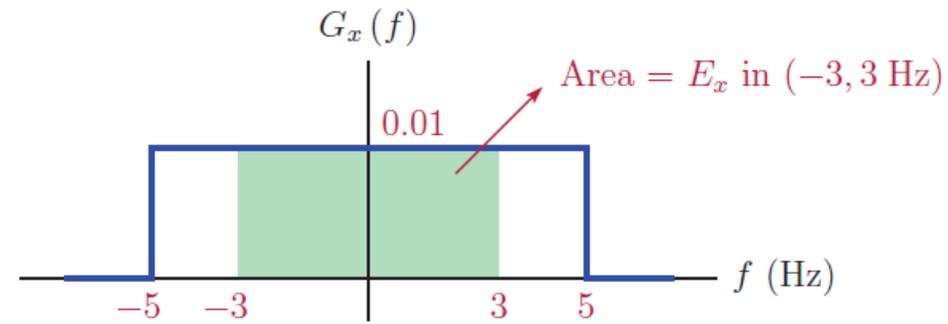
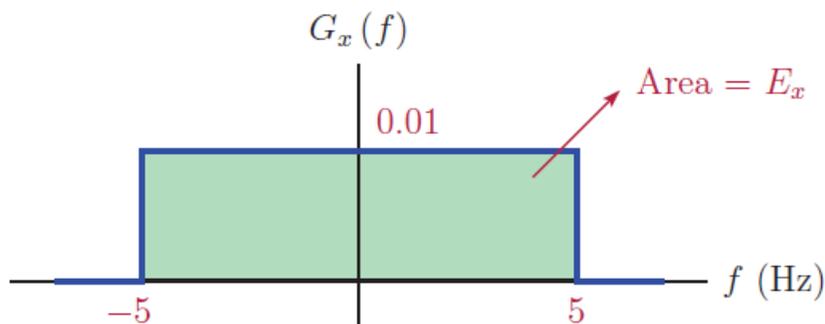
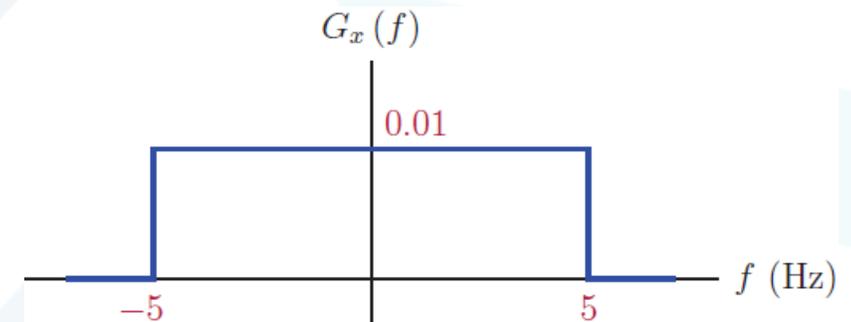
- **Example 24:** Energy spectral density of the sinc function

Determine the energy spectral density of $x(t) = \text{sinc}(10t)$. Afterwards, compute the total energy, and the energy in the sinc pulse at frequencies up to 3 Hz.

$$X(f) = \frac{1}{10} \Pi\left(\frac{f}{10}\right), \quad G_x(f) = |X(f)|^2 = \frac{1}{100} \Pi\left(\frac{f}{10}\right)$$

$$E_x = \int_{-\infty}^{\infty} G_x(f) df = \int_{-5}^5 \frac{1}{100} df = 0.1$$

$$E_x \text{ in } (-3, 3 \text{ Hz}) = \int_{-3}^3 G_x(f) df = \int_{-3}^3 \frac{1}{100} df = 0.06$$



Autocorrelation

- For an energy signal $x(t)$ the **autocorrelation function** is defined as:

$$r_{xx}(\tau) = \int_{-\infty}^{\infty} x(t)x(t + \tau)dt$$

- For a periodic power signal $\tilde{x}(t)$ with period T_0 , the corresponding definition of the autocorrelation function is:

$$\tilde{r}_{xx}(\tau) = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} \tilde{x}(t)\tilde{x}(t + \tau)dt$$

- The energy spectral density is the FT of the autocorrelation function:

$$\mathcal{F}\{r_{xx}(\tau)\} = G_x(f)$$

- The power spectral density is the FT of the autocorrelation function:

$$\mathcal{F}\{\tilde{r}_{xx}(\tau)\} = S_x(f)$$

- **Example 25:** Power spectral density of a sinusoidal signal revisited

$$\tilde{x}(t) = 5 \cos(200\pi t)$$

$$\tilde{r}_{xx}(\tau) = \frac{1}{0.01} \int_{-0.005}^{0.005} 25 \cos(200\pi t) \cos(200\pi[t + \tau]) dt = \frac{25}{2} \cos(200\pi\tau)$$

$$S_x(f) = F\{\tilde{r}_{xx}(\tau)\} = \frac{25}{4} \delta(f + 100) + \frac{25}{4} \delta(f - 100)$$

Properties of the autocorrelation function

- $r_{xx}(0) \geq |r_{xx}(\tau)|$ for all τ
- $r_{xx}(-\tau) = r_{xx}(\tau)$ for all τ , that is, the autocorrelation function has even symmetry.
- If the signal $x(t)$ is periodic with period T , then its autocorrelation function $\tilde{r}_{xx}(\tau)$ is also periodic with the same period.

5. Transfer Function Concept

- In **time-domain** analysis two distinct description forms for CTLTI systems:
 1. A **linear constant-coefficient differential equation** that describes the relationship between the input and the output signals.
 2. An **impulse response** which can be used with the **convolution operation** for determining the response of the system to an arbitrary input signal.
- The concept of **Transfer function** will be introduced as the third method for describing the characteristics of a system.

$$H(\omega) = \mathcal{F}\{h(t)\} = \int_{-\infty}^{\infty} h(t)e^{-j\omega t} dt$$

- **Note:** The transfer function concept is **valid** for LTI systems only.
- In general, $H(\omega)$ is a complex function of ω , $H(\omega) = |H(\omega)|e^{j\Theta(\omega)}$

- **Example 26:** Transfer function for the RC circuit

$$h(t) = \frac{1}{RC} e^{-t/RC} u(t)$$

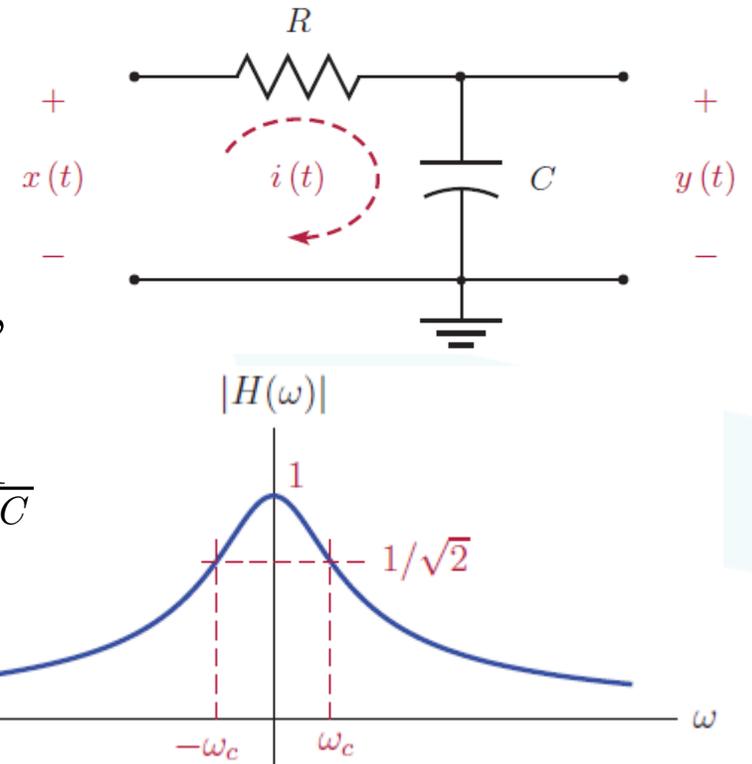
$$H(\omega) = \int_0^{\infty} \frac{1}{RC} e^{-t/RC} e^{-j\omega t} dt = \frac{1}{1 + j\omega RC} = \frac{1}{1 + j(\omega/\omega_c)},$$

$$|H(\omega)| = \frac{1}{\sqrt{1 + (\omega/\omega_c)^2}}, \quad \Theta(\omega) = -\tan^{-1}(\omega/\omega_c), \quad \omega_c = \frac{1}{RC}$$

$$H(\omega_c) = \frac{1}{1 + j}, \quad |H(\omega_c)| = \frac{1}{\sqrt{2}}$$

- ω_c represents the frequency at which the magnitude of the transfer function is 3 decibels below its peak value at $\omega = 0$,

$$20 \log_{10} \frac{|H(\omega_c)|}{|H(0)|} = 20 \log_{10} \frac{1}{\sqrt{2}} \approx -3 \text{ dB}$$



- The frequency ω_c is often referred to as the 3 dB **cutoff frequency** of the system.

Obtaining the transfer function from the differential equation

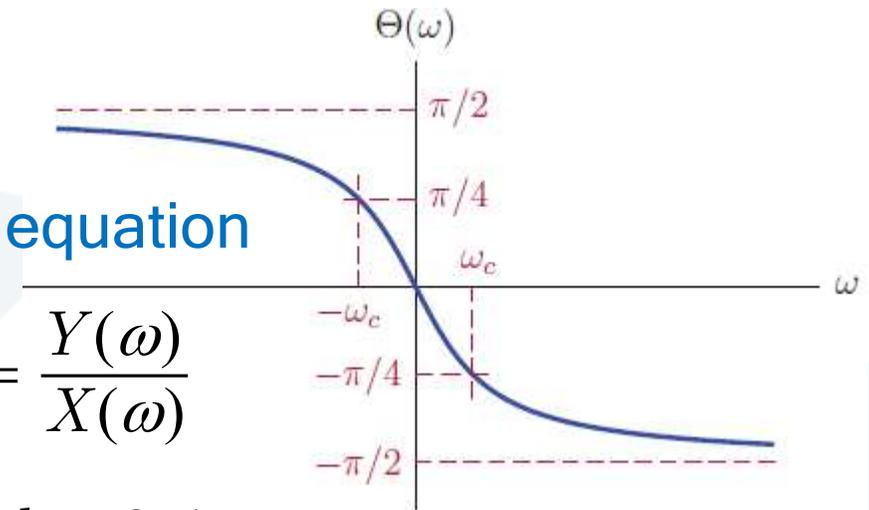
$$y(t) = h(t) * x(t) \xleftrightarrow{\mathcal{F}} Y(\omega) = H(\omega)X(\omega) \Rightarrow H(\omega) = \frac{Y(\omega)}{X(\omega)}$$

$$\frac{d^k y(t)}{dt^k} \xleftrightarrow{\mathcal{F}} (j\omega)^k Y(\omega), \quad \frac{d^k x(t)}{dt^k} \xleftrightarrow{\mathcal{F}} (j\omega)^k X(\omega), \quad k = 0, 1, \dots$$

- **Example 27:** Transfer function from the DE $\frac{d^2 y(t)}{dt^2} + 2 \frac{dy(t)}{dt} + 26y(t) = x(t)$

$$(j\omega)^2 Y(\omega) + 2(j\omega) Y(\omega) + 26Y(\omega) = X(\omega)$$

$$[(26 - \omega^2) + j2\omega] Y(\omega) = X(\omega) \Rightarrow H(\omega) = \frac{1}{(26 - \omega^2) + j2\omega}$$



6. CTLTI Systems with Periodic Input Signals

$$\tilde{x}(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(k\omega_0 t) + \sum_{k=1}^{\infty} b_k \sin(k\omega_0 t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

Response of a CTLTI system to complex exponential signal $\tilde{x}(t) = e^{j\omega_0 t}$

$$\begin{aligned} y(t) &= h(t) * \tilde{x}(t) = \int_{-\infty}^{\infty} h(\tau) \tilde{x}(t - \tau) d\tau = \int_{-\infty}^{\infty} h(\tau) e^{j\omega_0(t-\tau)} d\tau \\ &= e^{j\omega_0 t} \int_{-\infty}^{\infty} h(\tau) e^{-j\omega_0 \tau} d\tau = e^{j\omega_0 t} H(\omega_0) = |H(\omega_0)| e^{j[\omega_0 t + \Theta(\omega_0)]} \end{aligned}$$

- That is, $e^{j\omega t}$ is an **eigenfunction** of a LTI system and $H(\omega)$ is the corresponding **eigenvalue**. We refer to H as the **frequency response** of the system.

Response of a CTLTI system to sinusoidal signal $\tilde{x}(t) = \cos(\omega_0 t)$

$$\tilde{x}(t) = \cos(\omega_0 t) = \frac{1}{2} e^{j\omega_0 t} + \frac{1}{2} e^{-j\omega_0 t}$$

Response of a CTLTI system to sinusoidal signal $\tilde{x}(t) = \cos(\omega_0 t)$

$$\begin{aligned} y(t) &= \frac{1}{2} e^{j\omega_0 t} H(\omega_0) + \frac{1}{2} e^{-j\omega_0 t} H(-\omega_0) \\ &= \frac{1}{2} e^{j\omega_0 t} |H(\omega_0)| e^{j\Theta(\omega_0)} + \frac{1}{2} e^{-j\omega_0 t} |H(-\omega_0)| e^{-j\Theta(\omega_0)} \end{aligned}$$

If the impulse response $h(t)$ is real-valued:

$$\begin{aligned} |H(-\omega_0)| &= |H(\omega_0)|, \quad \Theta(-\omega_0) = -\Theta(\omega_0) \\ y(t) &= \frac{1}{2} |H(\omega_0)| e^{j[\omega_0 t + \Theta(\omega_0)]} + \frac{1}{2} |H(\omega_0)| e^{-j[\omega_0 t + \Theta(\omega_0)]} \\ &= |H(\omega_0)| \cos(\omega_0 t + \Theta(\omega_0)) \end{aligned}$$

- **Example 28:** Steady-state response of RC circuit for single-tone input

Let the component values be chosen to yield a 3 dB cutoff frequency of $\omega_c = 160\pi$ rad/s, or equivalently $f_c = 80$ Hz. Let the input signal be in the form $\tilde{x}(t) = 5 \cos(2\pi f t)$.

Compute the steady-state output signal for the cases $f_1 = 20$ Hz, $f_2 = 100$ Hz, $f_3 = 200$ Hz, and $f_4 = 500$ Hz.

$$H(f) = \frac{1}{1 + j(f/80)} \Rightarrow |H(f)| = \frac{1}{\sqrt{1 + (f/80)^2}}, \quad \Theta(f) = -\tan^{-1}(f/80)$$

$$|H(20)| = \frac{1}{\sqrt{1 + (20/80)^2}} = 0.9701, \quad \Theta(20) = -\tan^{-1}(20/80) = -0.245 \text{ rad}$$

$$y_1(t) = 5(0.9701) \cos(40\pi t - 0.245) = 4.8507 \cos(40\pi(t - 0.0019))$$

Response of a CTLTI system to periodic input signal

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

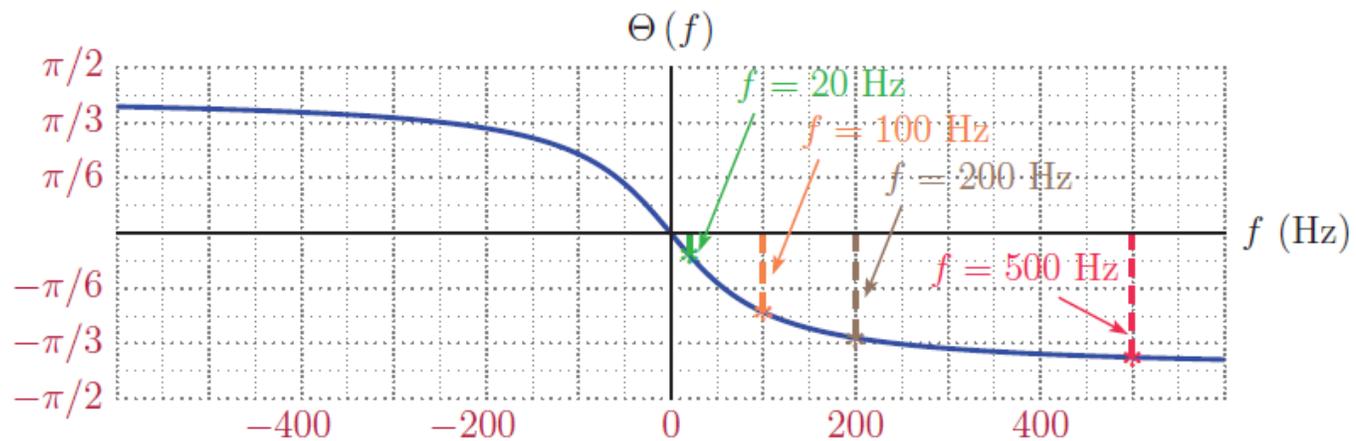
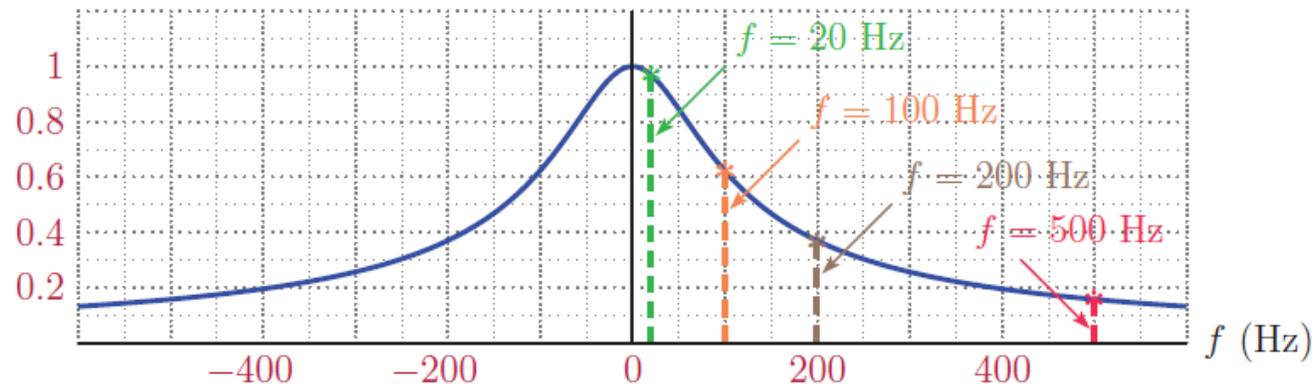
$$T\{\tilde{x}(t)\} = T\left\{\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}\right\} = \sum_{k=-\infty}^{\infty} T\{c_k e^{jk\omega_0 t}\} = \sum_{k=-\infty}^{\infty} c_k T\{e^{jk\omega_0 t}\} = \sum_{k=-\infty}^{\infty} c_k H(k\omega_0) e^{jk\omega_0 t}$$



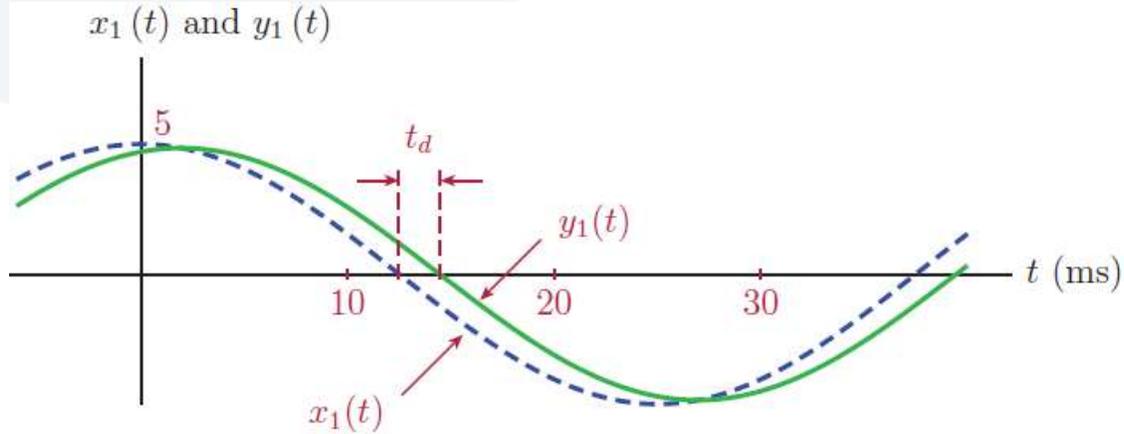
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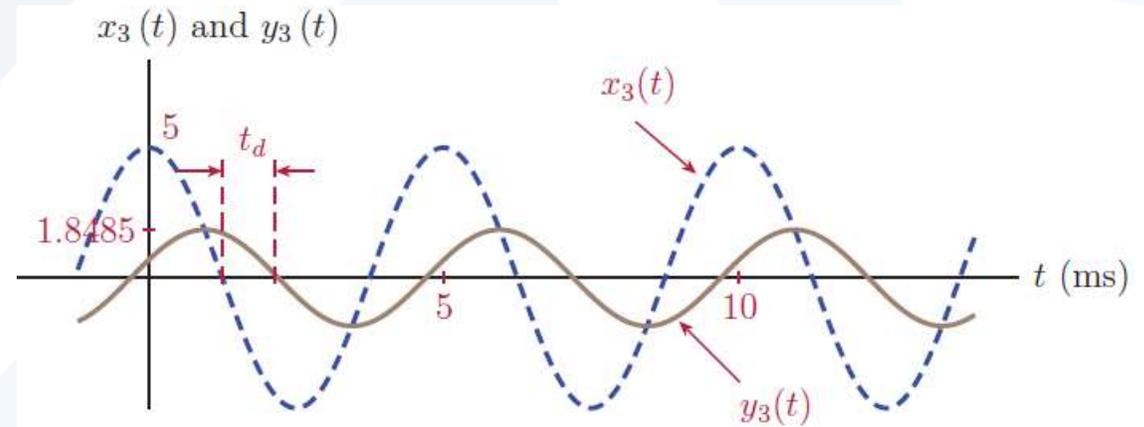
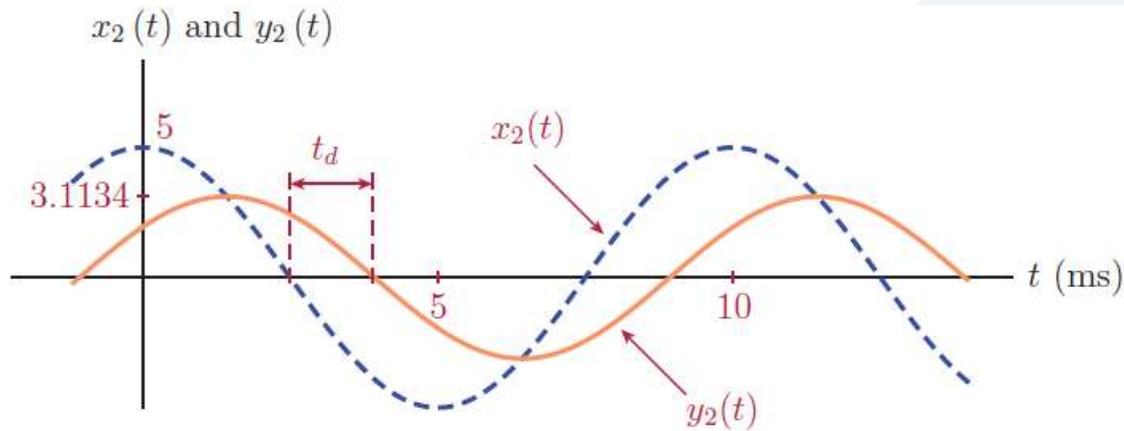
$|H(f)|$

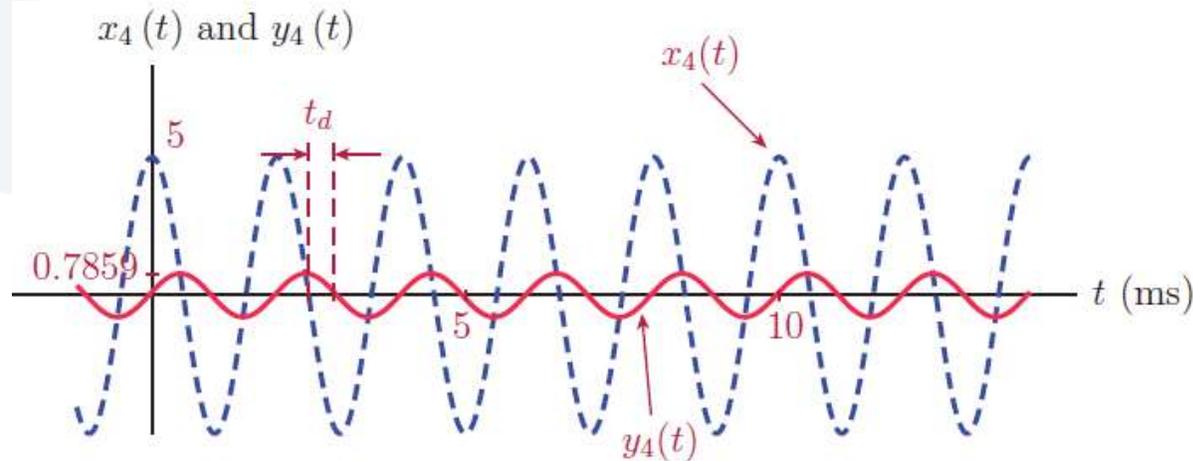


The phase shift of -0.245 radians translates to a time-delay of about $t_d = 1.9 \text{ ms}$



f (Hz)	$ H(f) $	$\Theta(f)$ (rad)	t_d (ms)
20	0.9701	-0.2450	1.95
100	0.6247	-0.8961	1.43
200	0.3714	-1.1903	0.94
500	0.1580	-1.4121	0.45



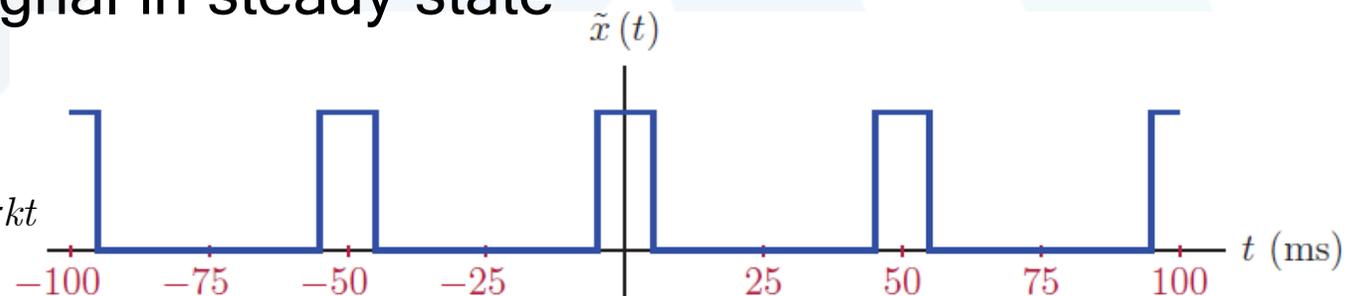


- Example 29:** RC circuit with pulse-train input

Let the input signal be a pulse train with period $T_0 = 50$ ms and duty cycle $d = 0.2$. Determine the output signal in steady state

$$c_k = 0.2 \operatorname{sinc}(0.2k) \Rightarrow$$

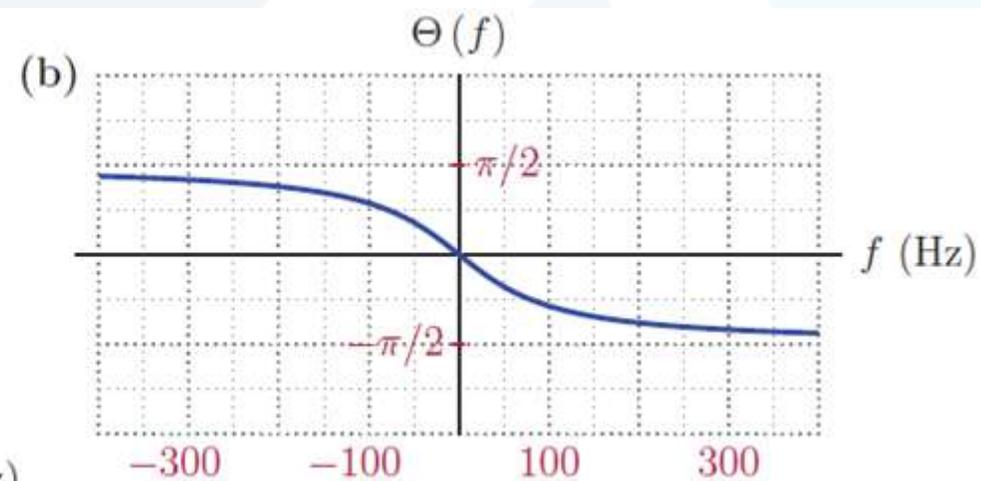
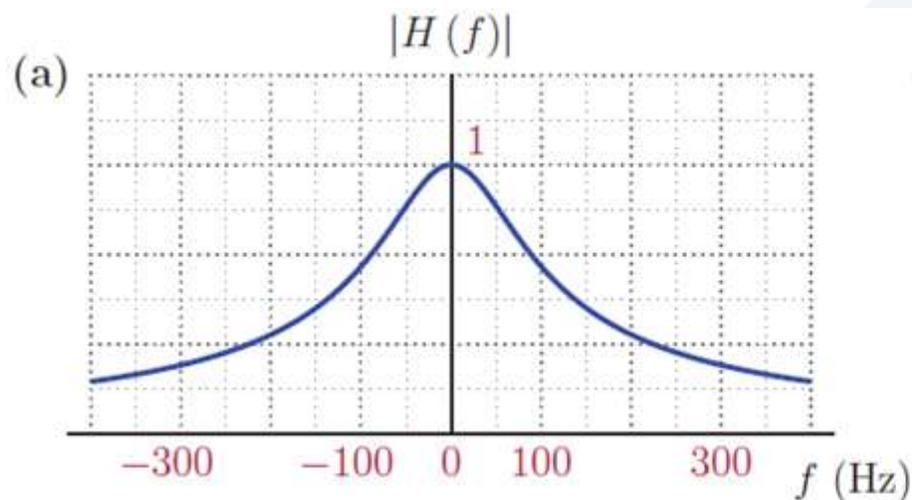
$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} 0.2 \operatorname{sinc}(0.2k) e^{j40\pi kt}$$

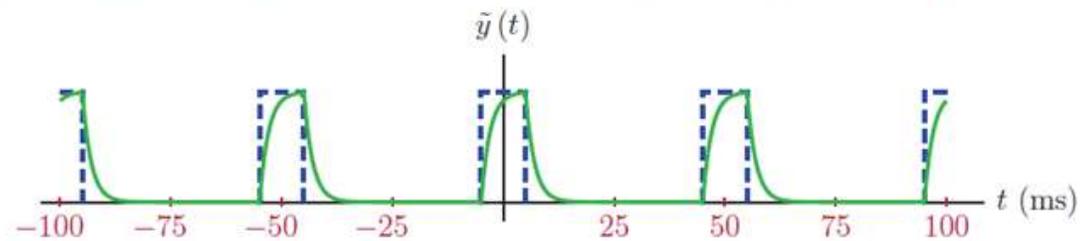
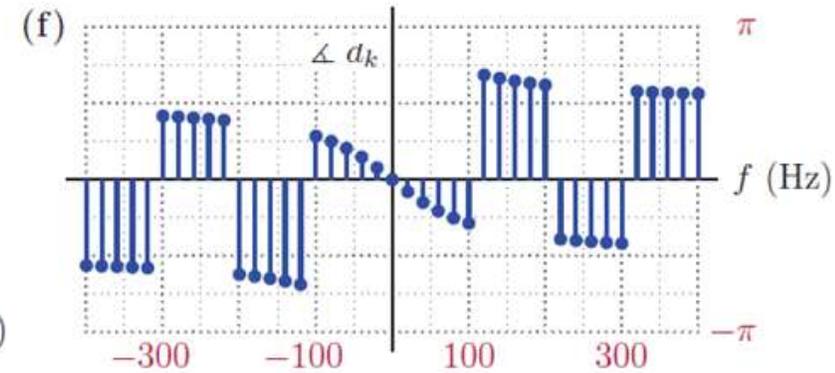
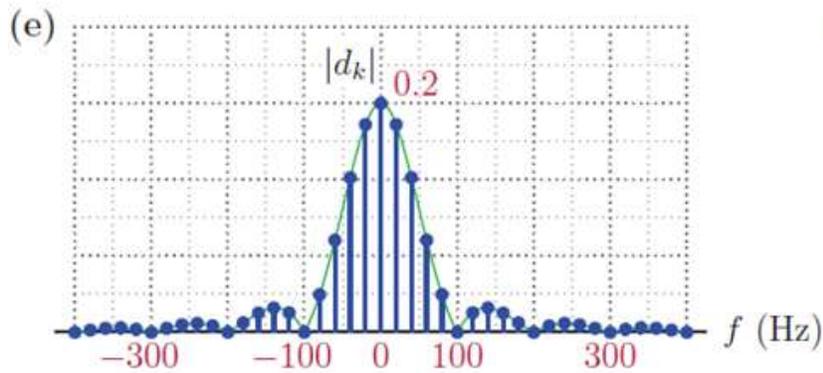
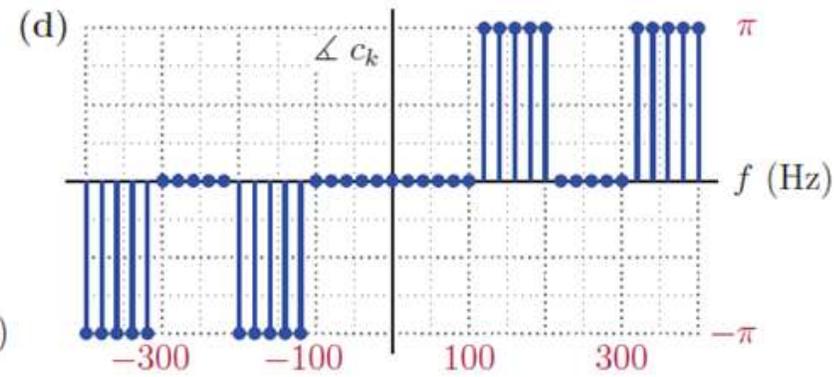
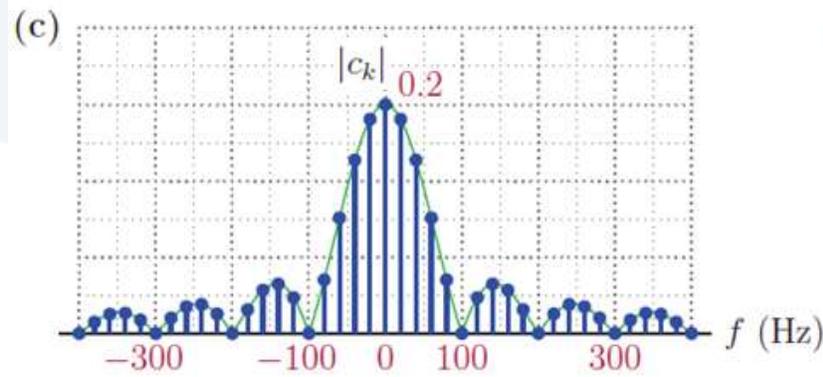




$$\tilde{y}(t) = \sum_{k=-\infty}^{\infty} \frac{c_k}{1 + j(20k/80)} e^{j40\pi kt} = \sum_{k=-\infty}^{\infty} d_k e^{j40\pi kt}$$

$$|d_k| = |c_k| |H(kf_0)| = \frac{|c_k|}{\sqrt{1 + (20k/80)^2}}, \quad \angle d_k = \angle c_k - \tan^{-1}(20k/80)$$





7. CTLTI Systems with Non-Periodic Input Signals

$$y(t) = h(t) * x(t) \Rightarrow Y(\omega) = H(\omega)X(\omega)$$

$$|Y(\omega)| = |H(\omega)||X(\omega)|, \quad \angle Y(\omega) = \angle X(\omega) + \Theta(\omega)$$

- **Example 30:** Pulse response of RC circuit revisited

Consider again the RC circuit. Let $f_c = 1/RC = 80$ Hz. Determine the FT of the response of the system to the unit-pulse input signal $x(t) = \Pi(t)$.

$$H(f) = \frac{1}{1 + j(f/f_c)}, \quad X(f) = \text{sinc}(f), \quad Y(f) = \frac{1}{1 + j(f/80)} \text{sinc}(f)$$

$$|Y(f)| = \frac{1}{\sqrt{1 + (f/80)^2}} |\text{sinc}(f)|, \quad \angle Y(f) = -\tan^{-1}(f/80) + \angle[\text{sinc}(f)]$$

