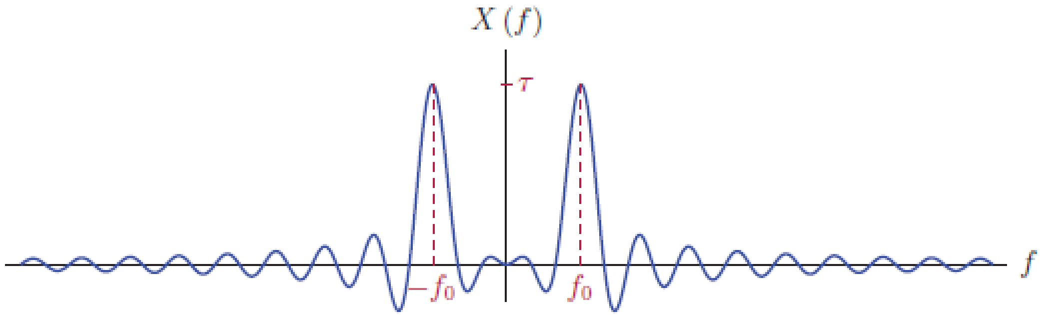
CEDC403: Signals and Systems

Lecture Notes 7: Fourier Analysis for Discrete Time Signals and Systems



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Chapter 5

Fourier Analysis for Discrete Time Signals and Systems

1. Analysis of Periodic Discrete-Time Signals

- A continuous-time periodic signal of period can be represented as a trigonometric/Exponential Fourier series (CTFS).
- A discrete-time periodic signal can be represented by a discrete-time Fourier series (DTFS) using a parallel development.
- One fundamental difference between DTFS and its CT counterpart, the exponential Fourier series, is regarding the number of series terms needed.
- A CT periodic signal may have an infinite range of frequencies, and therefore may require an infinite number of harmonically related basis functions.
- We will see that a discrete-time signal with a period of N samples will require at most N basis functions.

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Discrete-Time Fourier Series (DTFS)

- Consider a discrete-time signal $\tilde{x}[n]$ periodic with a period of N samples, that is, $\tilde{x}[n] = \tilde{x}[n+N]$ for all n. $\tilde{x}[n] = \sum_k \tilde{c}_k \phi_k[n] = \sum_k \tilde{c}_k e^{j\Omega_k n}$

  Two important questions need to be answered:

  1. How should the angular frequencies Ω_k be chosen?
  2. How many basis functions are needed?

- Since the period of $\tilde{x}[n]$ is N, the basis functions must also be periodic with N.

$$\phi_k[n+N] = \phi_k[n] \Rightarrow e^{j\Omega_k(n+N)} = e^{j\Omega_k n} \Rightarrow \Omega_k = 2\pi k/N \Rightarrow \phi_k[n] = e^{j(2\pi/N)kn}$$

$$\tilde{x}[n] = \sum_k \tilde{c}_k e^{j(2\pi/N)kn}$$

$\phi_{k+N}[n] = e^{j(2\pi/N)(k+N)n} = \phi_k[n] \Rightarrow$ we only need N terms $\tilde{x}[n] = \sum_{k=0}^{N-1} \tilde{c}_k e^{j(2\pi/N)kn}$

- This summation is referred to as the discrete-time Fourier series (DTFS) expansion of the periodic signal $\tilde{x}[n]$.
- Example 1: DTFS for a discrete-time sinusoidal signal $\tilde{x}[n] = \cos(0.2\pi n)$

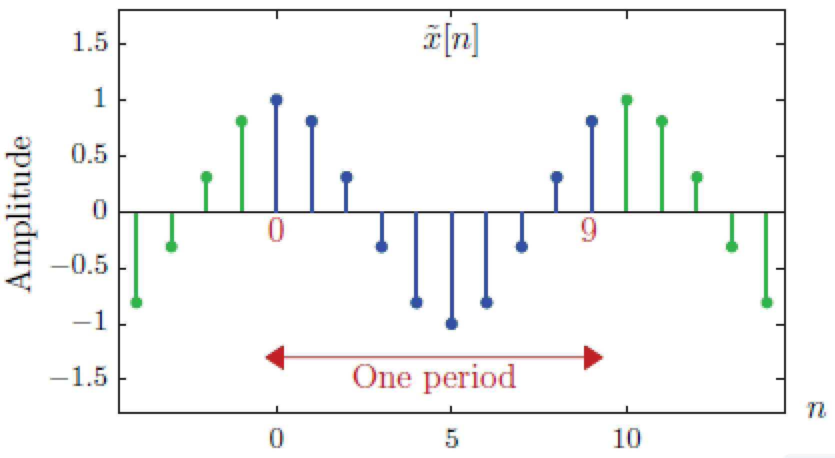
$\Omega_0 = 0.2\pi \text{ rad}, \quad F_0 = \frac{\Omega_0}{2\pi} = \frac{1}{10}$ periodic with a period of $N = 1/F_0 = 10$

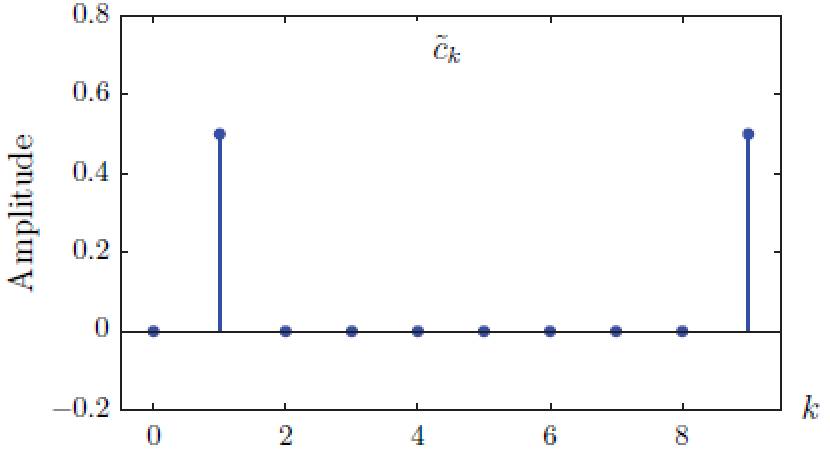
$$\tilde{x}[n] = \tfrac{1}{2}e^{j0.2\pi n} + \tfrac{1}{2}e^{-j0.2\pi n} = \tfrac{1}{2}e^{j(2\pi/10)n} + \tfrac{1}{2}e^{-j(2\pi/10)n} \Rightarrow \tilde{c}_1 = \tilde{c}_{-1} = \tfrac{1}{2}$$

$$\phi_{-1}[n] = e^{-j(2\pi/10)n} = e^{-j(2\pi/10)n}e^{j2\pi n} = e^{j(18\pi/10)n} = \phi_9[n]$$

$$\tilde{x}[n] = \tilde{c}_1 e^{j(2\pi/10)n} + \tilde{c}_9 e^{j(18\pi/10)n}$$

$$\tilde{c}_k = \begin{cases} \frac{1}{2}, & k = 1 \text{ or } k = 9 \\ 0, & \text{otherwise} \end{cases}$$





- Example 2: DTFS for a multi-tone signal $\tilde{x}[n] = 1 + \cos(0.2\pi n) + 2\sin(0.3\pi n)$

  $\Omega_1 = 0.2\pi$ and $\Omega_2 = 0.3\pi$ rad $\Rightarrow F_1 = 0.1$ and $F_2 = 0.15$ respectively.

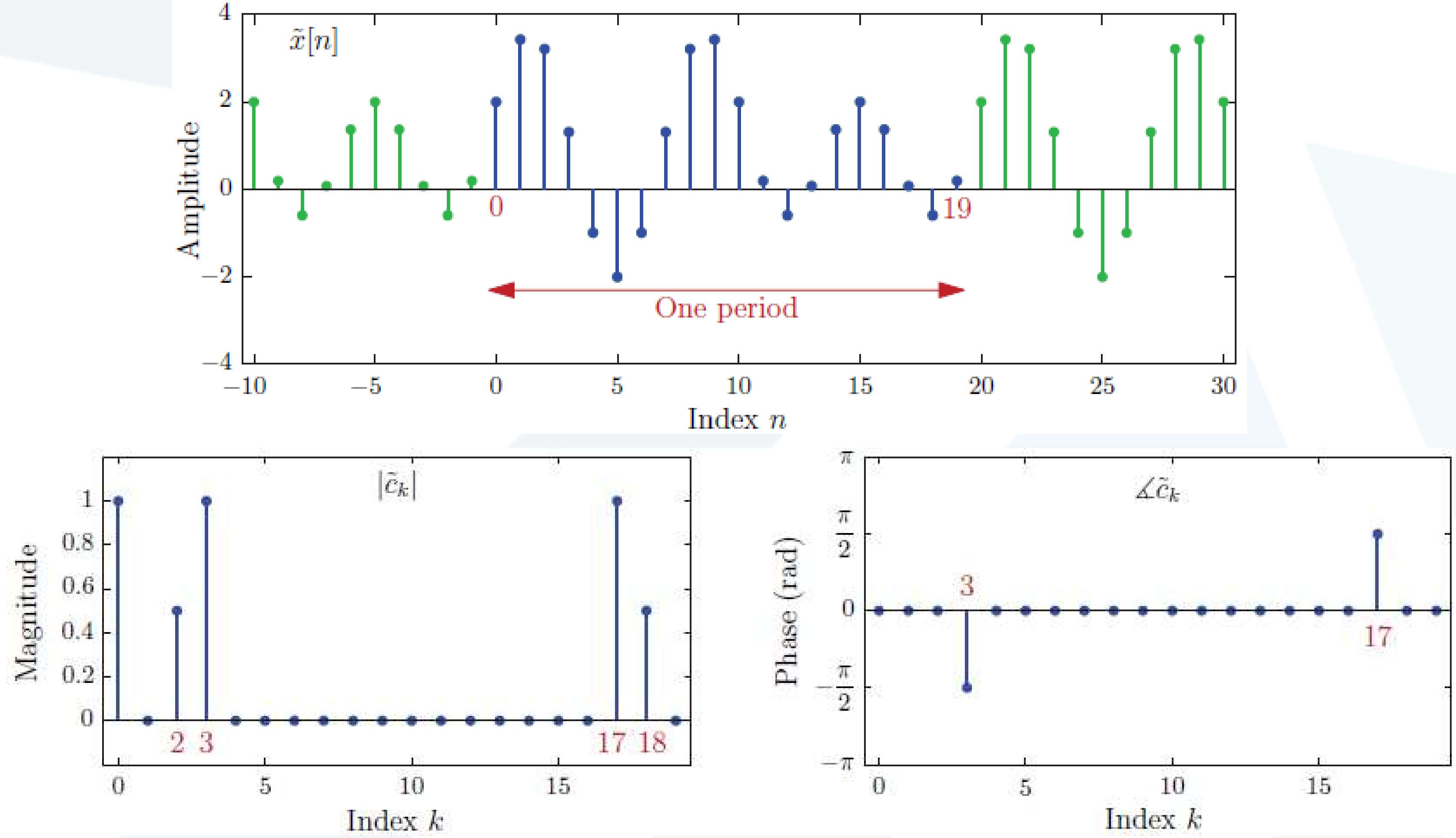
  The normalized fundamental frequency of $\tilde{x}[n]$ is $F_0 = 0.05$ ($N = 20$).

  $$\tilde{x}[n] = 1 + \cos(0.2\pi n) + 2\sin(0.3\pi n)$$

  $$\tilde{x}[n] = 1 + \frac{1}{2}e^{j(2\pi/20)2n} + \frac{1}{2}e^{-j(2\pi/20)2n} + \frac{1}{j}e^{j(2\pi/20)3n} - \frac{1}{j}e^{-j(2\pi/20)3n}$$

  $$\tilde{x}[n] = 1 + \frac{1}{2}\phi_2[n] + \frac{1}{2}\phi_{-2}[n] + e^{-j\pi/2}\phi_3[n] + e^{j\pi/2}\phi_{-3}[n]$$

$$\tilde{c}_0 = 1,\ \tilde{c}_{-2} = \tilde{c}_{18} = \tilde{c}_2 = \frac{1}{2},\ \tilde{c}_{-3} = \tilde{c}_{17} = e^{j\pi/2},\ \tilde{c}_3 = e^{-j\pi/2}$$



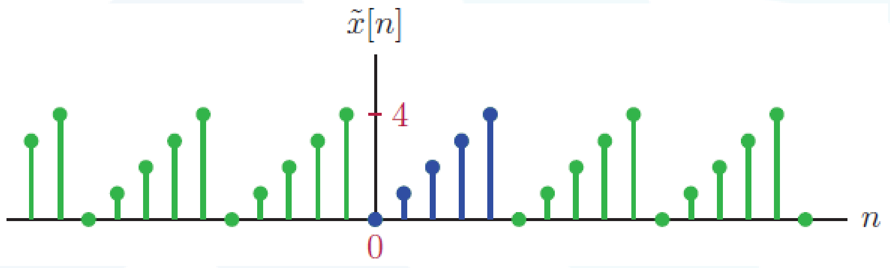
Discrete-time Fourier series (DTFS):

1. Synthesis equation: $\tilde{x}[n] = \sum_{k=0}^{N-1} \tilde{c}_k e^{j(2\pi/N)kn}, \quad$ all n

2. Analysis equation: $\tilde{c}_k = \frac{1}{N} \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j(2\pi/N)kn}$

- Example 3: DTFS representation of $\tilde{x}[n] = n$, $n = 0, 1, 2, 3, 4$ and $\tilde{x}[n] = \tilde{x}[n+5]$

$$\tilde{c}_k = \frac{1}{5} \sum_{n=0}^{4} \tilde{x}[n] e^{-j(2\pi/5)kn} = \frac{1}{5} e^{-j2\pi k/5} + \frac{2}{5} e^{-j4\pi k/5} + \frac{3}{5} e^{-j6\pi k/5} + \frac{4}{5} e^{-j8\pi k/5}$$



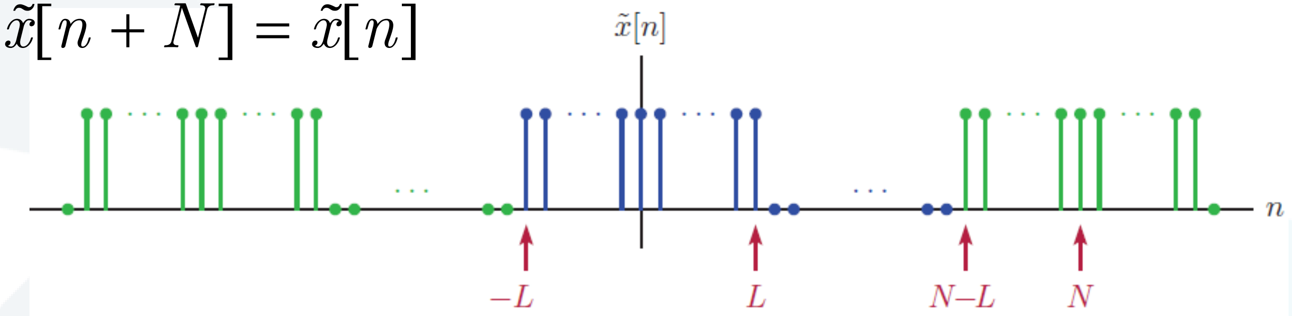
$$\tilde{c}_0 = 2,\ \tilde{c}_1 = -0.5 + j0.6882,\ \tilde{c}_2 = -0.5 + j0.1625,$$
$$\tilde{c}_3 = -0.5 - j0.1625,\ \tilde{c}_4 = -0.5 - j0.6882$$

$$\tilde{x}[n] = 2 + (-0.5 + j0.6882)e^{j2\pi n/5} + (-0.5 + j0.1625)e^{j4\pi n/5} + (-0.5 - j0.1625)e^{j6\pi n/5} + (-0.5 - j0.6882)e^{j8\pi n/5}$$

- Example 4: DTFS for discrete-time pulse train

$$\tilde{x}[n] = \begin{cases} 1, & -L \le n \le L \\ 0, & L < n < N - L \end{cases} \quad \text{and} \quad \tilde{x}[n+N] = \tilde{x}[n]$$

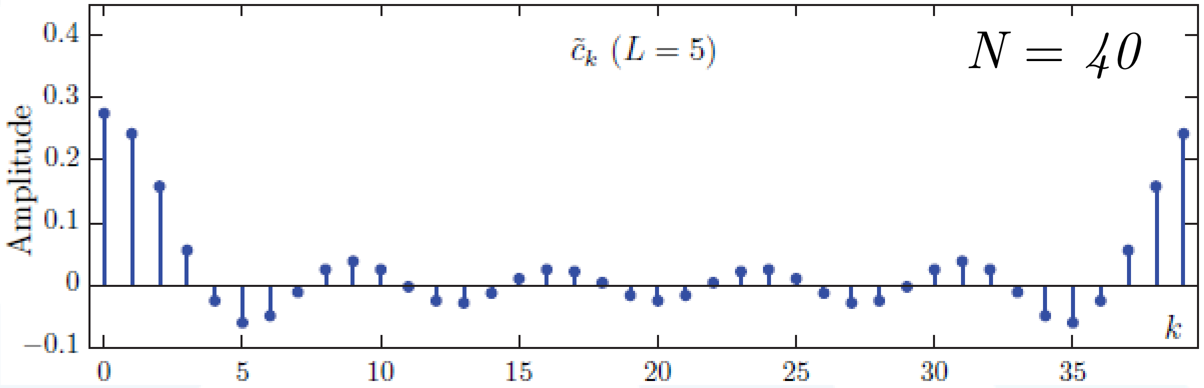
where $N > 2L + 1$.



$$\tilde{c}_k = \frac{1}{N}\sum_{n=-L}^{L} e^{-j(2\pi/N)kn} = \frac{1}{N}\,\frac{e^{j(2\pi/N)Lk} - e^{-j(2\pi/N)(L+1)k}}{1 - e^{-j(2\pi/N)k}} \times \frac{e^{-j\pi k/N}}{e^{-j\pi k/N}}$$

$$\tilde{c}_k = \frac{\sin\left(\frac{\pi k}{N}(2L+1)\right)}{N\sin\left(\frac{\pi k}{N}\right)}, \; k = 0, 1, \cdots, N-1$$

$$\tilde{x}[n] = \sum_{k=0}^{N-1} \frac{\sin\left(\frac{\pi k}{N}(2L+1)\right)}{N\sin\left(\frac{\pi k}{N}\right)} e^{j(2\pi/N)kn}$$



Properties of the DTFS

Periodicity $\tilde{x}[n] = \tilde{x}[n + rN]$ all integer $r \Rightarrow \tilde{c}_k = \tilde{c}_{k+rN},$ all integer r

Linearity $\tilde{x}_1[n] \xleftrightarrow{DTFS} \tilde{c}_k$ and $\tilde{x}_2[n] \xleftrightarrow{DTFS} \tilde{d}_k$

$$\Rightarrow \alpha_1\tilde{x}_1[n] + \alpha_2\tilde{x}_1[n] \xleftrightarrow{DTFS} \alpha_1\tilde{c}_k + \alpha_2\tilde{d}_k$$

Symmetry of DTFS coefficients

$\tilde{x}[n]$: real, $\text{Im}\{\tilde{x}[n]\} = 0 \Rightarrow \tilde{c}_k^* = \tilde{c}_{N-k},$ $\tilde{x}[n]$: imag, $\text{Re}\{\tilde{x}[n]\} = 0 \Rightarrow \tilde{c}_k^* = -\tilde{c}_{N-k}$

DTFS spectra of even and odd signals

- If the real-valued signal $\tilde{x}[n]$ is an even function of index n, the resulting DTFS spectrum $\tilde{c}_k$ is real-valued for all k.

$$\tilde{x}[-n] = \tilde{x}[n], \text{ for all } n \Rightarrow \text{Im}\{\tilde{c}_k\} = 0, \text{ for all } k$$

- If the real-valued signal $\tilde{x}[n]$ is an odd function of index n, the resulting DTFS spectrum $\tilde{c}_k$ is purely imaginary for all k.

$$\tilde{x}[-n] = -\tilde{x}[n], \text{ for all } n \;\Rightarrow \text{Re}\{\tilde{c}_k\} = 0, \text{ for all } k$$

Periodic convolution

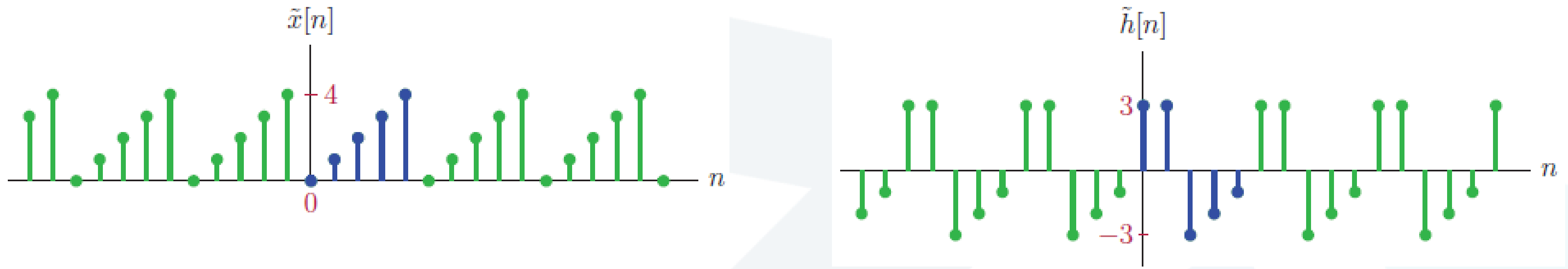
- The convolution of two DT signals: $y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$

  This summation fail to converge if both signals $x[n]$ and $h[n]$ are periodic with periods N. For such a case, a periodic convolution operator can be defined as:

$$\tilde{y}[n] = \tilde{x}[n] \otimes \tilde{h}[n] = \sum_{k=0}^{N-1} \tilde{x}[k]\tilde{h}[n-k], \quad \text{all } n$$

  If both $\tilde{x}[n]$ and $\tilde{h}[n]$ have the same period $N \Rightarrow \tilde{y}[n]$ also periodic with the same period.

- Example 5: Determine the periodic convolution $\tilde{y}[n] = \tilde{x}[n] \otimes \tilde{h}[n]$



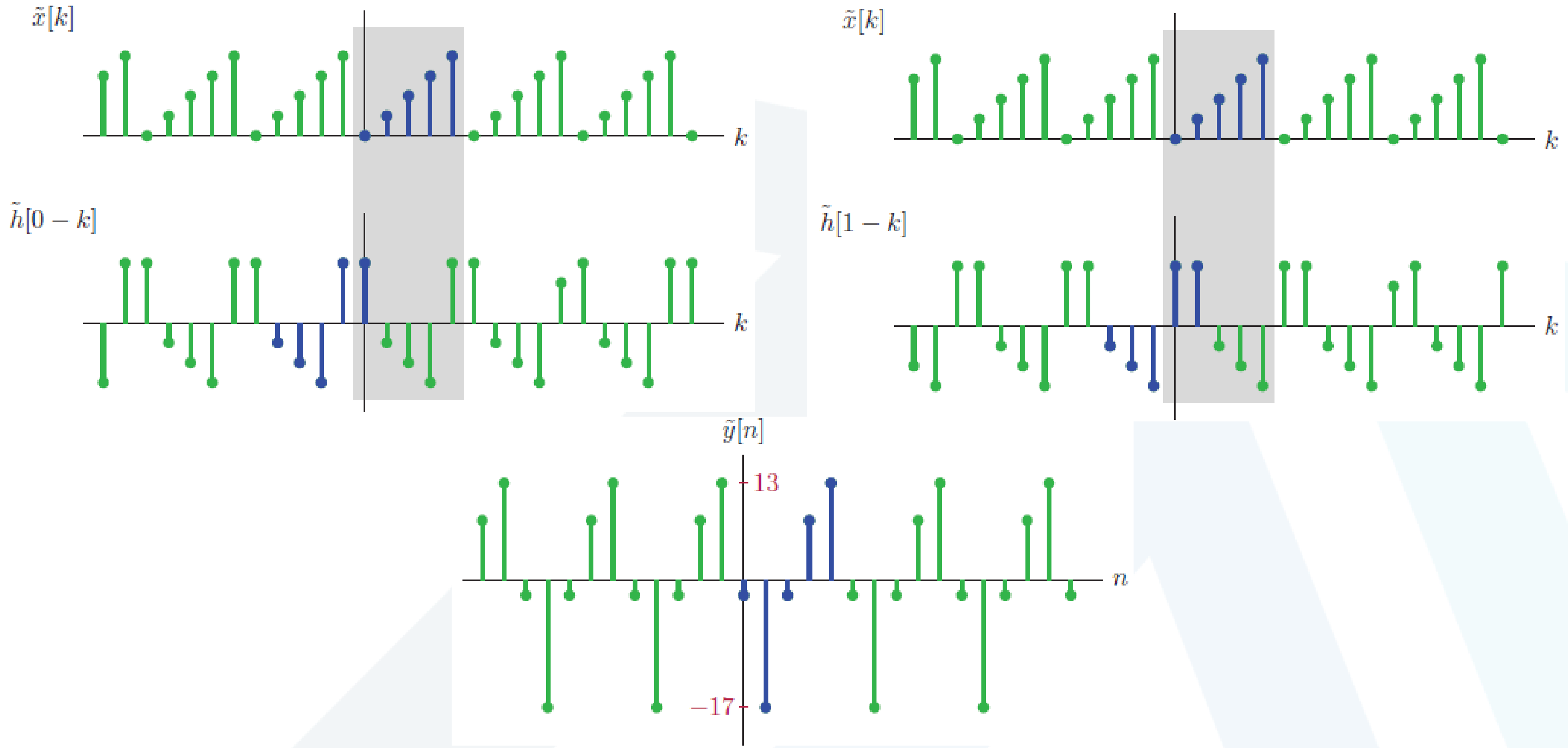
$$\tilde{x}[n] = \{\cdots, \underset{\uparrow}{0}, 1, 2, 3, 4, \cdots\} \qquad \tilde{h}[n] = \{\cdots, \underset{\uparrow}{3}, 3, -3, -2, -1, \cdots\}$$

$$\tilde{y}[n] = \sum_{k=0}^{4} \tilde{x}[k]\tilde{h}[n-k]$$

$$\tilde{y}[0] = \sum_{k=0}^{4} \tilde{x}[k]\tilde{h}[-k] = \tilde{x}[0]\tilde{h}[0] + \tilde{x}[1]\tilde{h}[4] + \tilde{x}[2]\tilde{h}[3] + \tilde{x}[3]\tilde{h}[2] + \tilde{x}[4]\tilde{h}[1] = -2$$

$$\tilde{y}[1] = \sum_{k=0}^{4} \tilde{x}[k]\tilde{h}[1-k] = \tilde{x}[0]\tilde{h}[1] + \tilde{x}[1]\tilde{h}[0] + \tilde{x}[2]\tilde{h}[4] + \tilde{x}[3]\tilde{h}[3] + \tilde{x}[4]\tilde{h}[2] = -17$$

$$\tilde{y}[2] = \sum_{k=0}^{4} \tilde{x}[k]\tilde{h}[2-k] = -2, \quad \tilde{y}[3] = \sum_{k=0}^{4} \tilde{x}[k]\tilde{h}[3-k] = 8, \quad \tilde{y}[4] = \sum_{k=0}^{4} \tilde{x}[k]\tilde{h}[4-k] = 13$$



$$\tilde{y}[n] = \{\cdots,\ \underset{\uparrow}{-2},\ -17,\ -2,\ 8,\ 13,\ \cdots\}$$

The periodic convolution property

Let $\tilde{x}[n]$ and $\tilde{h}[n]$ periodic with the same period $\tilde{x}[n] \xleftrightarrow{DTFS} \tilde{c}_k$ and $\tilde{h}[n] \xleftrightarrow{DTFS} \tilde{d}_k$

$$\Rightarrow \tilde{x}[n] \otimes \tilde{h}[n] \xleftrightarrow{DTFS} N\tilde{c}_k\tilde{d}_k$$

- Example 6: Periodic convolution

  Refer to $\tilde{x}[n]$, $\tilde{h}[n]$ and $\tilde{y}[n]$ of Example 6. The DTFS coefficients of $\tilde{x}[n]$ were determined in Example 4. Find the DTFS coefficients of $\tilde{h}[n]$ and $\tilde{y}[n]$. Verify the convolution property.

  Let $\tilde{c}[n]$, $\tilde{d}[n]$ and $\tilde{e}[n]$ represent the DTFS coef. of $\tilde{x}[n]$, $\tilde{h}[n]$ and $\tilde{y}[n]$ respectively.

k	$\tilde{c}_k$	$\tilde{d}_k$	$\tilde{e}_k$
0	$2.0000+j\,0.0000$	$0.0000+j\,0.0000$	$0.0000+j\,0.0000$
1	$-0.5000+j\,0.6882$	$1.5326-j\,0.6433$	$-1.6180+j\,6.8819$
2	$-0.5000+j\,0.1625$	$-0.0326-j\,0.6604$	$0.6180+j\,1.6246$
3	$-0.5000-j\,0.1625$	$-0.0326+j\,0.6604$	$0.6180-j\,1.6246$
4	$-0.5000-j\,0.6882$	$1.5326+j\,0.6433$	$-1.6180-j\,6.8819$

  It can easily be verified that: $\tilde{e}_k = 5\tilde{c}_k\tilde{d}_k,\ k = 0, \ldots, 4$

2. Analysis of Non-Periodic Discrete-Time Signals

Discrete-time Fourier transform (DTFT)

1. Synthesis equation: (Inverse transform) $x[n] = \mathcal{F}^{-1}\{X(\Omega)\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\Omega)\, e^{j\Omega n} d\Omega$

2. Analysis equation: (Forward transform) $X(\Omega) = \mathcal{F}\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}$

Existence of the DTFT

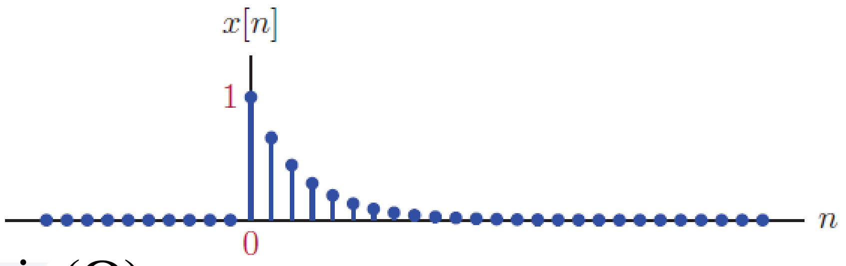
- A sufficient condition for the convergence of DTFT for the signal $x[n]$ be absolute summable, $\sum_{n=-\infty}^{\infty} |x[n]| < \infty$

- Alternatively, it is also sufficient for $x[n]$ to be square-summable: $\sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty$

DTFT of some signals

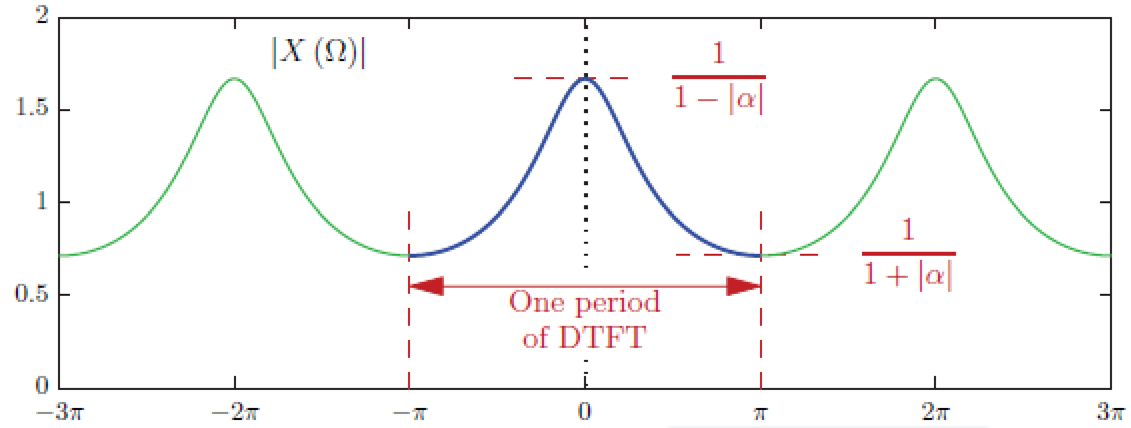
- Example 7: DTFT of right-sided exponential signal

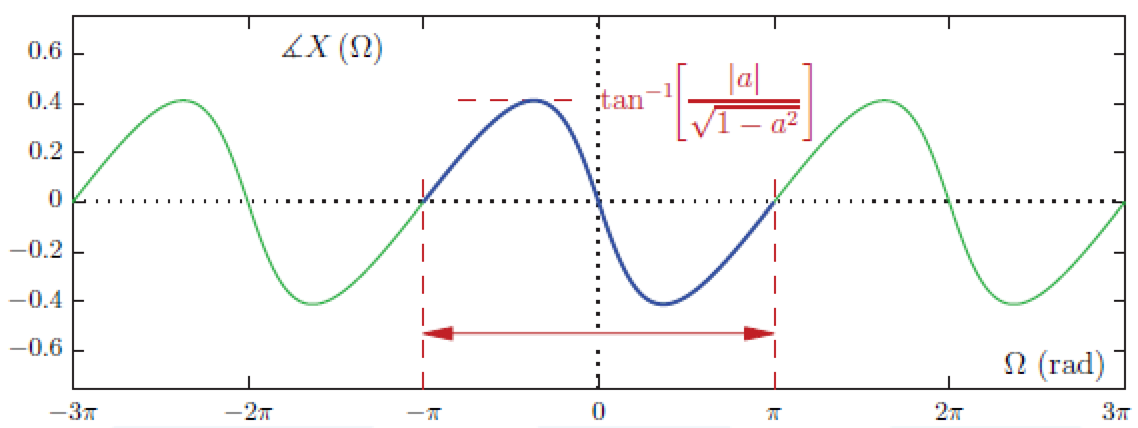
$$x[n] = \alpha^n u[n], \quad |\alpha| < 1$$



$$X(\Omega) = \sum_{n=0}^{\infty} \alpha^n e^{-j\Omega n} = \frac{1}{1 - \alpha e^{-j\Omega}}$$

$$|X(\Omega)| = \frac{1}{\sqrt{1 + \alpha^2 - 2\alpha \cos(\Omega)}}, \quad \measuredangle X(\Omega) = -\tan^{-1} \frac{\alpha \sin(\Omega)}{1 - \alpha \cos(\Omega)}$$



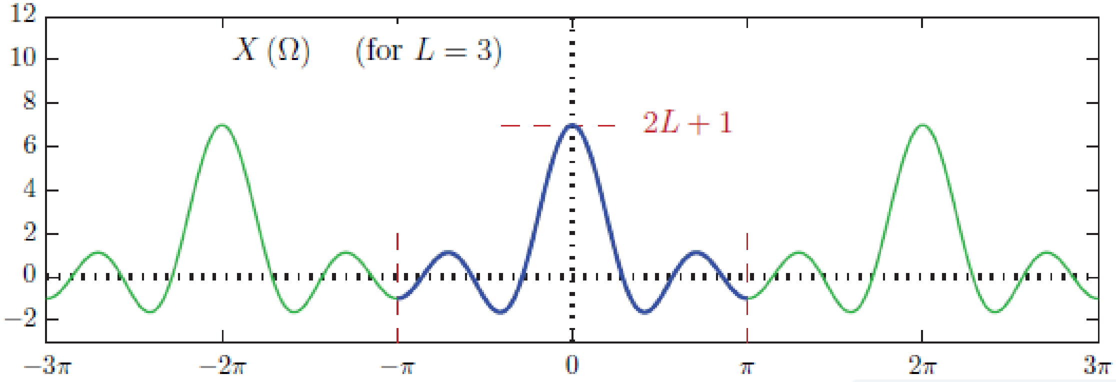


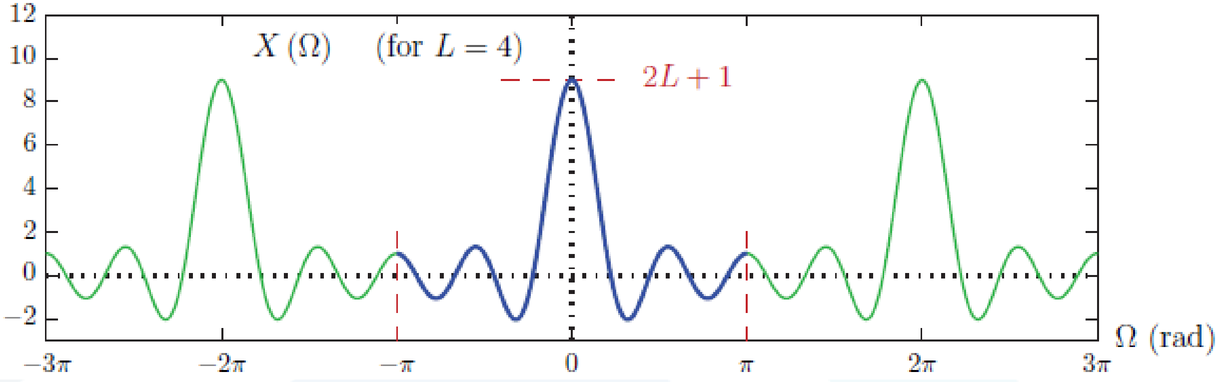
- Example 8: DTFT of unit-impulse signal

$$\mathcal{F}\{\delta[n]\} = \sum_{n=-\infty}^{\infty} \delta[n] e^{-j\Omega n} = 1$$

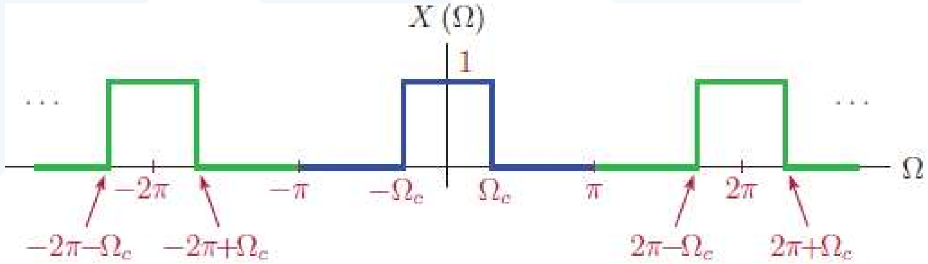
- Example 9: DTFT for discrete-time pulse

$$x[n] = \begin{cases} 1, & -L \le n \le L \\ 0, & \text{otherwise} \end{cases} \quad X(\Omega) = \sum_{n=-L}^{L} (1)\, e^{-j\Omega n} = \frac{e^{j\Omega L} - e^{-j\Omega(L+1)}}{1 - e^{-j\Omega}} = \frac{\sin\left(\frac{\Omega}{2}(2L+1)\right)}{\sin\left(\frac{\Omega}{2}\right)}$$

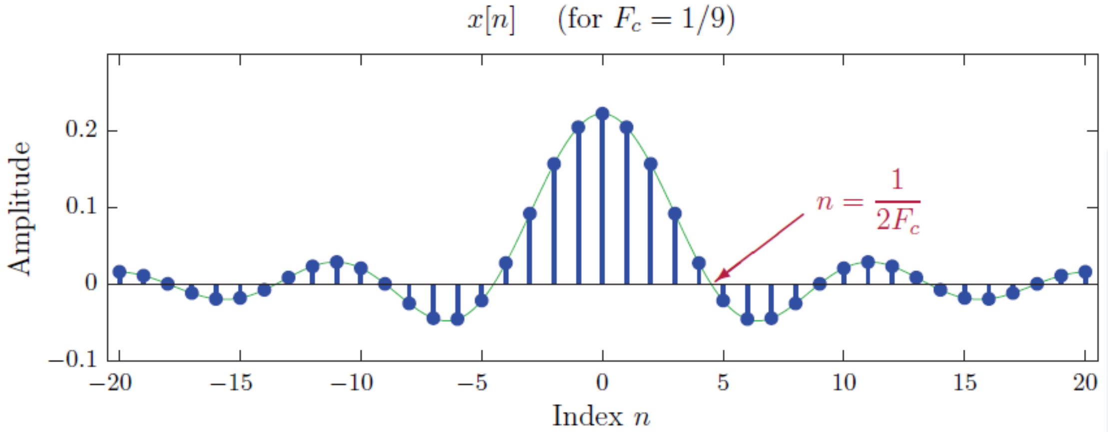




- Example 10: Inverse DTFT of rectangular spectrum defined in the range $-\pi < \Omega < \pi$ by: $X[\Omega] = \begin{cases} 1, & -\Omega_c < \Omega < \Omega_c \\ 0, & \text{otherwise} \end{cases}$



$X(\Omega)$ must be 2π-periodic: $X(\Omega) = X(\Omega + 2\pi)$



$$x[n] = \frac{1}{2\pi}\int_{-\Omega_c}^{\Omega_c} e^{j\Omega n} d\Omega = \frac{\sin(\Omega_c n)}{\pi n}$$
$$= 2F_c \,\text{sinc}(2F_c n)$$

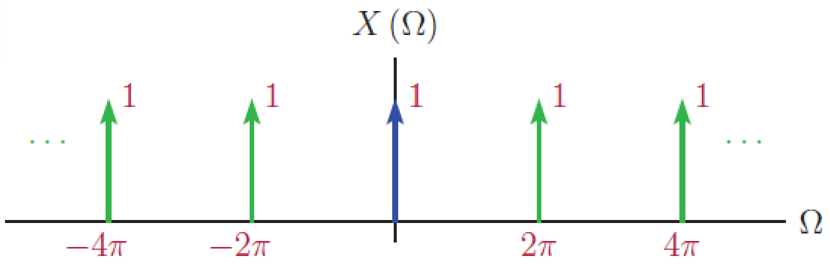
- Example 11: Inverse DTFT of the unit-impulse function

Find the signal of which the DTFT is $X(\Omega) = \delta(\Omega)$ in the range $-\pi < \Omega < \pi$.

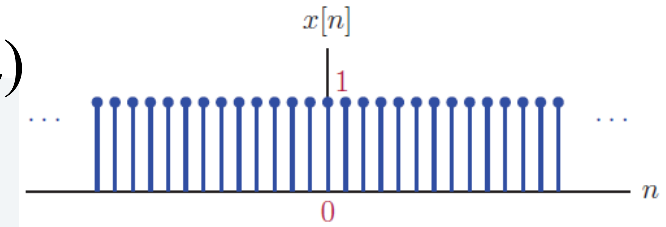
$X(\Omega)$ must be 2π-periodic: $X(\Omega) = \sum_{m=-\infty}^{\infty} \delta(\Omega - 2\pi m)$

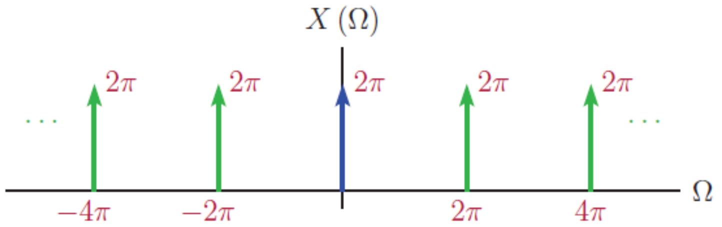
$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \delta(\Omega)\, e^{j\Omega n} d\Omega = \frac{1}{2\pi}, \quad \text{all } n$$

$$\frac{1}{2\pi} \overset{\mathcal{F}}{\longleftrightarrow} \sum_{m=-\infty}^{\infty} \delta(\Omega - 2\pi m)$$



$$\mathcal{F}\{1\} \stackrel{\mathcal{F}}{\longleftrightarrow} 2\pi \sum_{m=-\infty}^{\infty} \delta(\Omega - 2\pi m)$$





Properties of the DTFT

Periodicity

$X(\Omega + 2\pi r) = X(\Omega)$ for all integer r

Linearity

$$x_1[n] \stackrel{\mathcal{F}}{\longleftrightarrow} X_1(\Omega) \quad \text{and} \quad x_1[n] \stackrel{\mathcal{F}}{\longleftrightarrow} X_2(\Omega)$$

$$\Rightarrow \alpha_1 x_1[n] + \alpha_2 x_2[n] \stackrel{\mathcal{F}}{\longleftrightarrow} \alpha_1 X_1(\Omega) + \alpha_2 X_2(\Omega)$$

Time shifting

$$x[n] \stackrel{\mathcal{F}}{\longleftrightarrow} X(\Omega) \Rightarrow x[n-m] \stackrel{\mathcal{F}}{\longleftrightarrow} X(\Omega) e^{-j\Omega m}$$

Time reversal and Conjugation

$$x[n] \stackrel{\mathcal{F}}{\longleftrightarrow} X(\Omega) \Rightarrow x[-n] \stackrel{\mathcal{F}}{\longleftrightarrow} X(-\Omega)$$

$$x[n] \stackrel{\mathcal{F}}{\longleftrightarrow} X(\Omega) \Rightarrow x^*[n] \stackrel{\mathcal{F}}{\longleftrightarrow} X^*(-\Omega)$$

Symmetry of the DTFT

$x[n]$: Real, $\text{Im}\{x[n]\} = 0$ implies that $X^*(\Omega) = X(-\Omega)$.

$x[n]$: Imag, Re $\{x[n]\} =$ implies that $X^*(\Omega) = -X(-\Omega)$.

DTFST of even and odd signals

- If the real-valued signal $x[n]$ is an even function of time, the resulting DTFT is real-valued $X(\Omega)$ for all Ω.

$$x[-n] = x[n] \text{ , for all } n \text{ implies that } \text{Im}\{X(\Omega)\} = 0 \text{ , all } \Omega$$

- If the real-valued signal $x[n]$ is an odd function of time, the resulting DTFT $X(\Omega)$ is purely imaginary.

$$x[-n] = -x[n] \text{ , for all } n \text{ implies that } \text{Re}\{X(\Omega)\} = 0 \text{ , all } \Omega$$

Frequency shifting $x[n] \overset{\mathcal{F}}{\longleftrightarrow} X(\Omega) \quad \Rightarrow \quad x[n]e^{j\Omega_0 n} \overset{\mathcal{F}}{\longleftrightarrow} X(\Omega - \Omega_0)$

Modulation property

$$x[n] \overset{\mathcal{F}}{\longleftrightarrow} X(\Omega) \quad \Rightarrow$$

$$x[n]\cos(\Omega_0 n) \overset{\mathcal{F}}{\longleftrightarrow} \tfrac{1}{2}\left[X(\Omega-\Omega_0)+X(\Omega+\Omega_0)\right]$$

$$x[n]\sin(\Omega_0 t) \overset{\mathcal{F}}{\longleftrightarrow} \tfrac{1}{2}\left[X(\Omega-\Omega_0)e^{-j\pi/2}+X(\Omega+\Omega_0)e^{j\pi/2}\right]$$

Convolution property

$$x_1[n] \overset{\mathcal{F}}{\longleftrightarrow} X_1(\Omega) \quad \text{and} \quad x_2[n] \overset{\mathcal{F}}{\longleftrightarrow} X_2(\Omega)$$

$$\Rightarrow \quad x_1[n] * x_2[n] \overset{\mathcal{F}}{\longleftrightarrow} X_1(\Omega)X_2(\Omega)$$

Differentiation in the frequency domain

$$x[n] \overset{\mathcal{F}}{\longleftrightarrow} X(\Omega) \Rightarrow n^m x[n] \overset{\mathcal{F}}{\longleftrightarrow} j^m \frac{d^m X(\Omega)}{d\Omega^m}$$

- **Example 12:** Convolution using the DTFT

$$h[n]=(2/3)^n u[n] \text{ and } x[n]=(3/4)^n u[n]$$

Determine the convolution $y[n]=h[n]*x[n]$ using the DTFT

$$H(\Omega) = \frac{1}{1 - \frac{2}{3}e^{-j\Omega}}, \qquad X(\Omega) = \frac{1}{1 - \frac{3}{4}e^{-j\Omega}}$$

$$Y(\Omega) = H(\Omega)X(\Omega) = \frac{-8}{1 - \frac{2}{3}e^{-j\Omega}} + \frac{9}{1 - \frac{3}{4}e^{-j\Omega}} \qquad \Rightarrow y[n] = -8(2/3)^n u[n] + 9(3/4)^n u[n]$$

Multiplication of two signals

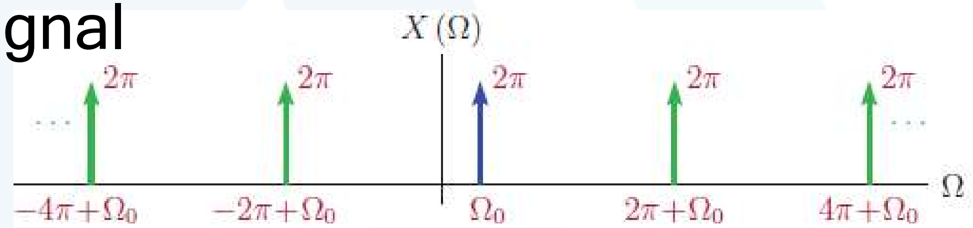
$x_1[n] \overset{\mathcal{F}}{\longleftrightarrow} X_1(\Omega)$ and $x_2[n] \overset{\mathcal{F}}{\longleftrightarrow} X_2(\Omega)$

$$\Rightarrow x_1[n]x_2[n] \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(\tau)X_2(\Omega - \tau)d\tau$$

Applying DTFT to periodic signals

- Example 13: DTFT of complex exponential signal

$$x[n] = e^{j\Omega_0 n}, \qquad -\pi < \Omega_0 < \pi$$



$$\mathcal{F}\{1\} = 2\pi\sum_{m=-\infty}^{\infty}\delta(\Omega - 2\pi m) \Rightarrow \mathcal{F}(e^{j\Omega_0 n}) = 2\pi\sum_{m=-\infty}^{\infty}\delta(\Omega - \Omega_0 - 2\pi m)$$

- Example 14: DTFT of sinusoidal signal

$$x[n] = \cos(\Omega_0 n), \quad -\pi < \Omega_0 < \pi$$

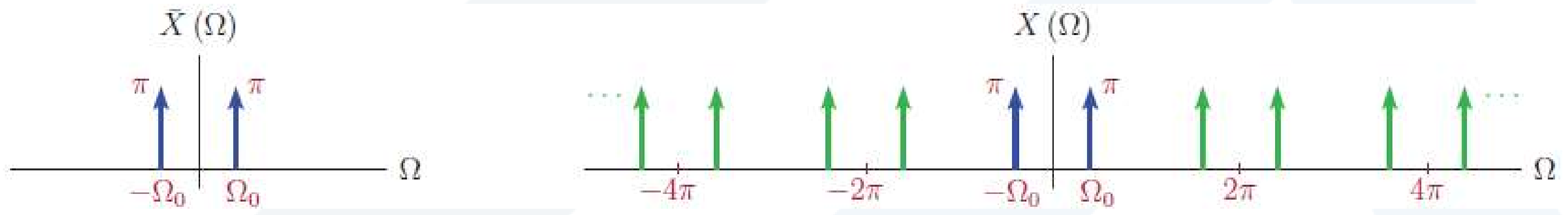
$$\mathcal{F}\{1\} = 2\pi\sum_{m=-\infty}^{\infty} \delta(\Omega - 2\pi m) \Rightarrow$$

$$\mathcal{F}\{\cos(\Omega_0 n)\} = \pi\sum_{m=-\infty}^{\infty} \delta(\Omega - \Omega_0 - 2\pi m) + \pi\sum_{m=-\infty}^{\infty} \delta(\Omega + \Omega_0 - 2\pi m)$$

Let $\bar{X}(\Omega)$ represent the part of the transform in the range $-\pi < \Omega_0 < \pi$.

$$\bar{X}(\Omega) = \pi\delta(\Omega - \Omega_0) + \pi\delta(\Omega + \Omega_0)$$

The DTFT can now be expressed as $X(\Omega) = \sum_{m=-\infty}^{\infty} \bar{X}(\Omega - 2\pi m)$



- In general for a periodic DT signal $\tilde{x}[n]$: $\tilde{x}[n] = \sum_{k=0}^{N-1} \tilde{c}_k e^{j(2\pi/N)kn}$

$$X(\Omega) = \sum_{n=-\infty}^{\infty} \tilde{x}[n] e^{-j\Omega n} = \sum_{n=-\infty}^{\infty} \left[\sum_{k=0}^{N-1} \tilde{c}_k e^{j(2\pi/N)kn} \right] e^{-j\Omega n} = \sum_{k=0}^{N-1} \tilde{c}_k \left[\sum_{n=-\infty}^{\infty} e^{j(2\pi/N)kn} e^{-j\Omega n} \right]$$

The expression in square brackets is the DTFT of the signal $e^{j(2\pi/N)kn}$

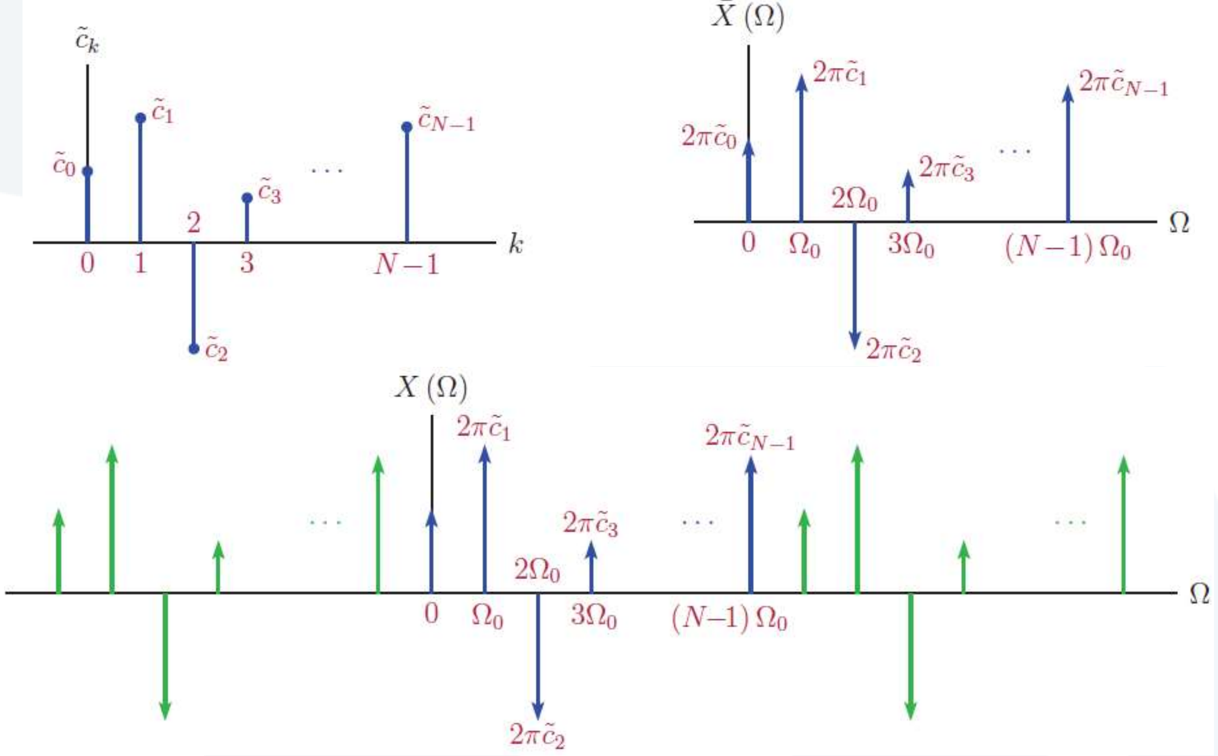
$2\pi/N = \Omega_0$ is the fundamental angular frequency for the periodic signal $\tilde{x}[n]$:

$$\sum_{n=-\infty}^{\infty} e^{j(2\pi/N)kn} e^{-j\Omega n} = 2\pi \sum_{m=-\infty}^{\infty} \delta\left(\Omega - \frac{2\pi k}{N} - 2\pi m\right) = 2\pi \sum_{m=-\infty}^{\infty} \delta(\Omega - k\Omega_0 - 2\pi m)$$

$$X(\Omega) = 2\pi \sum_{k=0}^{N-1} \tilde{c}_k \sum_{m=-\infty}^{\infty} \delta(\Omega - k\Omega_0 - 2\pi m)$$

The part of the transform in the range $0 < \Omega < 2\pi$ is found by setting $m = 0$

$$\bar{X}(\Omega) = 2\pi \sum_{k=0}^{N-1} \tilde{c}_k \delta(\Omega - k\Omega_0) \qquad X(\Omega) = \sum_{m=-\infty}^{\infty} \bar{X}(\Omega - 2\pi m)$$



- Example 15: DTFT of the unit step sequence $u[n]$

$$u[n] = x[n] + y[n], \quad x[n] = \frac{1}{2},\ y[n] = \begin{cases} \frac{1}{2}, & n \geq 0 \\ -\frac{1}{2}, & n < 0 \end{cases} \qquad X(\Omega) = \pi \sum_{k=-\infty}^{\infty} \delta(\Omega - 2\pi k)$$

$$y[n]-y[n-1]=\delta[n]\Rightarrow Y(\Omega)(1-e^{-j\Omega})=1\Rightarrow Y(\Omega)=\frac{1}{1-e^{-j\Omega}}$$

$$\mathcal{F}\{u[n]\}=\pi\sum_{k=-\infty}^{\infty}\delta(\Omega-2\pi k)+\frac{1}{1-e^{-j\Omega}}$$

- Example 16: DTFT of an Accumulator System

$$x[n]*u[n]=\sum_{k=-\infty}^{\infty}x[k]u[n-k]=\sum_{k=-\infty}^{n}x[k]$$

$$\mathcal{F}\left\{\sum_{k=-\infty}^{n}x[k]\right\}=\mathcal{F}\{x[n]\}\mathcal{F}\{u[n]\}=X(\Omega)\left(\pi\sum_{k=-\infty}^{\infty}\delta(\Omega-2\pi k)+\frac{1}{1-e^{-j\Omega}}\right)$$

Because of 2π periodicity, $X(0)=X(2\pi k)$. Moreover, $X(\Omega)\delta(\Omega-2\pi k)=X(2\pi k)\,\delta(\Omega-2\pi k)=X(0)\delta(\Omega-2\pi k)$. Hence,

$$\mathcal{F}\left\{\sum_{k=-\infty}^{n}x[k]\right\}=\pi X(0)\sum_{k=-\infty}^{\infty}\delta(\Omega-2\pi k)+\frac{1}{1-e^{-j\Omega}}X(\Omega)$$

Parseval's theorem

- For a periodic power signal $\tilde{x}[n]$ with period N and DTFS coefficients $\{\tilde{c}_k,\ k = 0, \ldots, N-1\}$:

$$P_x = \frac{1}{N}\sum_{n=0}^{N-1}\left|\tilde{x}[n]\right|^2 = \sum_{k=0}^{N-1}\left|\tilde{c}_k\right|^2$$

- For a non-periodic energy signal $x[n]$ with DTFT, $X(\Omega)$:

$$E_x = \sum_{n=-\infty}^{\infty}\left|x[n]\right|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|X(\Omega)\right|^2 d\Omega$$

- Example 17: DTFT of an Accumulator System

  Find the energy of $x[n] = \mathrm{sinc}(\Omega_c n)$, assuming $\Omega_c < \pi$

$$\mathcal{F}\{\mathrm{sinc}(\Omega_c n)\} = \frac{\pi}{\Omega_c}\mathrm{rect}\left(\frac{\Omega}{2\Omega_c}\right) \Rightarrow E_x = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\frac{\pi}{\Omega_c}\mathrm{rect}\left(\frac{\Omega}{2\Omega_c}\right)\right]^2 d\Omega = \frac{\pi}{\Omega_c}$$

3. Transfer Function Concept

- In time-domain analysis of systems two distinct descriptions for DTLTI systems:
  1. A linear constant-coefficient difference equation that describes the relationship between the input and the output signals.
  2. An impulse response which can be used with the convolution operation for determining the response of the system to an arbitrary input signal.
- The concept of Transfer function will be introduced as the third method for describing the characteristics of a system.

$$H(\Omega) = \mathcal{F}\{h[n]\} = \sum_{n=-\infty}^{\infty} h[n] e^{-j\Omega n}$$

- The transfer function concept is valid for LTI systems only.
- In general, $H(\Omega)$ is a complex function of Ω, $H(\Omega) = |H(\Omega)| e^{j\Theta(\omega)}$.

Obtaining the transfer function from the difference equation

$$y[n] = h[n] * x[n] \xleftrightarrow{\mathcal{F}} Y(\Omega) = H(\Omega)X(\Omega) \Rightarrow H(\Omega) = \frac{Y(\Omega)}{X(\Omega)}$$

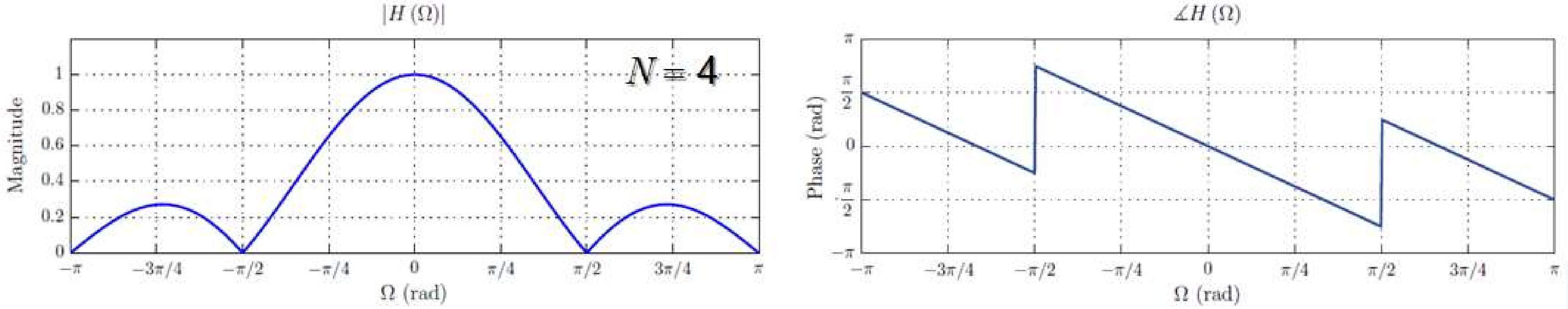
$$y[n-m] \xleftrightarrow{\mathcal{F}} e^{-j\Omega m}\, Y(\Omega),\ m = 0, 1, \cdots \qquad x[n-m] \xleftrightarrow{\mathcal{F}} e^{-j\Omega m}\, X(\Omega),\ m = 0, 1, \cdots$$

- Example 18: Transfer function for length-N moving average filter

$$y[n] = \frac{1}{N}\sum_{k=0}^{N-1} x[n-k]$$

$$Y(\Omega) = \frac{1}{N}\sum_{k=0}^{N-1} e^{-j\Omega k} X(\Omega) \Rightarrow H(\Omega) = \frac{1}{N}\sum_{k=0}^{N-1} e^{-j\Omega k} = \frac{1}{N}\frac{1-e^{-j\Omega N}}{1-e^{-j\Omega}}$$

$$H(\Omega) = \frac{\sin(\Omega N/2)}{N\sin(\Omega/2)} e^{j\Omega(N-1)/2}$$



4. DTLTI Systems with Periodic Input Signals

$$\tilde{x}[n] = \sum_{k=0}^{N-1} \tilde{c}_k e^{j(2\pi/N)kn}$$

Response of a DTLTI system to complex exponential signal

$$\tilde{x}[n] = e^{j\Omega_0 n}$$

$$y[n] = h[n] * \tilde{x}[n] = \sum_{k=-\infty}^{\infty} h[k]\tilde{x}[n-k] = \sum_{k=-\infty}^{\infty} h[k]e^{j\Omega_0(n-k)} = e^{j\Omega_0 n}\sum_{k=-\infty}^{\infty} h[k]e^{-j\Omega_0 k}$$

$$\tilde{y}[n] = e^{j\Omega_0 n}H(\Omega_0) = |H(\Omega_0)|\, e^{j[\Omega_0 n + \Theta(\Omega_0)]}$$

- That is, $e^{j\Omega t}$ is an eigenfunction of a LTI system for the eigenvalue $H(\Omega)$.

Response of a DTLTI system to sinusoidal signal

$$\tilde{x}[n] = \cos(\Omega_0 n)$$

$$\tilde{x}(t) = \cos(\Omega_0 n) = \tfrac{1}{2} e^{j\Omega_0 n} + \tfrac{1}{2} e^{-j\Omega_0 n}$$

$$\tilde{y}[n] = \tfrac{1}{2} e^{j\Omega_0 n} H(\Omega_0) + \tfrac{1}{2} e^{-j\Omega_0 n} H(-\Omega_0)$$

$$\tilde{y}[n] = \tfrac{1}{2} e^{j\Omega_0 n} \left|H(\Omega_0)\right| e^{j\Theta(\Omega_0)} + \tfrac{1}{2} e^{-j\Omega_0 n} \left|H(-\Omega_0)\right| e^{j\Theta(-\Omega_0)}$$

If the imp. response $h[n]$ is real-valued: $\left|H(-\Omega_0)\right| = \left|H(\Omega_0)\right|,\ \Theta(-\Omega_0) = -\Theta(\Omega_0)$

$$\tilde{y}[n] = \tfrac{1}{2} \left|H(\Omega_0)\right| e^{j[\Omega_0 n + \Theta(\Omega_0)]} + \tfrac{1}{2} \left|H(\Omega_0)\right| e^{-j[\Omega_0 n + \Theta(\Omega_0)]} = \left|H(\Omega_0)\right| \cos(\Omega_0 n + \Theta(\Omega_0))$$

- Example 19: Steady-state response of DTLTI system to sinusoidal input

$$y[n] - 0.9y[n-1] + 0.36y[n-2] = x[n] - 0.2x[n-1]$$

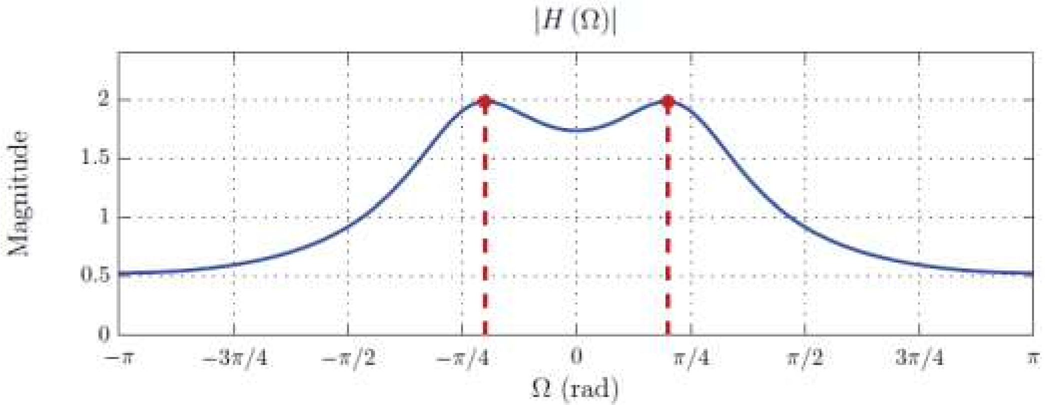
Find the response of the system to the sinusoidal input signal $\tilde{x}(t) = 5\cos(\frac{\pi n}{5})$

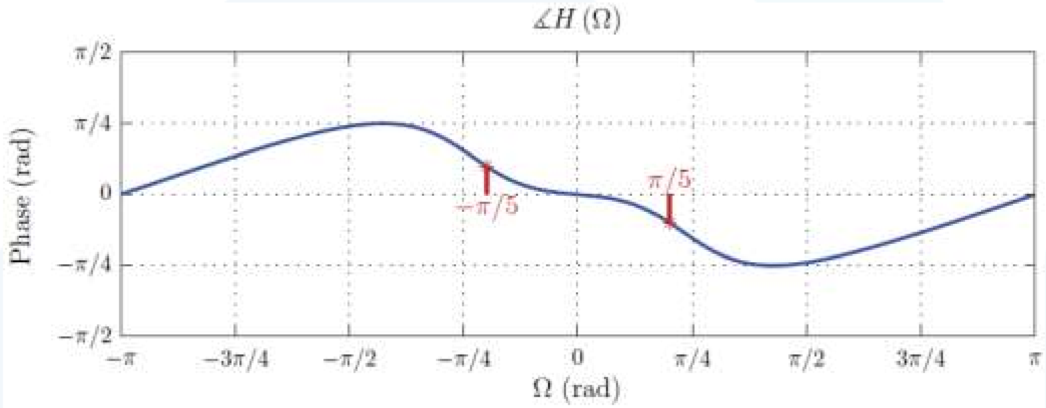
$$H(\Omega) = \frac{Y(\Omega)}{X(\Omega)} = \frac{1 - 0.2e^{-j\Omega}}{1 - 0.9e^{-j\Omega} + 0.36e^{-j2\Omega}}$$

$$H(\pi/5) = \frac{1 - 0.2e^{-j\pi/5}}{1 - 0.9e^{-j\pi/5} + 0.36e^{-j2\pi/5}} = 1.8890 - j0.6133 = 1.9861e^{-j0.3139}$$

The steady-state response of the system to the specified input signal $\tilde{x}[n]$ is:

$$\tilde{y}[n] = 5\left|H(\pi/5)\right|\cos(\pi n/5 + \Theta(\pi/5)) = 9.9305\cos(\pi n/5 - 0.3139)$$





Response of a DTLTI system to periodic input signal

$$\tilde{x}[n] = \sum_{k=0}^{N-1} \tilde{c}_k e^{j(2\pi/N)kn}$$

$$T\{\tilde{x}[n]\} = T\left\{\sum_{k=0}^{N-1} \tilde{c}_k e^{j(2\pi/N)kn}\right\} = \sum_{k=0}^{N-1} \tilde{c}_k T\left\{e^{j(2\pi/N)kn}\right\} T\{\tilde{x}[n]\} = \sum_{k=0}^{N-1} \tilde{c}_k H\left(\frac{2\pi k}{N}\right) e^{j(2\pi/N)kn}$$

5. DTLTI Systems with Non Periodic Input Signals

For a non-periodic signal $x[n]$ as input to a DTLTI system. The output of the system $y[n]$ is given by: $y[n] = h[n] * x[n]$

- Let us assume that The system is stable ensuring that $H(\Omega)$ converges, and the DTFT of the input signal also converges.

$$Y(\Omega) = H(\Omega)X(\Omega) \qquad |Y(\Omega)| = |H(\Omega)||X(\Omega)|, \quad \measuredangle Y(\Omega) = \measuredangle X(\Omega) + \Theta(\Omega)$$