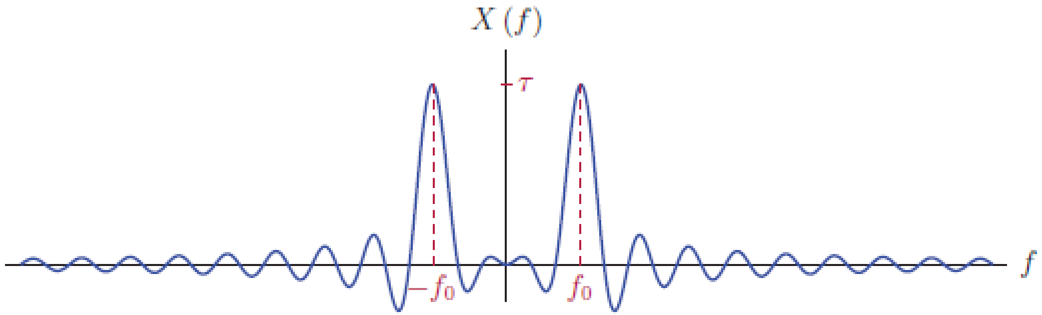
CEDC403: Signals and Systems

Lecture Notes 9 & 10: Laplace Transform for Continuous-Time Signals and Systems



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Chapter 7

Laplace Transform for Continuous-Time Signals and Systems

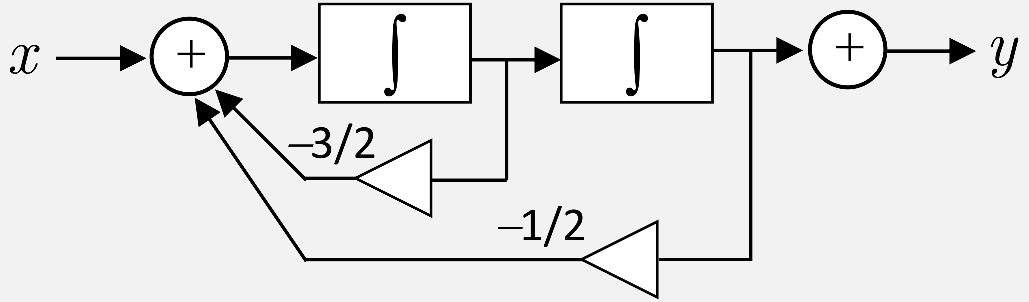
Pierre Simon Laplace
(1749-1827)

Born 23 March 1749 in Beaumont-en-Auge, Normandy, Kingdom of France

Died 05 March 1827 (aged 77) in Paris, Kingdom of France

Concept Map for CT Systems

Block Diagram



Impulse Response

$$h(t) = 2(e^{-t/2} - e^{-t})u(t)$$

Differential Equation

$$2\ddot{y}(t) + 3\dot{y}(t) + y(t) = 2x(t)$$

Transfer Function

$$\frac{Y(s)}{X(s)} = \frac{2}{2s^2 + 3s + 1}$$

Introduction

- The Laplace transform (LT) can be viewed as a generalization of the FT.
- The LT representation exists for some functions that do not have a Fourier transform representation. So, we can handle some functions with the LT that cannot be handled with the Laplace transform ($x(t) = tu(t)$).
- Certain characteristics of CT systems can only be studied via the Laplace transform. Such is the case of stability, transient and steady-state responses.

1. Laplace Transform

- The Laplace transform of a continuous-time signal $x(t)$ is defined as:

$$\mathcal{L}\{x(t)\} = X(s) = \int x(t)e^{-st}dt$$

where $s = \sigma + j\omega$, σ: damping factor, ω: frequency variable.

- There are two important variants:

Unilateral (or one-sided): $X(s) = \int_{0^-}^{\infty} x(t)e^{-st}dt;$

Bilateral (or two sided): $X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt;$

- The default Laplace transform is the bilateral Laplace transform.

Relationship Between LT and Continuous-Time FT

$$X(\sigma + j\omega) = \int_{-\infty}^{\infty} x(t)e^{-(\sigma+j\omega)t}dt = \int_{-\infty}^{\infty} [x(t)e^{-\sigma t}]e^{-j\omega t}dt = \mathcal{F}\{e^{-\sigma t}x(t)\}$$

$$X(j\omega) = \left[\int_{-\infty}^{\infty} x(t)e^{-st}dt\right]\Bigg|_{s=j\omega} = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \mathcal{F}\{x(t)\}$$

- Example 1: Laplace transform of the unit impulse

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt = \int_{-\infty}^{\infty} \delta(t)e^{-st}dt = 1$$

- Example 2: Laplace transform of the unit-step signal

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt = \int_{-\infty}^{\infty} u(t)e^{-st}dt = \int_{0}^{\infty} e^{-st}dt = \frac{1}{s}, \qquad \mathrm{Re}\{s\} > 0$$

Regions of Convergence

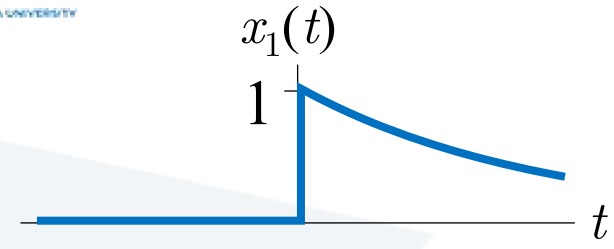
- The region of convergence (ROC), for the Laplace transform, $X(s)$, is the set of values of s in the complex plane for which the integral converges.
- For the LT $X(s)$ to exist: $\left|\int_{-\infty}^{\infty} x(t)e^{-st}dt\right| = \left|\int_{-\infty}^{\infty} x(t)e^{-\sigma t}e^{-j\omega t}dt\right| \le \int_{-\infty}^{\infty} \left|x(t)e^{-\sigma t}\right| dt < \infty$
- $x(t)e^{-\sigma t}$ must be absolutely integrable. This may be possible by choosing an appropriate σ. The value chosen for σ determines the ROC of $X(s)$.
- Note: The frequency does not affect the ROC.

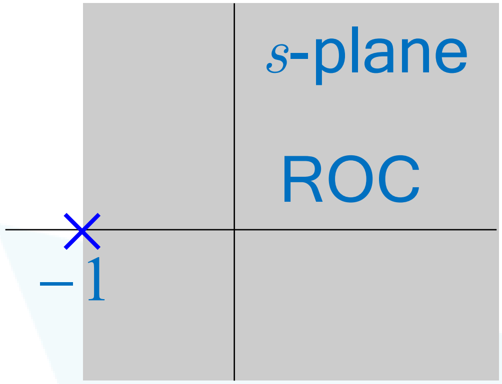
Poles and Zeros and the Region of Convergence

- Typically, $X(s)$ is rational, a ratio of two polynomials $X(s) = N(s)/D(s)$. The roots of $N(s)$ are called zeros, and the roots of $D(s)$ are called poles.
- If $\{\sigma_i\}$ are the real parts of the poles of $X(s)$, the ROC corresponding to different types of signals is determined from its poles as follows:
- For a causal signal $x(t)$, the region of convergence of its Laplace transform $X(s)$ is a plane to the right of the poles, $R_c = \{(\sigma, \omega): \sigma > \max\{\sigma_i\}, -\infty < \omega < \infty\}$
- For a anticausal signal $x(t)$, the ROC of its Laplace transform $X(s)$ is a plane to the left of the poles, $R_{ac} = \{(\sigma, \omega): \sigma < \min\{\sigma_i\}, -\infty < \omega < \infty\}$
- For a noncausal signal $x(t)$, the ROC of its LT $X(s)$ is the intersection of the ROC of the causal component R_c and the anticausal component R_{ac}, $R_c \bigcap R_{ac}$

- Example 3: Find the LT of $x_1(t)$

$$x_1(t) = \begin{cases} e^{-t} & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

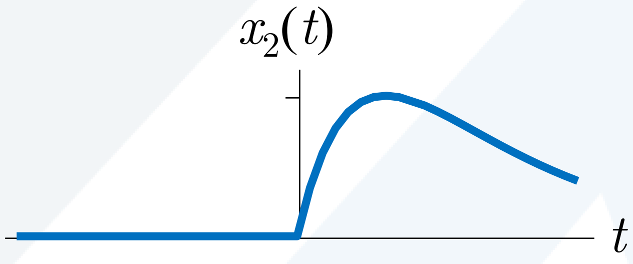


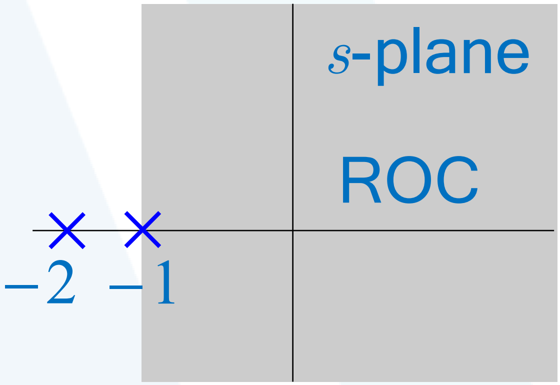


$$X_1(s) = \int_0^{\infty} e^{-t} e^{-st} dt = \left. \frac{e^{-(s+1)t}}{-(s+1)} \right|_0^{\infty} = \frac{1}{s+1}, \quad \mathrm{Re}\{s\} > -1$$

- Example 4: Find the LT of $x_2(t)$

$$x_2(t) = \begin{cases} e^{-t} - e^{-2t} & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

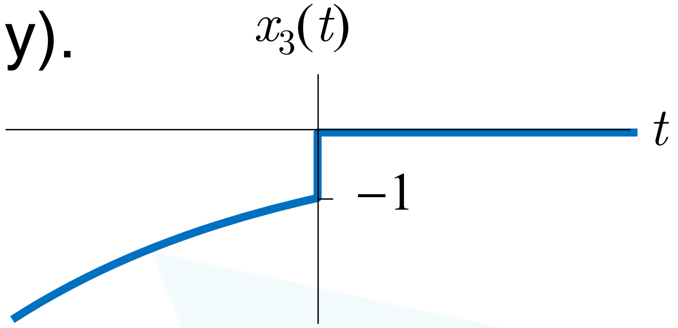




$$X_2(s) = \int_0^{\infty} (e^{-t} - e^{-2t}) e^{-st} dt = \frac{1}{(s+1)(s+2)}$$

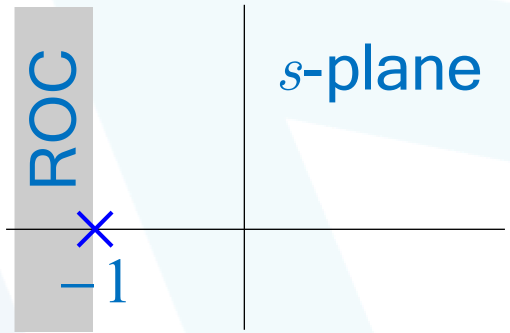
Re$\{s + 1\} > 0$ & Re$\{s + 2\} > 0 \Rightarrow$ Re$\{s\} > -1$.

- Note: Left-sided signals have left-sided LT (bilateral only).
- Example 5: LT of an anti-causal exponential signal



$$x_3(t) = \begin{cases} -e^{-t} & \text{if } t \le 0 \\ 0 & \text{otherwise} \end{cases}$$

$$X_3(s) = \int_{-\infty}^{0} -e^{-t} e^{-st} dt = \left. \frac{-e^{-(s+1)t}}{-(s+1)} \right|_{-\infty}^{0} = \frac{1}{s+1}, \quad \mathrm{Re}\{s\} < -1$$



Left and Right Sided ROCs

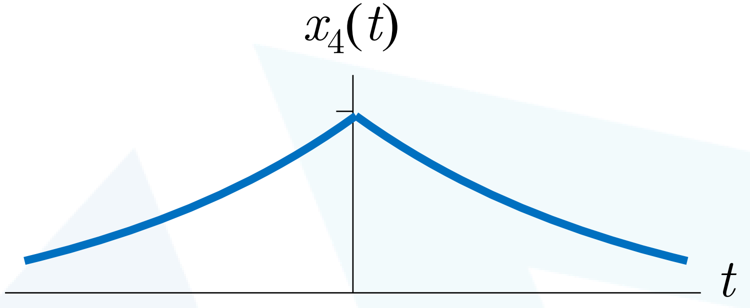
- It is possible for two different signals to have the same transform expression for $X(s)$.

$$\mathcal{L}\{e^{-t}u(t)\} = 1/(s+1), \quad \text{ROC: } \mathrm{Re}\{s\} > -1$$

$$\mathcal{L}\{-e^{-t}u(-t)\} = 1/(s+1), \quad \text{ROC: } \mathrm{Re}\{s\} < -1$$

In order for us to uniquely identify which signal among the two led to a particular transform, the ROC must be specified along with the transform.

- Example 6: Find the Laplace transform of $x_4(t)$



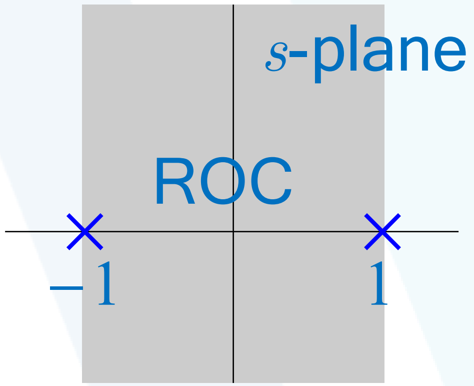
$$x_4(t) = e^{-|t|}$$

$$X_4(s) = \int_{-\infty}^{\infty} e^{-|t|} e^{-st} dt = \int_{-\infty}^{0} e^{(1-s)t} dt + \int_{0}^{\infty} e^{(1+s)t} dt$$

$$= \left.\frac{e^{(1-s)t}}{(1-s)}\right|_{-\infty}^{0} + \left.\frac{-e^{(1+s)t}}{-(1+s)}\right|_{0}^{\infty} = \underbrace{\frac{1}{1-s}}_{\mathrm{Re}<1} + \underbrace{\frac{1}{1+s}}_{\mathrm{Re}>-1} = \frac{2}{1-s^2}$$

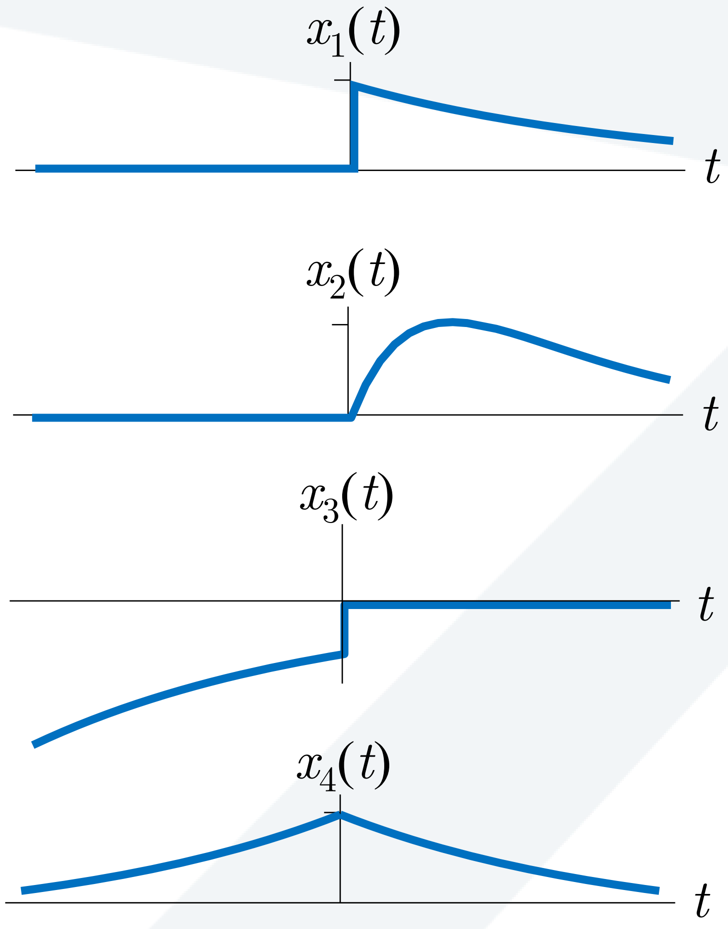
converge if: $\mathrm{Re}\{s\} < 1$ and $\mathrm{Re}\{s\} > -1$

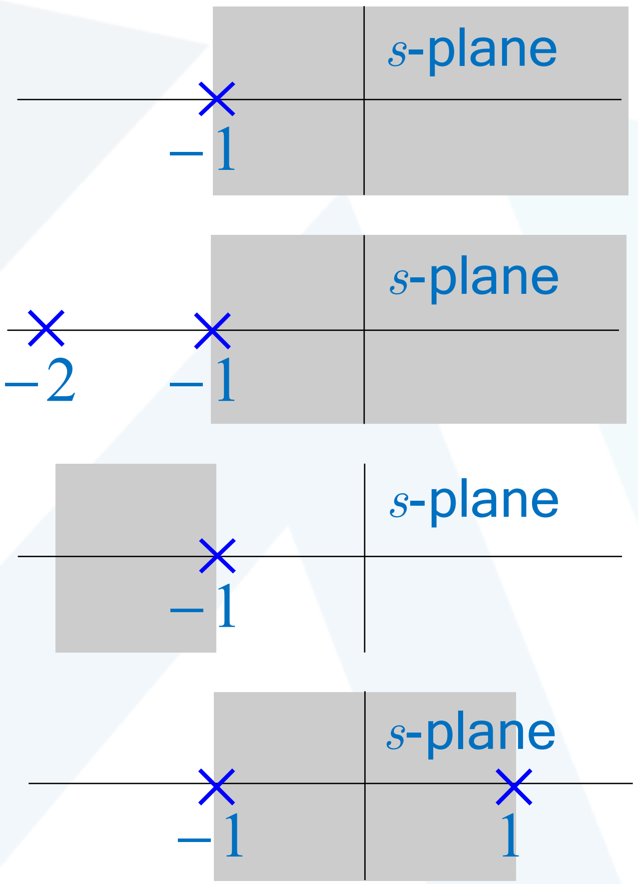
ROC: $-1 < \mathrm{Re}\{s\} < 1$



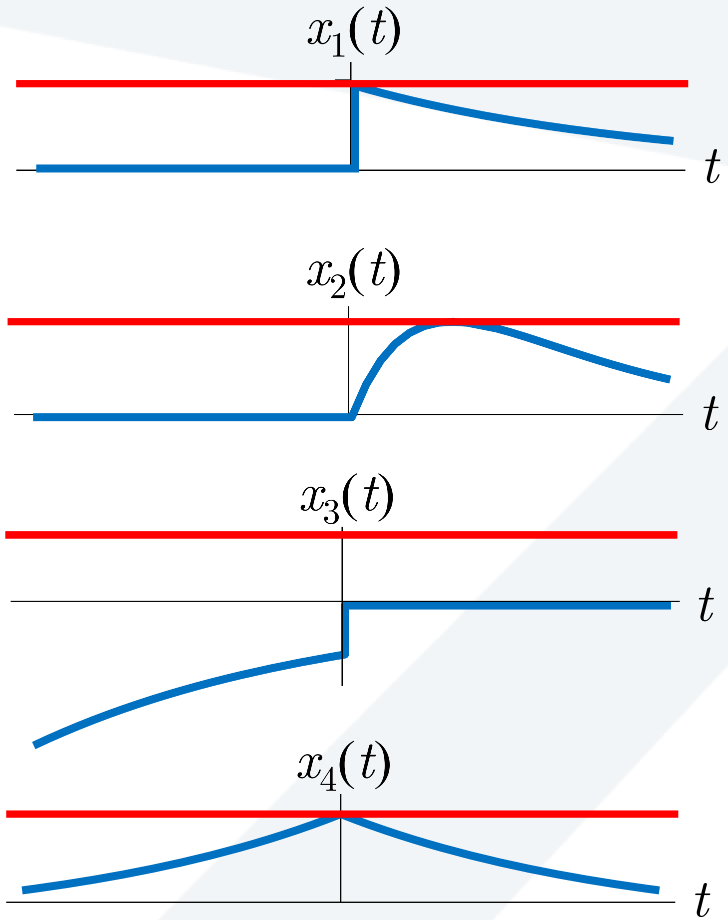
Time-Domain Interpretation of ROC

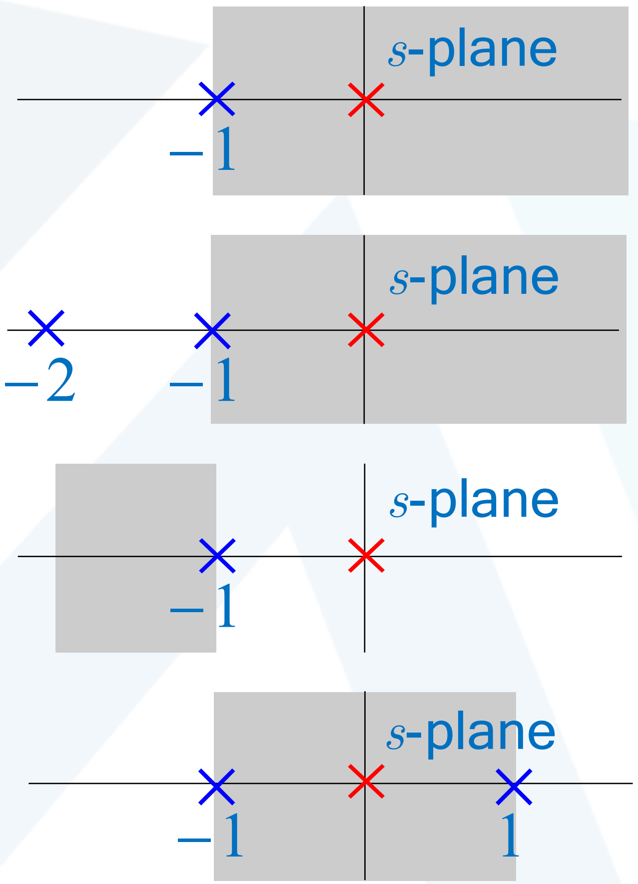
$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$



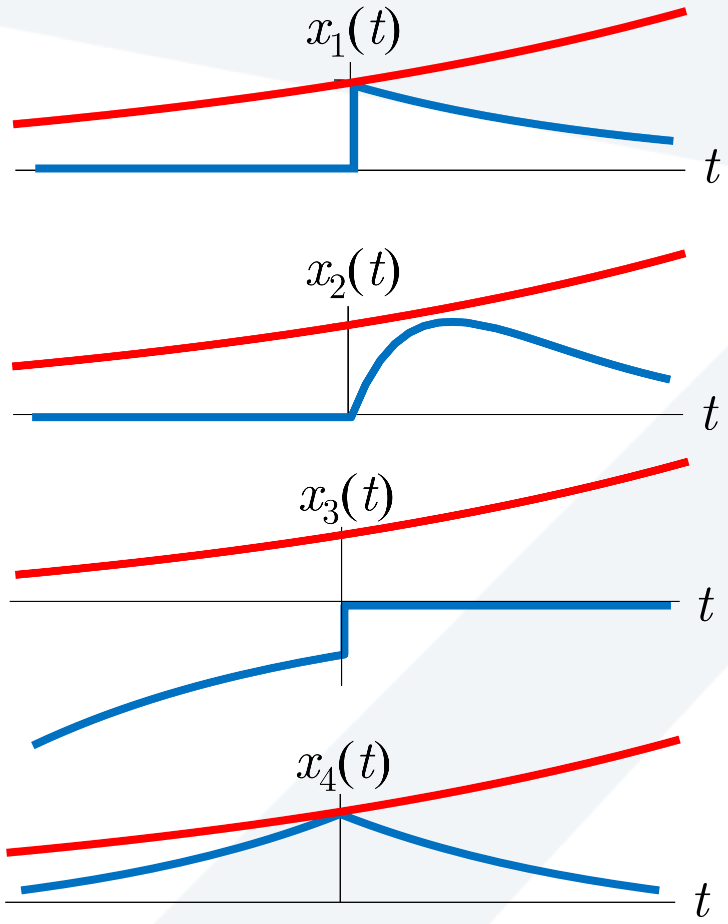


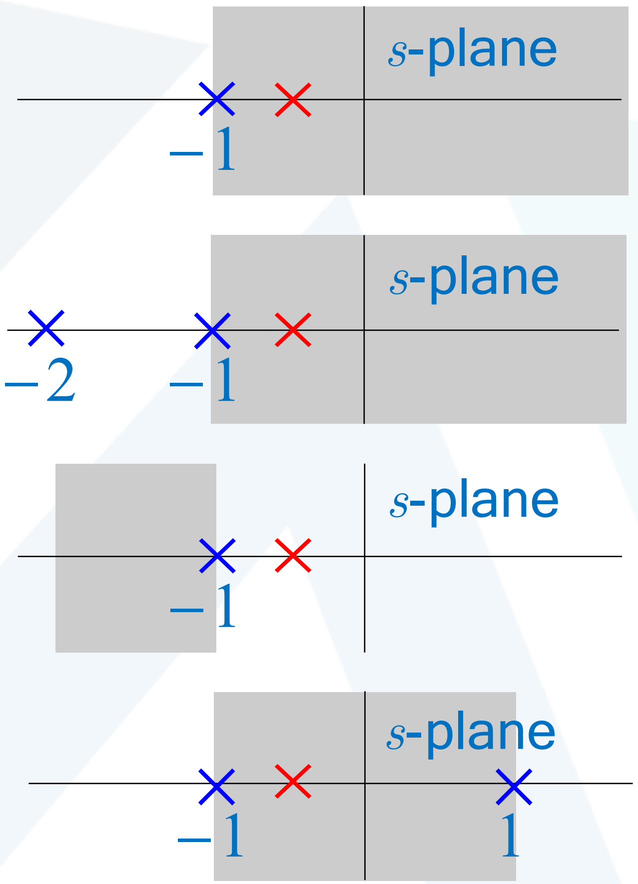
$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$



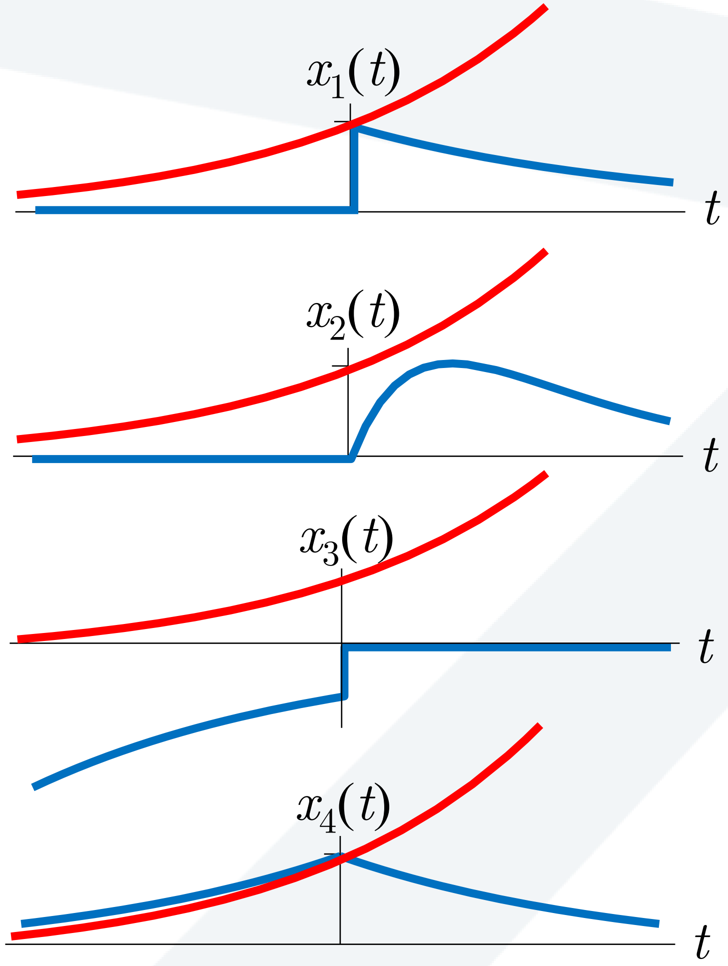


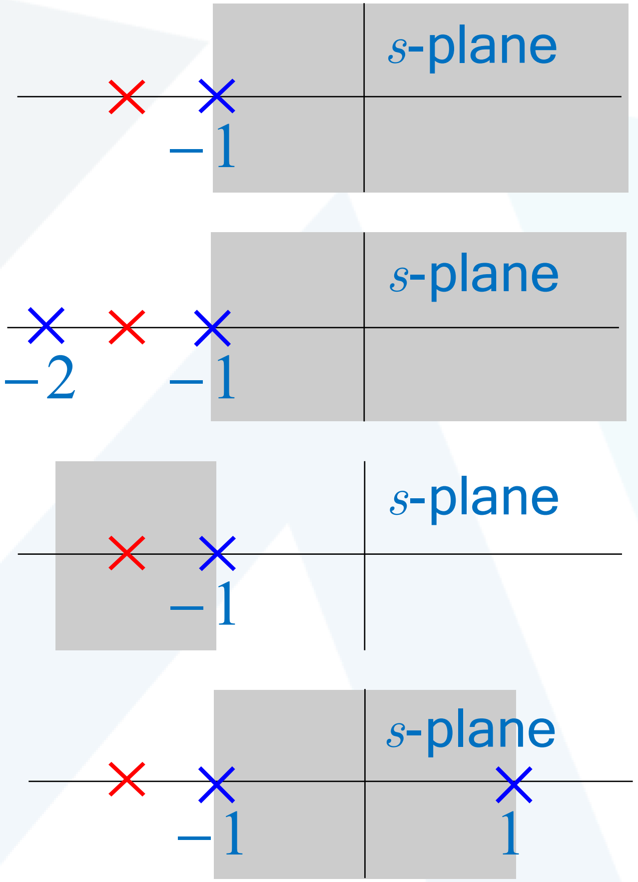
$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$



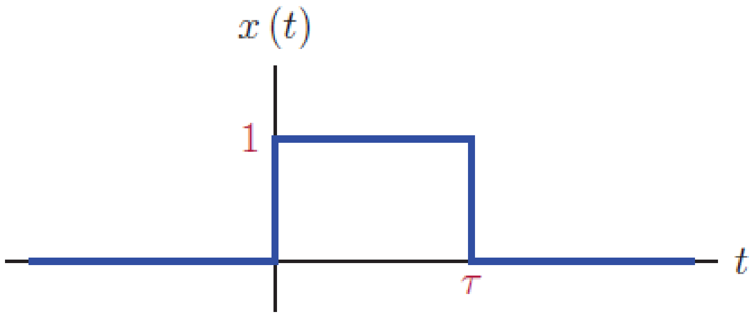


$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$





- Example 7: LT of a pulse signal $x(t) = \Pi\left(\dfrac{t - \tau/2}{\tau}\right)$



$$X(s) = \int_0^{\tau} (1)e^{-st}dt = \left.\frac{e^{-st}}{-s}\right|_0^{\tau} = \frac{1 - e^{-s\tau}}{s}$$

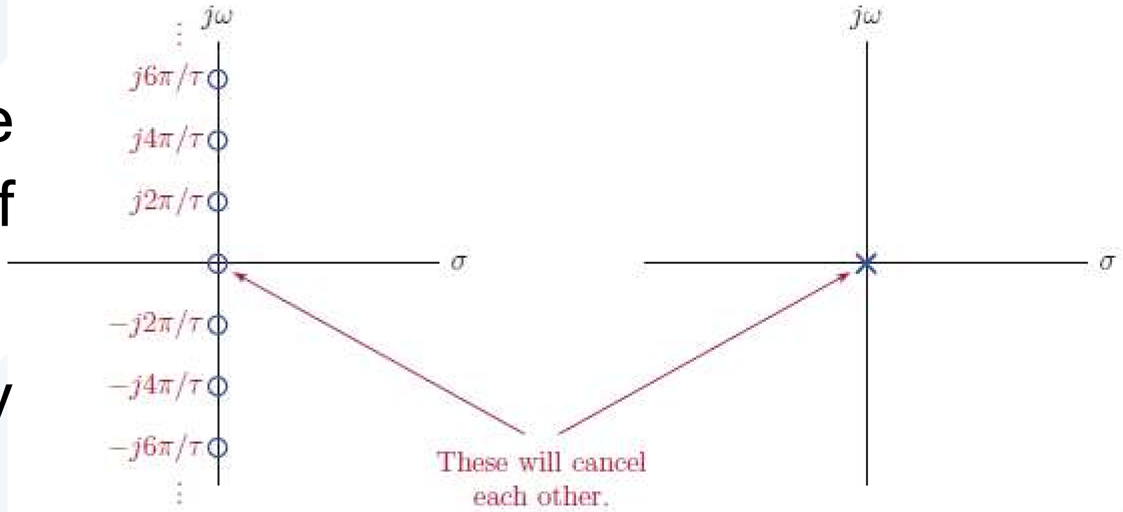
$$X(s) = \frac{1 - e^{-s\tau}}{s} = \tau - \frac{s\tau^2}{2!} + \frac{s^2\tau^3}{3!} - \cdots$$

$$\lim_{s\to 0} X(s) = \tau \Rightarrow X(s) \text{ converge at } s = 0$$

ROC is $\mathrm{Re}\{s\} > -\infty$

The numerator of $X(s)$ has an infinite number of roots, all on the $j\omega$-axis of the s-plane: $z_k = j2\pi k/\tau$, k integer.

The denominator of $X(s)$ has only one root at $s = 0$.



Properties of Laplace Transform

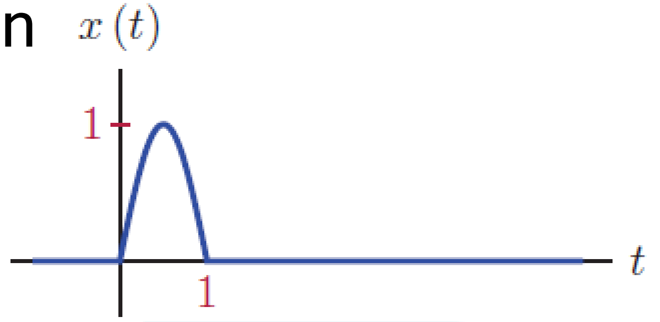
Property	$x(t)$	$X(s)$	ROC
Linearity	$ax_1(t) + bx_2(t)$	$aX_1(s) + bX_2(s)$	$\supset (R_1 \cap R_2)$
Delay by T	$x(t - T)$	$X(s)e^{-sT}$	R
Multiply by t	$tx(t)$	$-dX(s)/ds$	R
Multiply by $e^{-\alpha t}$	$x(t)e^{-\alpha t}$	$X(s + \alpha)$	Shift R by $-\alpha$
Scaling in t	$x(at)$	$\frac{1}{\|a\|}X(\frac{s}{a})$	aR
Differentiate in t	$dx(t)/dt$	$sX(s)$	$\supset R$
Integrate in t	$\int_{-\infty}^{t} x(\tau)d\tau$	$\frac{X(s)}{s}$	$\supset (R \cap (\mathrm{Re}(s) > 0))$
Convolve in t	$x_1(t) * x_2(t)$	$X_1(s)\ X_2(s)$	$\supset (R_1 \cap R_2)$

Laplace Transform Pairs

1	$\delta(t)$	1	All s
2	$u(t)$	$1/s$	$\text{Re}\{s\} > 0$
3	$-u(-t)$	$1/s$	$\text{Re}\{s\} < 0$
4	$t^n u(t)$	$\dfrac{n!}{s^{n+1}}$	$\text{Re}\{s\} > 0$
5	$-t^n u(-t)$	$\dfrac{n!}{s^{n+1}}$	$\text{Re}\{s\} < 0$
6	$e^{-at}u(t)$	$\dfrac{1}{s+a}$	$\text{Re}\{s\} > -a$
7	$-e^{-at}u(-t)$	$\dfrac{1}{s+a}$	$\text{Re}\{s\} < -a$

8	$t^n e^{-at} u(t)$	$\dfrac{n!}{(s+a)^{n+1}}$	$\text{Re}\{s\} > -a$
9	$-t^n e^{-at} u(-t)$	$\dfrac{n!}{(s+a)^{n+1}}$	$\text{Re}\{s\} < -a$
10	$[\cos\omega_0 t] u(t)$	$\dfrac{s}{s^2+\omega_0^2}$	$\text{Re}\{s\} > 0$
11	$[\sin\omega_0 t] u(t)$	$\dfrac{\omega_0}{s^2+\omega_0^2}$	$\text{Re}\{s\} > 0$
12	$[e^{-at}\cos\omega_0 t] u(t)$	$\dfrac{s+a}{(s+a)^2+\omega_0^2}$	$\text{Re}\{s\} > -a$
13	$[e^{-at}\sin\omega_0 t] u(t)$	$\dfrac{\omega_0}{(s+a)^2+\omega_0^2}$	$\text{Re}\{s\} > -a$

- Example 8: Laplace transform of a truncated sine function



$$x(t) = \begin{cases} \sin(\pi t), & 0 < t < 1 \\ 0, & \text{otherwise} \end{cases}$$

$$X(s) = \int_0^1 \sin(\pi t) e^{-st} dt = \frac{1}{2j}\int_0^1 (e^{j\pi t} - e^{-j\pi t}) e^{-st} dt = \frac{\pi(1+e^{-s})}{s^2+\pi^2}$$

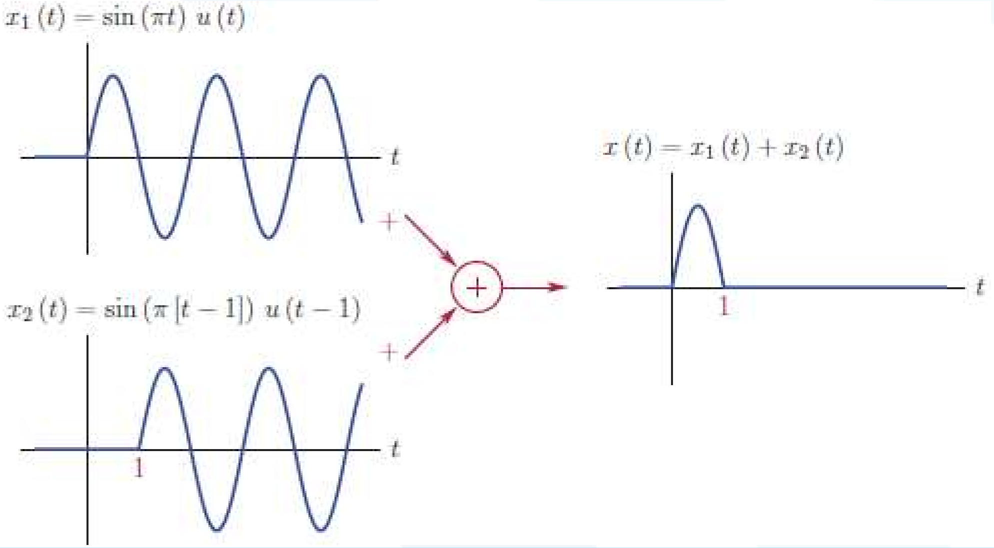
Another method

$$x(t) = \sin(\pi t)u(t) + \sin(\pi[t-1])u(t-1)$$

$$X(s) = \frac{\pi}{s^2+\pi^2} + \frac{\pi}{s^2+\pi^2}e^{-s} = \frac{\pi(1+e^{-s})}{s^2+\pi^2}$$

ROC: entire s-plane except points where

$\mathrm{Re}\{s\} \to -\infty$



- Example 9: Using the convolution property of the Laplace transform

$$x_1(t) = e^{-t}u(t), \quad x_2(t) = \delta(t) - e^{-2t}u(t)$$

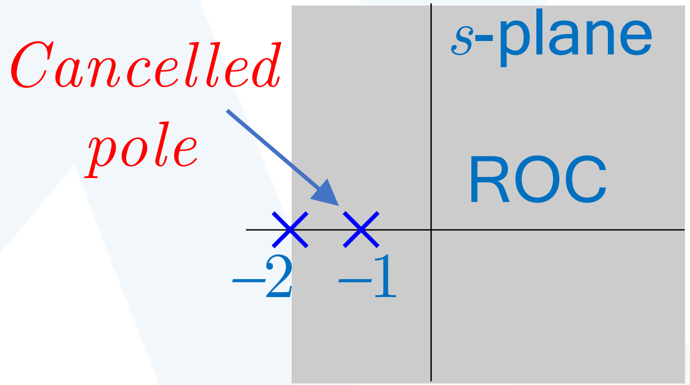
Determine $x(t) = x_1(t) * x_2(t)$ using Laplace transform techniques.

$$X_1(s) = \frac{1}{s+1}, \qquad \text{ROC: Re}\{s\} > -1$$

$$X_2(s) = 1 - \frac{1}{s+2} = \frac{s+1}{s+2}, \qquad \text{ROC: Re}\{s\} > -2$$

$$X(s) = X_1(s)X_2(s) = \frac{1}{s+2}, \qquad \text{ROC: Re}\{s\} > -2$$

$$x(t) = \mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} = e^{-2t}u(t)$$



Initial Value Theorem

For a function x with Laplace transform X, if x is causal and contains no impulses or higher order singularities at the origin, then:

$$x(0^+) = \lim_{s\to\infty} sX(s)$$

- When X is known but x is not, the initial value theorem eliminates the need to explicitly find x in order to evaluate $x(0^+)$.

- Example 10: Calculate the initial value of the function $x(t)$, whose LT is:

$$X(s) = \frac{2(s+1)}{(s+1)^2 + 5^2}$$

$$x(0^+) = \lim_{t\to 0^+} x(t) = \lim_{s\to\infty} sX(s) = \lim_{s\to\infty} \frac{2s(s+1)}{(s+1)^2 + 5^2} = 2$$

Verification: $x(t) = 2e^{-t}\cos(5t)\,u(t)$

Final Value Theorem

For a function x with Laplace transform X, if x is causal and $x(t)$ has a finite limit as $t \to \infty$, then:

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)$$

- When X is known but x is not, the final value theorem eliminates the need to explicitly find x in order to evaluate limit $\lim_{t\to\infty} x(t)$.

- Example 11: Calculate the final value of the function $x(t)$, whose Laplace transform is:

$$X(s) = \frac{s+2}{s(s+1)}$$

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s) = \lim_{s\to 0} \frac{s(s+2)}{s(s+1)} = \lim_{s\to 0} \frac{s+2}{s+1} = 2$$

Verification: $x(t) = (2 - e^{-t})\,u(t)$

Inverse Laplace Transform

The inverse LT x of X is given by $L^{-1}\{X(s)\} = x(t) = \dfrac{1}{2\pi j}\displaystyle\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}ds,$

where Re$(s) = \sigma$ is in the ROC of X.

- We do not usually compute the inverse Laplace transform directly using the above equation.
- For rational functions, the inverse Laplace transform can be more easily computed using partial fraction expansions (PFE).

- Example 12: Calculate the inverse LT of the function $H(s) = 1/(s + a)$

$h(t) = e^{-at}u(t)$ with ROC: Re$\{s\} > -a$

$h(t) = -e^{-at}u(-t)$ with ROC: Re$\{s\} < -a$

- **Example 13:** Using PFE with complex poles

The Laplace transform of a signal $x(t)$ is $X(s) = \dfrac{s+1}{s(s^2+9)}$

with the ROC specified as Re $\{s\} > 0$. Determine $x(t)$.

$$X(s) = \frac{k_1}{s} + \frac{k_2}{s+j3} + \frac{k_3}{s-j3} \Rightarrow k_1 = \frac{1}{9}, \quad k_2 = -\frac{1}{18} + j\frac{1}{6}, \quad k_3 = \frac{1}{18} - j\frac{1}{6}$$

$$x(t) = \tfrac{1}{9}u(t) - \tfrac{1}{18}[e^{-j3t} + e^{j3t}]u(t) + j\tfrac{1}{6}[e^{-j3t} - e^{j3t}]u(t) \quad (x(t) \text{ is causal})$$

$$x(t) = \tfrac{1}{9}u(t) - \tfrac{1}{9}\cos(3t)u(t) + \tfrac{1}{3}\sin(3t)u(t)$$

- **Example 14:** Multiple-order poles

A causal signal $x(t)$ has the Laplace transform $X(s) = \dfrac{s(s+1)}{(s+1)^3(s+2)}$

Determine $x(t)$ using partial fraction expansion.

$$X(s)=\frac{s(s+1)}{(s+1)^3(s+2)}=\frac{-3}{s+1}+\frac{3}{(s+1)^2}-\frac{2}{(s+1)^3}+\frac{3}{s+2}$$

$$\mathcal{L}\{e^{-t}u(t)\}=\frac{1}{s+1}, \quad \mathcal{L}\{te^{-t}u(t)\}=-\frac{d}{ds}\frac{1}{s+1}=\frac{1}{(s+1)^2}$$

$$\mathcal{L}\{t^2e^{-t}u(t)\}=-\frac{d}{ds}\frac{1}{(s+1)^2}=\frac{2}{(s+1)^3}$$

$$x(t)=-3e^{-t}u(t)+3te^{-t}u(t)-t^2e^{-t}u(t)+3e^{-2t}u(t)$$

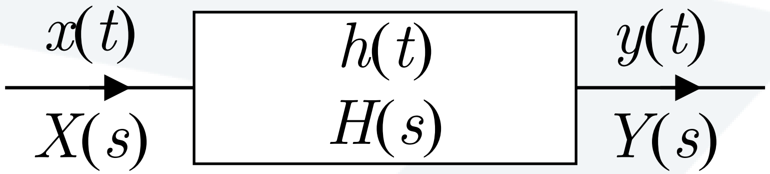
- Example 15: PFE when numerator order is not less than denominator order

The LT of a causal signal is $X(s)=\dfrac{s(s+1)}{(s+2)(s+3)}$, Determine $x(t)$.

$$X(s)=\frac{s(s+1)}{(s+2)(s+3)}=1+\frac{2}{s+2}-\frac{6}{s+3}\Rightarrow x(t)=\delta(t)+2e^{-2t}u(t)-6e^{-3t}u(t)$$

2. Using the Laplace Transform with CTLTI Systems

Transfer Function and LTI Systems



Zero-State Response

- Since $y(t) = x(t) * h(t)$, the system is characterized in the Laplace domain by $Y(s) = X(s)H(s)$.
- $H(s)$ is the transfer function (or system function) of the system.
- A LTI system is completely characterized by its transfer function H.

Relating the transfer function to the differential equation

- Many LTI systems of practical interest can be represented using an Nth-order linear differential equation with constant coefficients.

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$ where the a_k and b_k are complex constants and

$$\mathcal{L}\left\{\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k}\right\} = \mathcal{L}\left\{\sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}\right\} \Rightarrow \sum_{k=0}^{N} \mathcal{L}\left\{a_k \frac{d^k y(t)}{dt^k}\right\} = \sum_{k=0}^{M} \mathcal{L}\left\{b_k \frac{d^k x(t)}{dt^k}\right\}$$

$$\sum_{k=0}^{N} a_k \mathcal{L}\left\{\frac{d^k y(t)}{dt^k}\right\} = \sum_{k=0}^{M} b_k \mathcal{L}\left\{\frac{d^k x(t)}{dt^k}\right\}$$

$$\sum_{k=0}^{N} a_k s^k Y(s) = \sum_{k=0}^{M} b_k s^k X(s) \Rightarrow H(s) = \frac{Y(s)}{X(s)} = \frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k}$$

- The transfer function is always rational.
- The impulse response of the system $h(t) = \mathcal{L}^{-1}\{H(s)\}$.

- The transfer function concept is meaningful only for LTI systems that are both linear and time invariant.
- In determining the transfer function from the differential equation, all initial conditions must be assumed to be zero.
- Example 17: Finding the DE from input and output signals

  The unit-step response of a CTLTI system is $y(t) = T\{u(t)\} = (2 - 4e^{-t} + 2e^{-2t})\,u(t)$

  Find a differential equation for this system.

$$X(s) = \frac{1}{s}, \quad Y(s) = \frac{2}{s} - \frac{4}{s+1} + \frac{2}{s+2} = \frac{4}{s(s+1)(s+2)}$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{4}{(s+1)(s+2)} = \frac{4}{s^2 + 3s + 2} \Rightarrow \frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = 4x(t)$$

Response of a CTLTI system to a complex exponential signal $x(t) = e^{s_0 t}$

where s_0 represents a point in the s-plane within the ROC of the transfer function:

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\,d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau)e^{s_0(t-\tau)}d\tau = e^{s_0 t}\int_{-\infty}^{\infty} h(\tau)e^{-s_0\tau}d\tau = e^{s_0 t}H(s_0)$$

- Complex exponential are eigenfunctions of LTI systems.
- Example 18: Response to a complex exponential signal

  A CTLTI system is defined by the transfer function: $H(s) = \dfrac{s+6}{s^2+5s+6}$

  is driven by the complex exponential input signal $x(t) = e^{(-0.4+j4)t}$

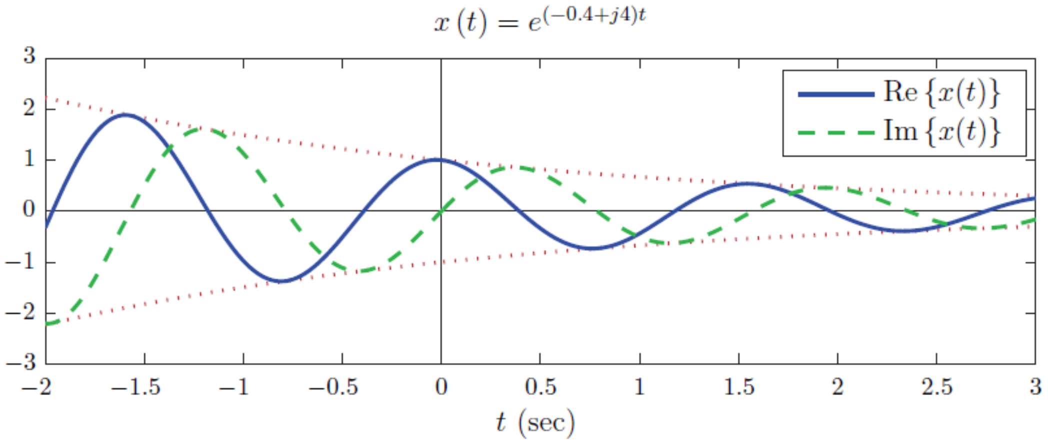
  Determine the steady-state response of the system.

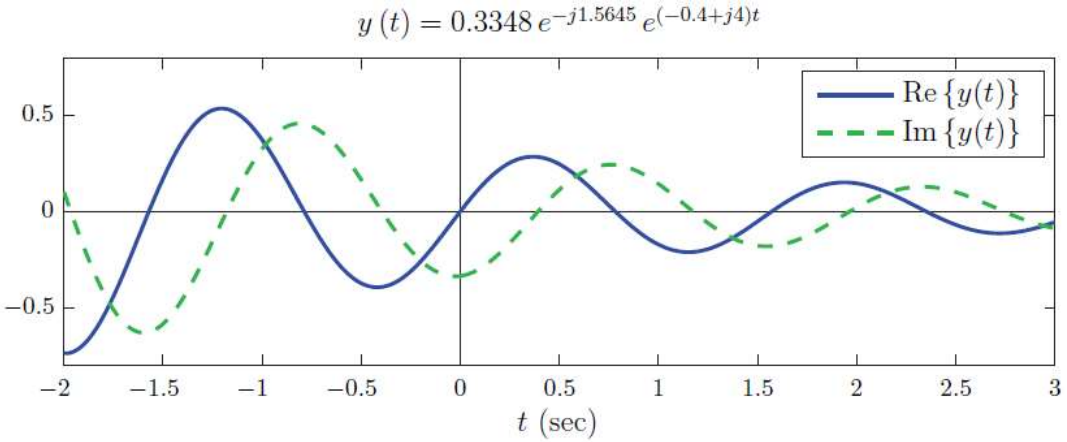
$$x(t) = e^{-0.4t}\left[\cos(4t) + j\sin(4t)\right]$$

$$H(-0.4+j4) = \frac{(-0.4+j4)+6}{(-0.4+j4)^2 + 5(-0.4+j4)+6} = 0.0021 - j0.3348 = 0.3348e^{-j1.5645}$$

$$y(t) = e^{(-0.4+j4)t}\left(0.3348e^{-j1.5645}\right) = 0.3348e^{-0.4t}e^{j(4t-1.5645)}$$

$$y(t) = 0.3348e^{-0.4t}\cos(4t-1.5645) + j0.3348e^{-0.4t}\sin(4t-1.5645)$$





Response of a CTLTI system to an exponential damped sinusoid $x(t) = e^{\sigma_0 t}\cos(\omega_0 t)$

$$x(t) = \tfrac{1}{2}e^{\sigma_0 t}e^{j\omega_0 t} + \tfrac{1}{2}e^{\sigma_0 t}e^{-j\omega_0 t}$$

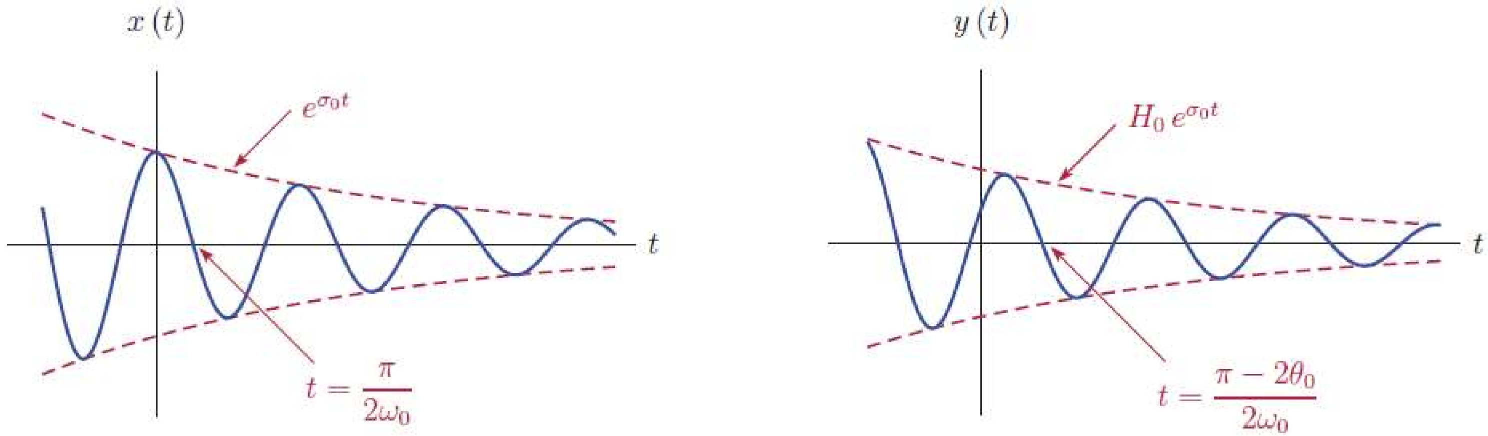
Let $s_0 = \sigma_0 + j\omega_0 \Rightarrow x(t) = \tfrac{1}{2}e^{s_0 t} + \tfrac{1}{2}e^{s_0^* t}$

$$y(t) = T\{\tfrac{1}{2}e^{s_0 t} + \tfrac{1}{2}e^{s_0^* t}\} = \tfrac{1}{2}T\{e^{s_0 t}\} + \tfrac{1}{2}T\{e^{s_0^* t}\} = \tfrac{1}{2}e^{s_0 t}H(s_0) + \tfrac{1}{2}e^{s_0^* t}H(s_0^*)$$

$H(s_0) = H_0 e^{j\theta_0}, \quad H(s_0^*) = H_0 e^{-j\theta_0}$ (for a real-valued impulse response)

$$y(t) = \tfrac{1}{2}e^{s_0 t}H_0 e^{j\theta_0} + \tfrac{1}{2}e^{s_0^* t}H_0 e^{-j\theta_0} = \tfrac{1}{2}e^{\sigma_0 t}e^{j\omega_0 t}H_0 e^{j\theta_0} + \tfrac{1}{2}e^{\sigma_0 t}e^{-j\omega_0 t}H_0 e^{-j\theta_0}$$

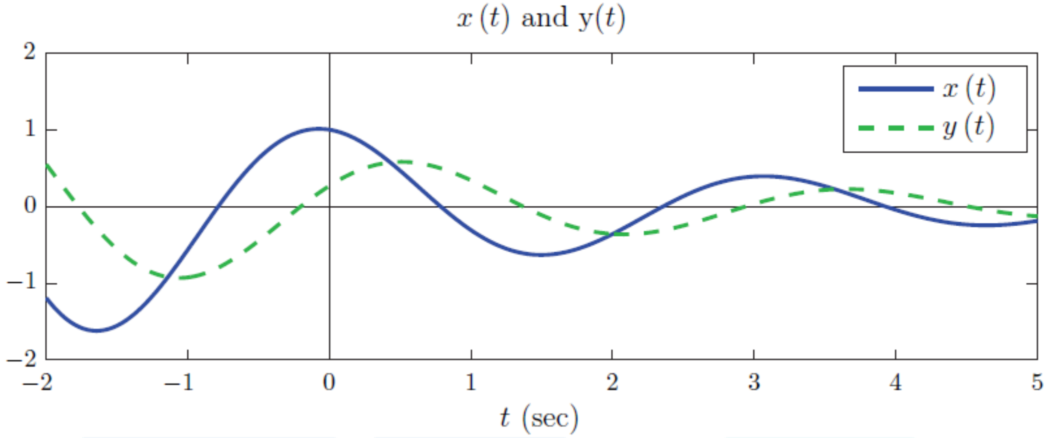
$$y(t) = \tfrac{1}{2}H_0 e^{\sigma_0 t}[e^{j(\omega_0 t+\theta_0)} + e^{-j(\omega_0 t+\theta_0)}] = H_0 e^{\sigma_0 t}\cos(\omega_0 t + \theta_0)$$



- Example 19: Determine the response of the system in Example 18 to the input signal $x(t) = e^{-0.3t}\cos(2t)$

$$H(-0.3 + j2) = \frac{(-0.3 + j2) + 6}{(-0.3 + j2)^2 + 5(-0.3 + j) + 6} = 0.6849e^{-j1.1664}$$

$$y(t) = 0.6849e^{-0.3t}\cos(2t - 1.1664)$$



Transfer function and causality

- A CTLTI system to be causal, $h(t) = 0,\ t < 0$: $H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}dt = \int_{0}^{\infty} h(t)e^{-st}dt$
- The ROC for the transfer function of a causal system is to the right of a vertical line in the s-plane ⇒ the transfer function must converge at Re$\{s\} \to \infty$.

$$H(s)=\frac{Y(s)}{X(s)}=\frac{b_M s^M+b_{M-1}s^{M-1}+\cdots+b_1 s+b_0}{a_N s^N+a_{N-1}s^{N-1}+\cdots+a_1 s+a_0}$$

To be causal we need: $\lim_{s\to\infty} H(s)=\lim_{s\to\infty}\frac{b_M}{a_N}s^{M-N}<\infty \Leftrightarrow M-N\le 0 \Rightarrow M\le N$

Causality condition:

- In the transfer function of a causal CTLTI system the order of the numerator must not be greater than the order of the denominator.

Transfer function and stability:

- For a CTLTI system to be stable: $\int_{-\infty}^{\infty}\left|h(t)\right|dt<\infty$
- The FT of a signal exists if it is absolute integrable. But the FT of the impulse response is equal to the s-domain TF evaluated on the $j\omega$-axis of the s-plane,

$H(\omega) = H(s)\big|_{s=j\omega}$ provided that the $j\omega$-axis of the s-plane is within the ROC.

Stability condition:

- For a CTLTI system to be stable, the ROC of its s-domain transfer function must include the imaginary axis.
- For a causal system to be stable, the transfer function must not have any poles on the imaginary axis or in the right half s-plane.
- For a anticausal system to be stable, the transfer function must not have any poles on the imaginary axis or in the left half s-plane.
- For a noncausal system the ROC for the transfer function, if it exists, is the region expressed in the form $\sigma_1 < \text{Re}\{s\} < \sigma_2$. For stability we need $\sigma_1 < 0$ and $\sigma_2 > 0$. The poles of the transfer function may be either:

a. On or to the left of the vertical line $\sigma = \sigma_1$

b. On or to the right of the vertical line $\sigma = \sigma_2$

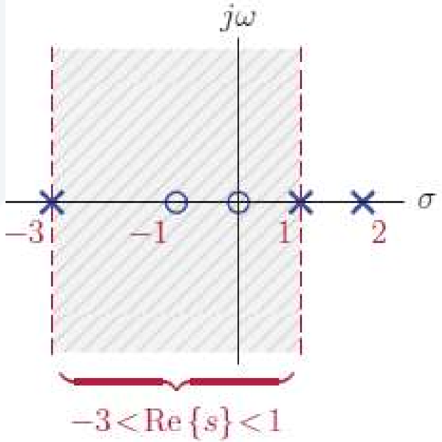
- Example 20: A stable system is characterized by: $H(s) = \dfrac{15s(s+1)}{(s+3)(s-1)(s-2)}$

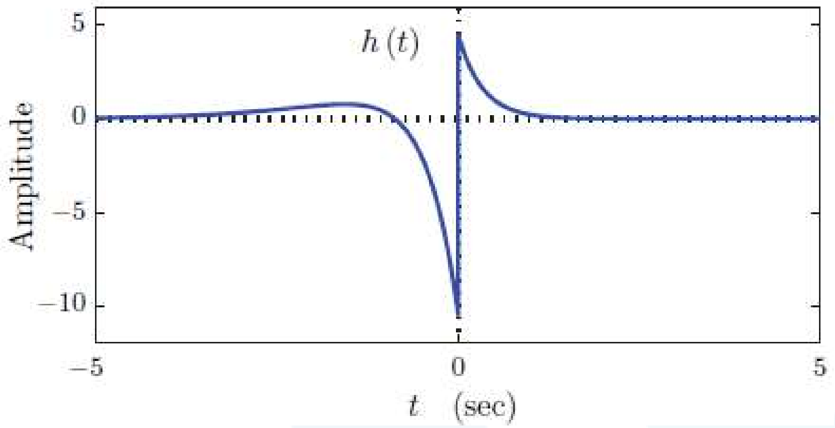
Determine the ROC of the TF. Afterwards find the impulse response.

The 3 poles are at $s = -3, 1, 2$. Since the system is known to be stable, its ROC must include the j-ω axis. The only possible choice is $-3 < \text{Re}\{s\} < 1$.

$$H(s) = \frac{4.5}{s+3} - \frac{7.5}{s-1} + \frac{18}{s-2}$$

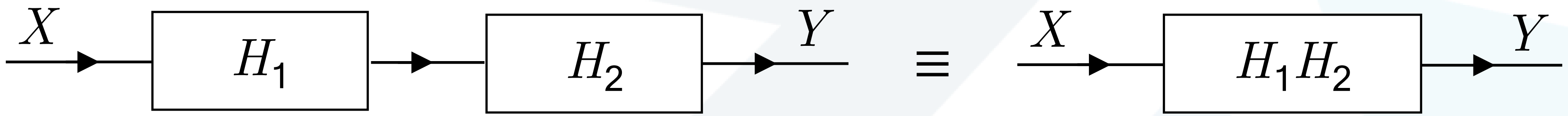
$$h(t) = 4.5e^{-3t}u(t) + 7.5e^{t}u(-t) - 18e^{2t}u(-t)$$



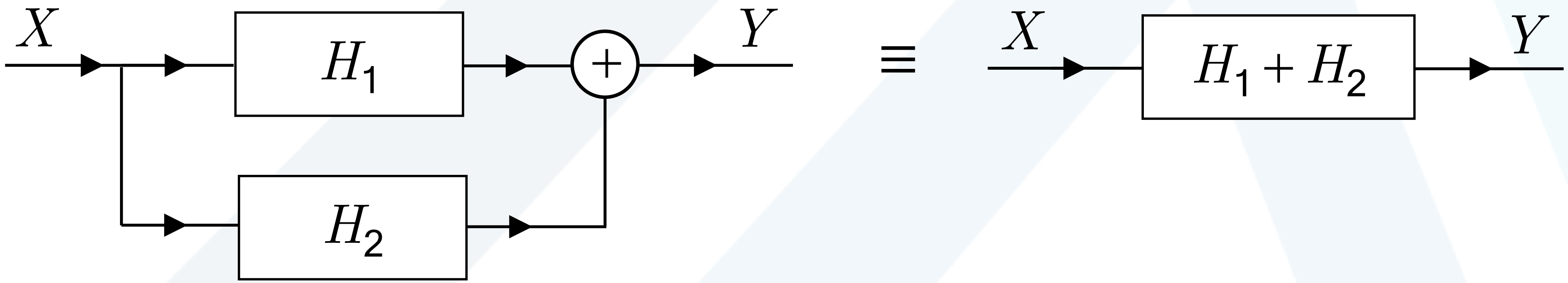


Interconnection of LTI Systems

- The series interconnection of the LTI systems with transfer functions H_1 and H_2 is the LTI system with transfer function $H_1 H_2$.

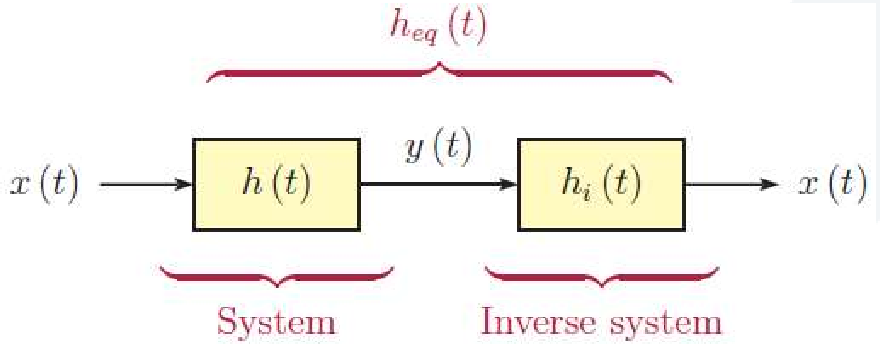


- The parallel interconnection of the LTI systems with transfer functions H_1 and H_2 is the LTI system with transfer function $H_1 + H_2$.



Inverse systems

- The inverse of a system is another system which, when connected in cascade with the original system, forms an identity system.
- Let the original system $h(t)$ and its inverse $h_i(t)$ be both CTLTI systems.



$$h_{eq}(t) = h(t) * h_i(t) = \delta(t)$$

$$H_{eq}(s) = H(s)H_i(s) \Rightarrow H_i(s) = \frac{1}{H(s)}$$

- Two important characteristics of the inverse system are causality and stability.

$$H(s) = \frac{B(s)}{A(s)} = \frac{a_M s^M + a_{M-1}s^{M-1} + \cdots + a_1 s + a_0}{b_N s^N + b_{N-1}s^{N-1} + \cdots + b_1 s + b_0}$$

$$H_i(s) = \frac{A(s)}{B(s)} = \frac{a_N s^N + a_{N-1} s^{N-1} + \cdots + a_1 s + a_0}{b_M s^M + b_{M-1} s^{M-1} + \cdots + b_1 s + b_0}$$

- If $H(s)$ is causal then $M \leq N$. Causality of the inverse system $H_i(s)$ requires $N \leq M$. Hence, we need $N = M$ if both the original system and its inverse are required to be causal.
- To analyze the stability of the inverse system, using $N = M$ we have:

$$H(s) = \frac{b_N (s - z_1)(s - z_2)\cdots(s - z_N)}{a_N (s - p_1)(s - p_2)\cdots(s - p_N)}$$ $H(s)$ in pole-zero form

- If the original system is both causal and stable, Re $\{p_k\} < 0$, $k = 1, ..., N$.

$$H_i(s) = \frac{a_N (s - p_1)(s - p_2)\cdots(s - p_N)}{b_N (s - z_1)(s - z_2)\cdots(s - z_N)}$$

- For the inverse system to be stable, both zeros and poles of the original system must be in the left half s-plane: Re $\{p_k\} < 0$, and Re $\{z_k\} < 0$, $k = 1, ..., N$.
- A causal CTLTI system that has all of its zeros and poles in the left half s-plane is referred to as a minimum-phase system. A minimum-phase system and its inverse are both causal and stable.
- Example 21: A causal CTLTI system is described by a differential equation:

$$\frac{dy(t)}{dt} + 2y(t) = \frac{dx(t)}{dt} + x(t)$$

Determine if a causal and stable inverse can be found for this system.

$$(s+2)Y(s) = (s+1)X(s) \Rightarrow H(s) = \frac{s+1}{s+2} \Rightarrow H_i(s) = \frac{s+2}{s+1}$$

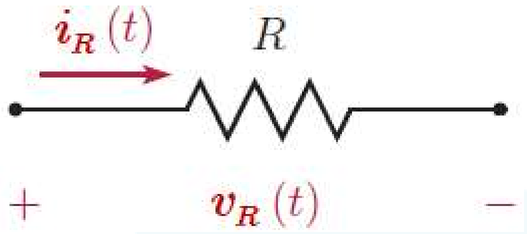
The inverse system is also causal and stable. $\frac{dy(t)}{dt} + y(t) = \frac{dx(t)}{dt} + 2x(t)$

Application: Electronic Circuit Analysis

Electronic Circuits

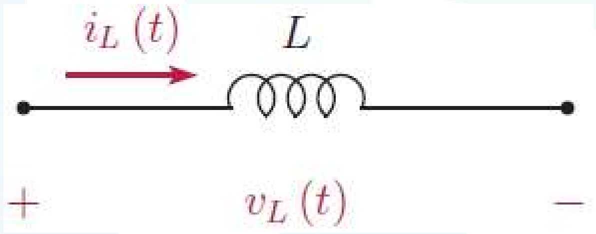
- A resistor $v_R(t) = Ri_R(t)$ or $i_R(t) = \frac{1}{R}v_R(t)$

  $V_R(s) = RI_R(s)$ or $I_R(s) = \frac{1}{R}V_R(s)$



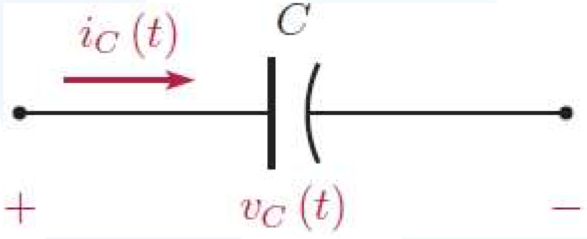
- An inductor $v_L(t) = L\frac{d}{dt}i_L(t)$ or $i_L(t) = \frac{1}{L}\int_{-\infty}^{t} v_L(\tau)d\tau$

  $V_L(s) = sLI_L(s)$ or $I_L(s) = \frac{1}{sL}V_L(s)$



- A capacitor $v_C(t) = \frac{1}{C}\int_{-\infty}^{t} i_C(\tau)d\tau$ or $i_C(t) = C\frac{d}{dt}v_C(t)$

  $V_C(s) = \frac{1}{sC}I_C(s)$ or $I_C(s) = sCV_C(s)$

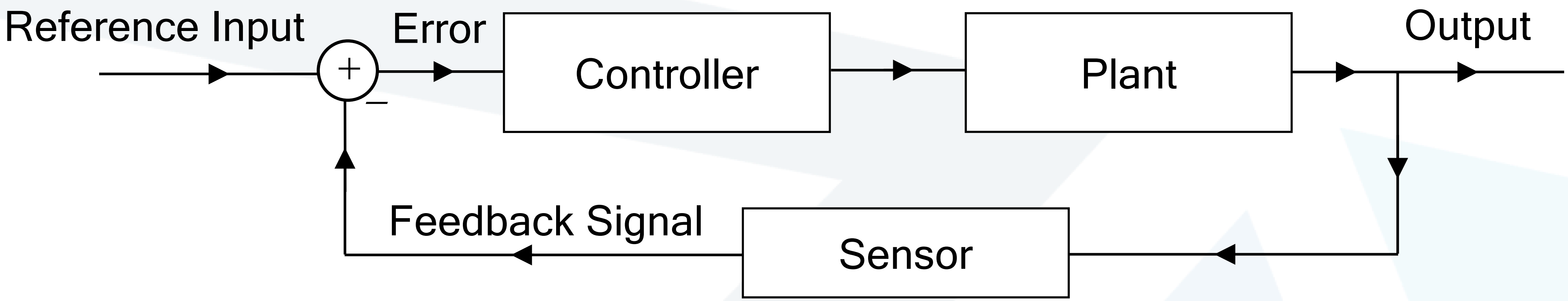


Application: Design and Analysis of Control Systems

Control Systems

- The desired values of the quantities being controlled are collectively viewed as the input of the control system.
- The actual values of the quantities being controlled are collectively viewed as the output of the control system.
- A control system whose behavior is not influenced by the actual values of the quantities being controlled is called an open loop (or non-feedback) system.
- A control system whose behavior is influenced by the actual values of the quantities being controlled is called a closed loop (or feedback) system.
- An example of a simple control system would be a thermostat system, which controls the temperature in a room or building.

Feedback Control Systems



- input: desired value of the quantity to be controlled.
- output: actual value of the quantity to be controlled.
- error: difference between the desired and actual values.
- plant: system to be controlled.
- controller: device that monitors the error and changes the input of the plant. with the goal of forcing the error to zero.

- sensor: device used to measure the actual output.

A control system includes two very important components:

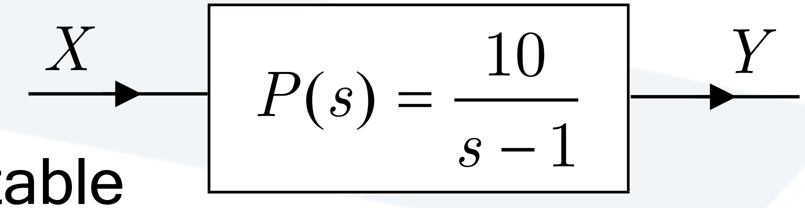
- Transducer: Since it is possible that the output signal $y(t)$ and the reference signal $x(t)$ might not be of the same type, a transducer is used to change $y(t)$ so it is compatible with the reference input $x(t)$.
- Actuator: A device that makes possible the execution of the control action on the plant, so that the output of the plant follows the reference input.

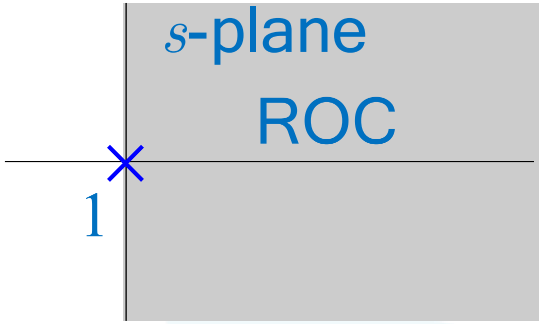
Stability Analysis of Feedback Systems

- Often, we want to ensure that a system is BIBO stable.
- The BIBO stability property is more easily characterized in the Laplace domain than in the time domain.

Example 22: Stabilization Example: Unstable Plant

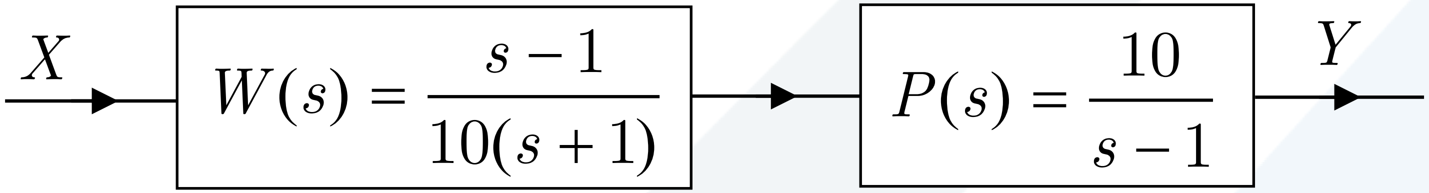
- Causal LTI plant
- System is not BIBO stable

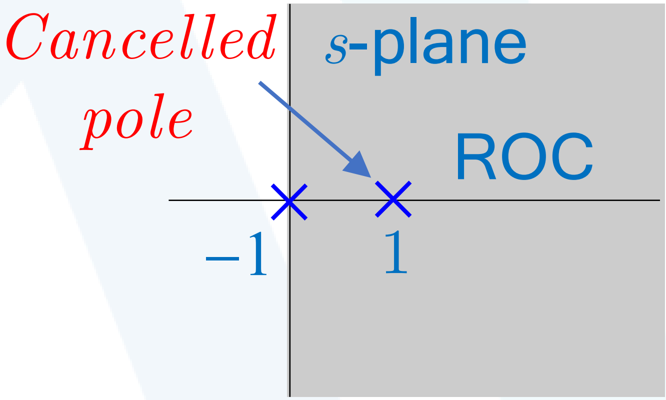




Example 23: Stabilization Example: Using Pole-Zero Cancellation

- System formed by series interconnection of plant and causal LTI compensator:



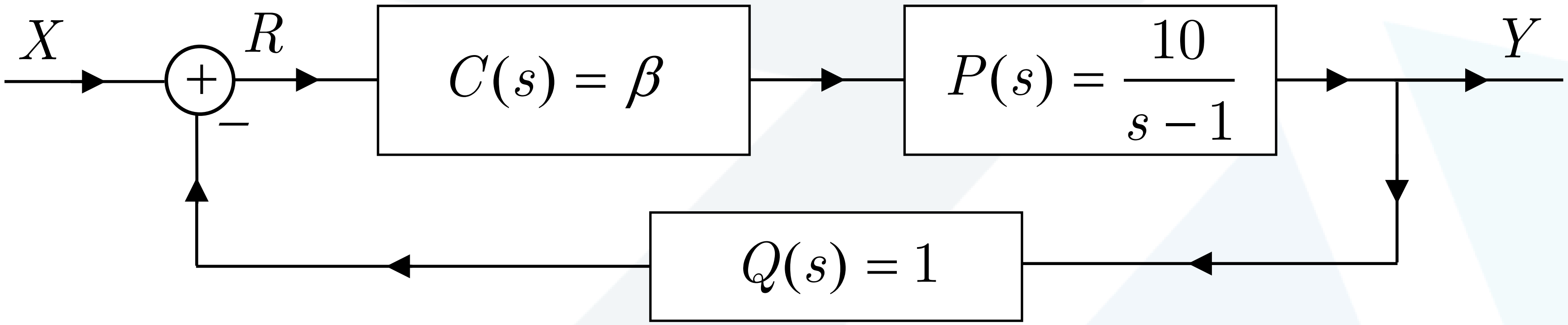


- Transfer function H of overall system (BIBO stable):

$$H(s) = W(s)P(s) = \frac{s-1}{10(s+1)}\frac{10}{s-1} = \frac{1}{(s+1)}$$

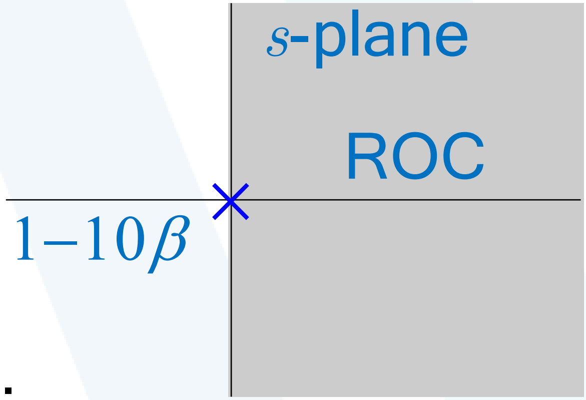
Example 24: Stabilization Example: Using Feedback

- Feedback system (with causal LTI compensator and sensor):



- Transfer function H of overall system:

$$H(s) = \frac{C(s)P(s)}{1 + C(s)P(s)Q(s)} = \frac{10\beta}{s - (1 - 10\beta)}$$



- Feedback system is BIBO stable if and only if $1-10\beta < 0$.

3. Bode Plots

- Bode plots of the frequency response are used in the analysis and design of feedback control systems. A Bode plot consists of the dB magnitude 20 $\log_{10}|H(\omega)|$ and the phase $\measuredangle H(\omega)$, each graphed as a function of $\log_{10}(\omega)$.

$$H(s) = K_1 \frac{(1 - s/z_1)(1 - s/z_2)\cdots(1 - s/z_M)}{(1 - s/p_1)(1 - s/p_2)\cdots(1 - s/p_N)}$$

- Let us write $H(s)$ as a cascade combination of $M + N$ subsystems:

$$H(s) = K_1 H_1(s) H_2(s) \cdots H_M(s) H_{M+1}(s) H_{M+2}(s) \cdots H_{M+N}(s)$$

with

$$H_i(s) = 1 - s/z_i, \quad i = 1, \cdots, M$$

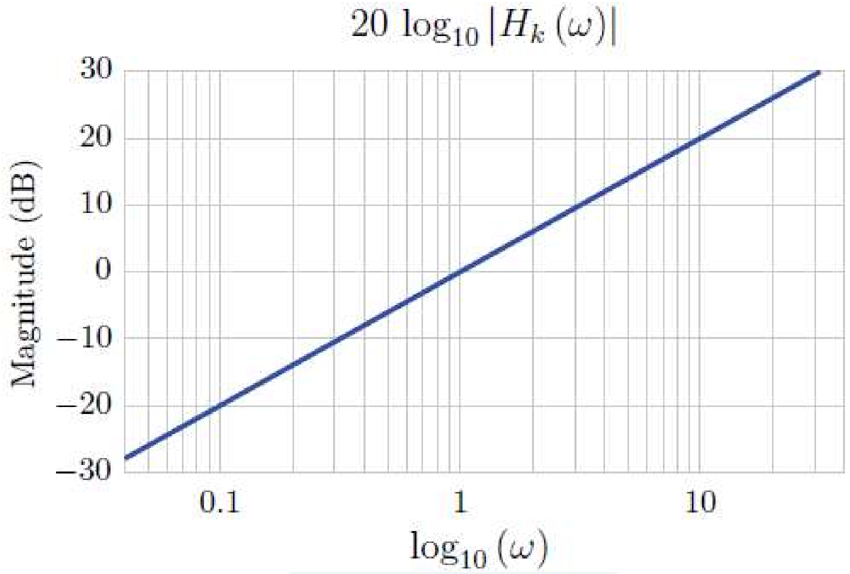
$$H_{M+i}(s) = \frac{1}{1 - s/p_i}, \quad i = 1, \cdots, N$$

- Zero at the origin

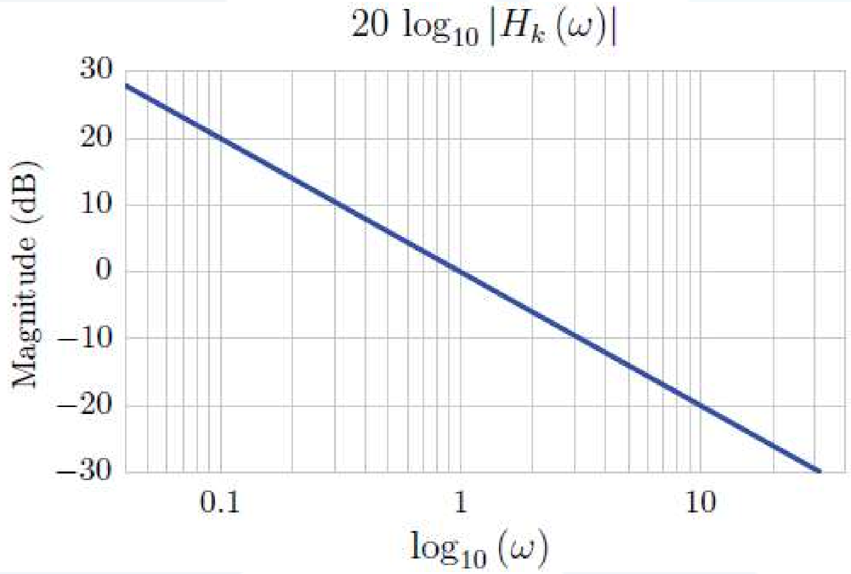
$$H_k(s) = s \Rightarrow 20\log_{10}\left|H_k(\omega)\right| = 20\log_{10}(\omega), \quad \measuredangle H_k(\omega) = 90^\circ$$

- Pole at the origin

$$H_k(s) = 1/s \Rightarrow 20\log_{10}\left|H_k(\omega)\right| = -20\log_{10}(\omega), \quad \measuredangle H_k(\omega) = -90^\circ$$



dB magnitude for $H_k(s) = s$



dB magnitude for $H_k(s) = 1/s$

- Single real zero

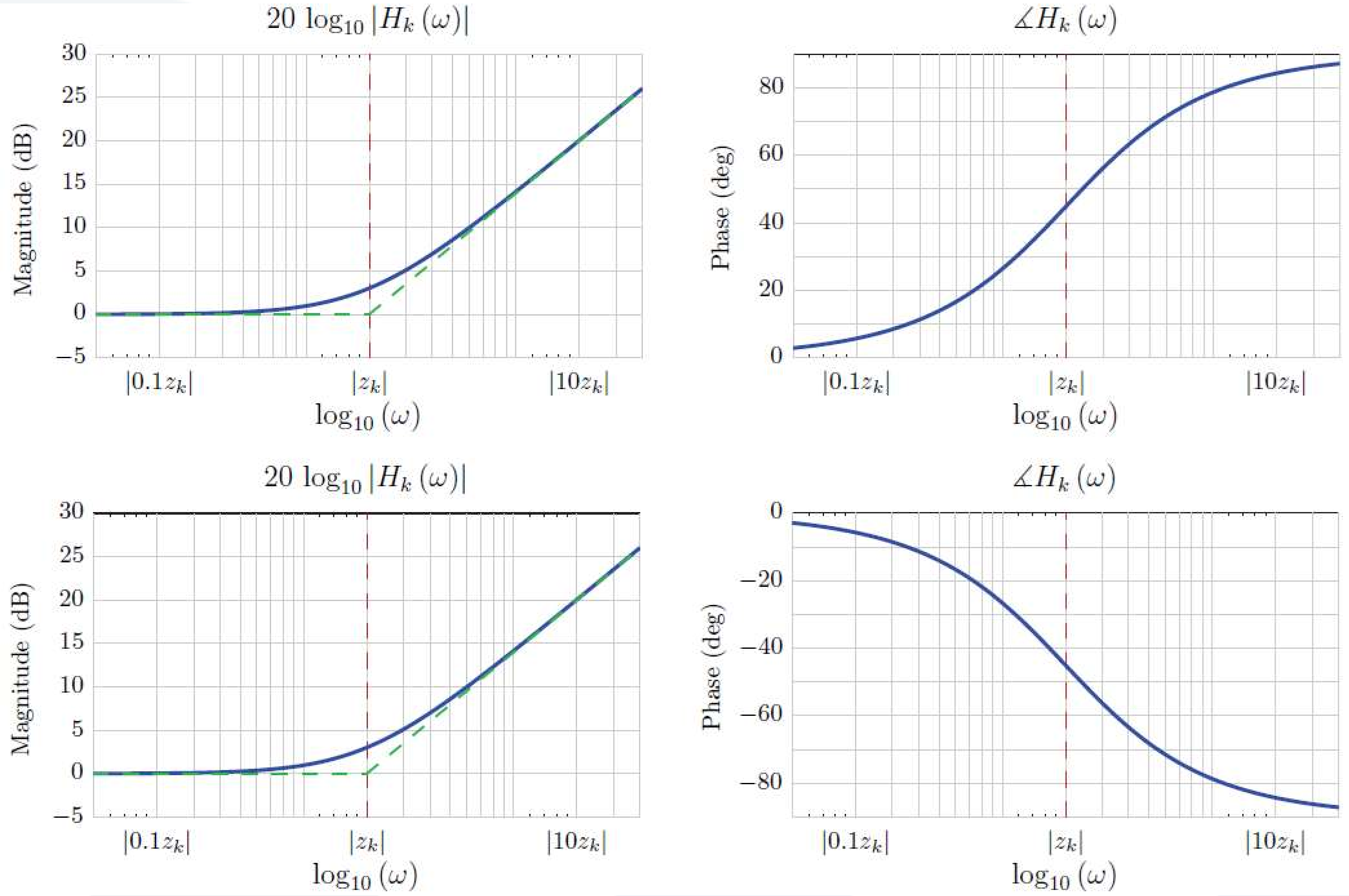
$$H_k(\omega) = H_k(s)\big|_{s=j\omega} = 1 - j\omega/z_k$$

$$20\log_{10}\left|H_k(\omega)\right| = 20\log_{10}\sqrt{1+\left(\omega/z_k\right)^2} = 10\log_{10}\left[1+\left(\omega/z_k\right)^2\right],$$

$$\measuredangle H_k(\omega) = \measuredangle\left(1 - j\omega/z_k\right) = \tan^{-1}\left(-\omega/z_k\right) = -\tan^{-1}\left(\omega/z_k\right)$$

Magnitude: For $\omega << |z_k|$ the magnitude is asymptotic to 0 dB. For $\omega >> |z_k|$ it becomes asymptotic to a straight line with a slope of 20 dB per decade. At $\omega = |z_k|$ it is approximately equal to 3 dB.

Phase: For $\omega << |z_k|$ the phase is asymptotic to 0°. For $\omega >> |z_k|$ the phase is 90° for $z_k < 0$ and –90° for $z_k > 0$. At $\omega = |z_k|$ the phase is 45° for $z_k < 0$ and –45° for $z_k > 0$.



$20\ \log_{10}|H_k(\omega)|$
Magnitude (dB)
$|0.1z_k|$ $|z_k|$ $|10z_k|$
$\log_{10}(\omega)$
$\measuredangle H_k(\omega)$
Phase (deg)
$|0.1z_k|$ $|z_k|$ $|10z_k|$
$\log_{10}(\omega)$

$z_k < 0$

$z_k > 0$

- Single real pole

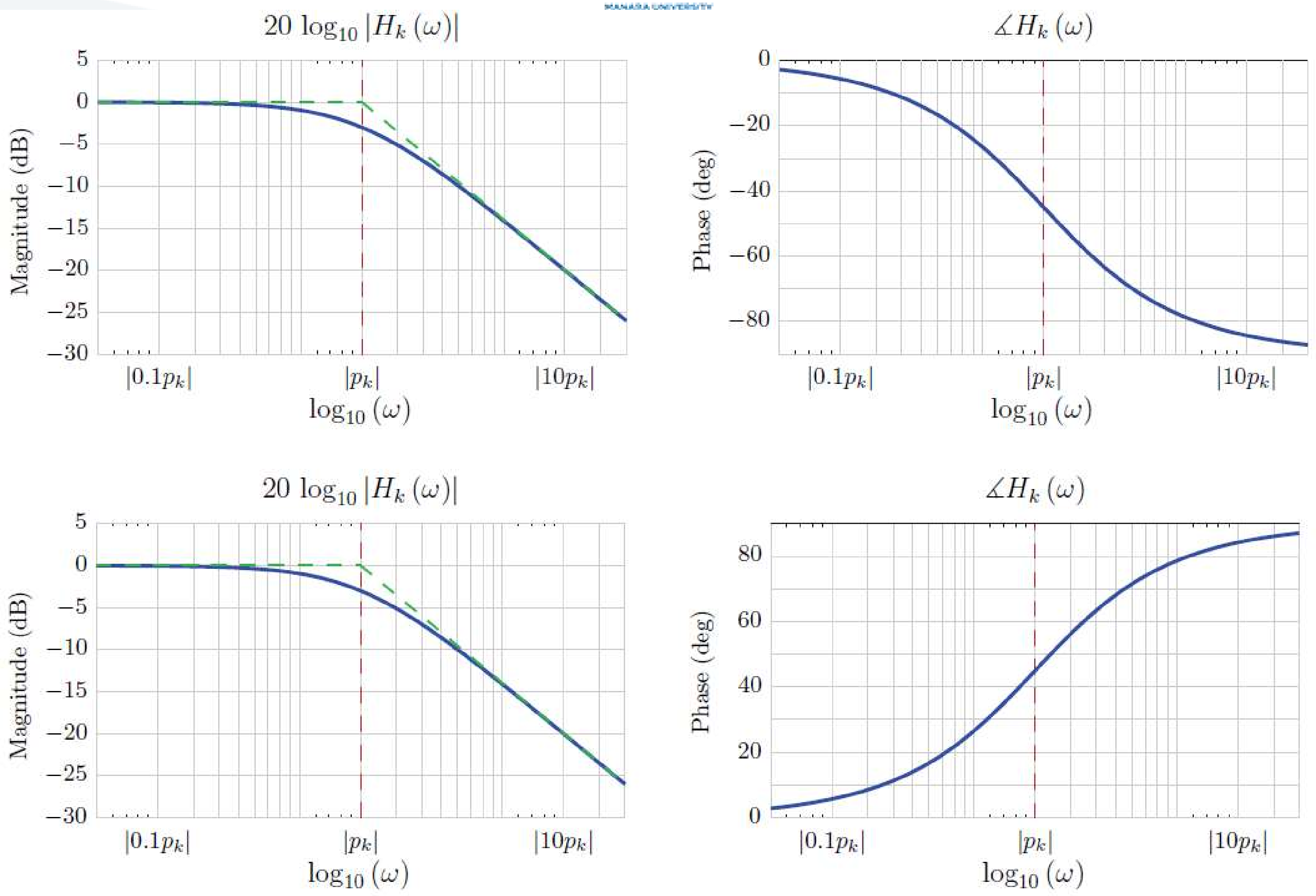
$$H_k(\omega) = H_k(s)\big|_{s=j\omega} = 1/(1 - j\omega/p_k)$$

$$20\log_{10}\left|H_k(\omega)\right| = 20\log_{10}\frac{1}{\sqrt{1+(\omega/p_k)^2}} = -10\log_{10}\left[1+(\omega/p_k)^2\right],$$

$$\measuredangle H_k(\omega) = \measuredangle 1/(1 - j\omega/p_k) = -\tan^{-1}(-\omega/p_k) = \tan^{-1}(\omega/p_k)$$

Magnitude: For $\omega << |p_k|$ the magnitude is asymptotic to 0 dB. For $\omega >> |p_k|$ it becomes asymptotic to a straight line with a slope of –20 dB per decade. At $\omega = |p_k|$ it is approximately equal to –3 dB.

Phase: For $\omega << |p_k|$ the phase is asymptotic to 0°. For $\omega >> |p_k|$ the phase is –90° for $p_k < 0$ and 90° for $p_k > 0$. At $\omega = |p_k|$ the phase is –45° for $z_k < 0$ and 45° for $z_k > 0$.



$20 \log_{10} |H_k(\omega)|$
$\measuredangle H_k(\omega)$
Magnitude (dB)
Phase (deg)
$|0.1 p_k|$
$|p_k|$
$|10 p_k|$
$\log_{10}(\omega)$
5
0
−5
−10
−15
−20
−25
−30
−20
−40
−60
−80
80
60
40
20

$p_k < 0$

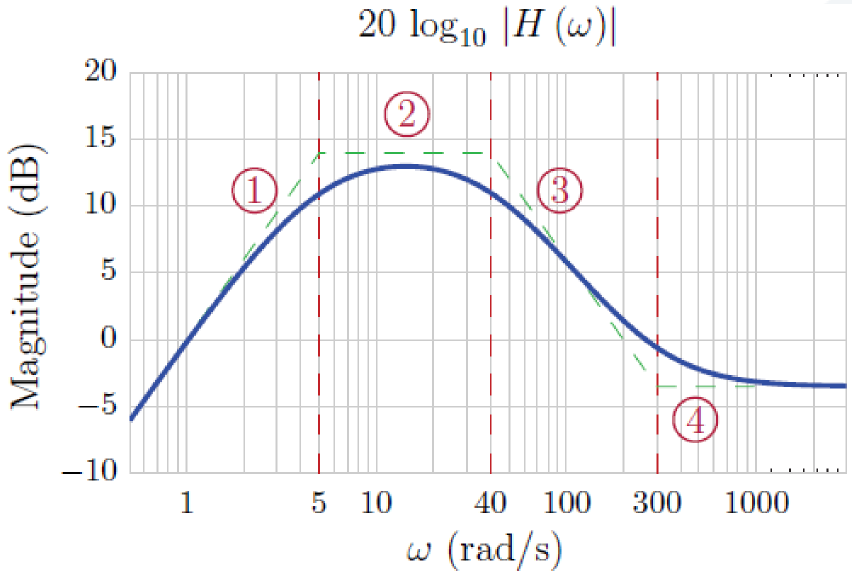
$p_k > 0$

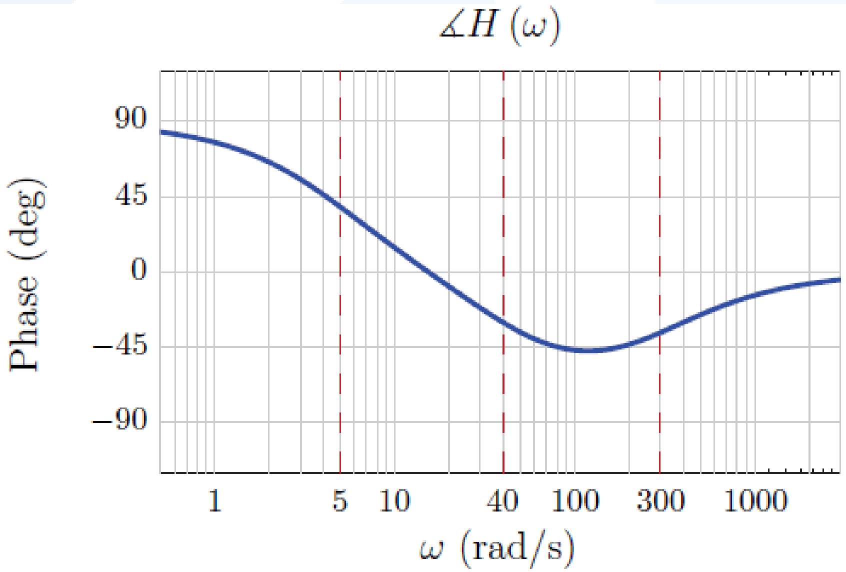
- Example 25: Constructing a Bode plot

$$H(s) = \frac{s(1 + s/300)}{(1 + s/5)(1 + s/40)}$$

$$H(s) = H_1(s)H_2(s)H_3(s)H_4(s)$$

$$H_1(s) = s, \quad H_2(s) = (1 + s/300), \quad H_3(s) = 1/(1 + s/5), \quad H_4(s) = 1/(1 + s/40)$$





- Conjugate pair of poles

Consider a causal and stable second-order system with a pair of complex conjugate poles, that is, $p_2 = p_1^*$

$$H(s) = \frac{1}{(1 - s/p_1)(1 - s/p_1^*)} = \frac{|p_1|^2}{(s - p_1)(s - p_1^*)}$$

Let us put $H(s)$ into the standard form $H(s) = \dfrac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$

$$\omega_0^2 = |p_1|^2, \quad \zeta = -\frac{\mathrm{Re}\{p_1\}}{|p_1|}$$

Since the system is causal and stable, Re $\{p_1\} < 0$. Consequently, when the poles of the system form a complex conjugate pair, we have $0 < \zeta < 1$.

- Analysis of the second-order system

$$H(s) = \frac{1}{(1 - s/p_1)(1 - s/p_2)} = \frac{p_1 p_2}{(s - p_1)(s - p_2)}$$

$$\omega_0^2 = p_1 p_2, \quad 2\zeta\omega_0 = -\mathrm{Re}\{p_1\} - \mathrm{Re}\{p_2\}$$

$$\omega_0 = \sqrt{p_1 p_2}, \quad \zeta = \frac{-\mathrm{Re}\{p_1\} - \mathrm{Re}\{p_2\}}{2\sqrt{p_1 p_2}}$$

The parameter ω_0 is called the natural undamped frequency of the system. The parameter ζ is called the damping ratio.

$$p_{1,2} = -\zeta\omega_0 \pm \omega_0\sqrt{\zeta^2 - 1}$$

$\zeta > 1$: The poles p_1 and p_2 are real-valued and distinct. The system is said to be overdamped.

$\zeta = 1$: The 2 poles are $p_1 = p_2 = -\zeta\omega_0$. The system is said to be critically damped.

$\zeta < 1$: The two poles are a complex conjugate pair:

$$p_{1,2} = -\zeta\omega_0 \pm j\omega_0\sqrt{1-\zeta^2} = -\zeta\omega_0 \pm j\omega_d$$

In this case the system is said to be underdamped.

$$H(\omega) = \frac{\omega_0^2}{(j\omega)^2 + 2\zeta\omega_0 j\omega + \omega_0^2} = \frac{1}{1-\left(\omega/\omega_0\right)^2 + j2\zeta\left(\omega/\omega_0\right)}$$

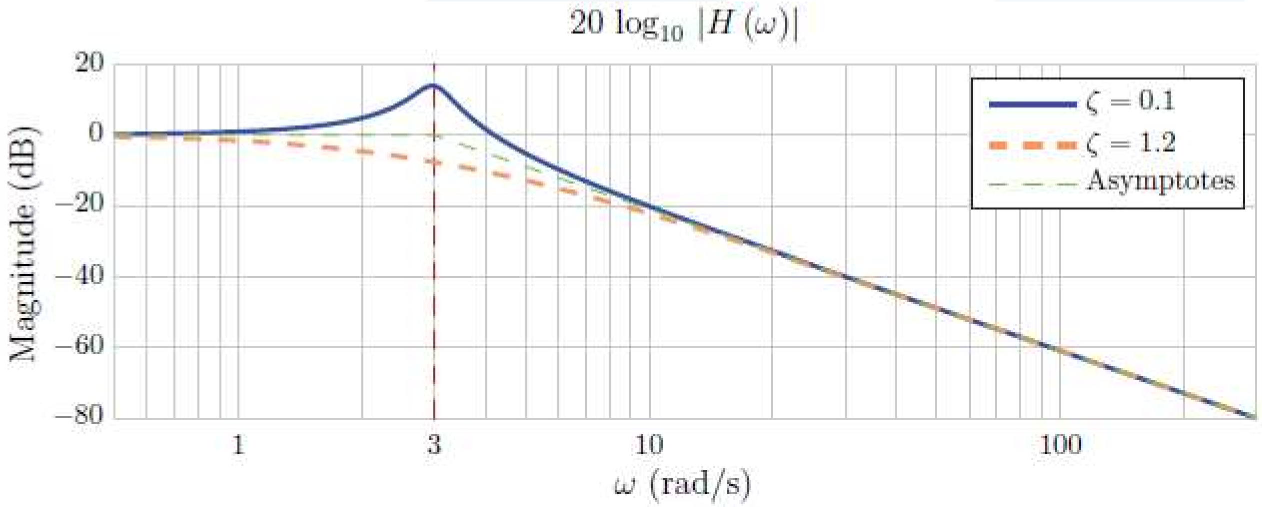
$$20\log_{10}\left|H(\omega)\right| = -10\log_{10}\left\{\left[1-\left(\omega/\omega_0\right)^2\right]^2 + \left[2\zeta\left(\omega/\omega_0\right)\right]^2\right\}$$

$$\measuredangle H(\omega) = -\tan^{-1}\left[\frac{2\zeta(\omega/\omega_0)}{1-(\omega/\omega_0)^2}\right]$$

Define the quality factor as $Q = 1/2\zeta$

Magnitude: For $\omega << \omega_0$ the magnitude is asymptotic to 0 dB. For $\omega >> \omega_0$ it becomes asymptotic to a straight line with a slope of –40 dB per decade. At $\omega = \omega_0$ the actual magnitude is 20 $\log_{10} Q = -20 \log_{10} (2\zeta)$.

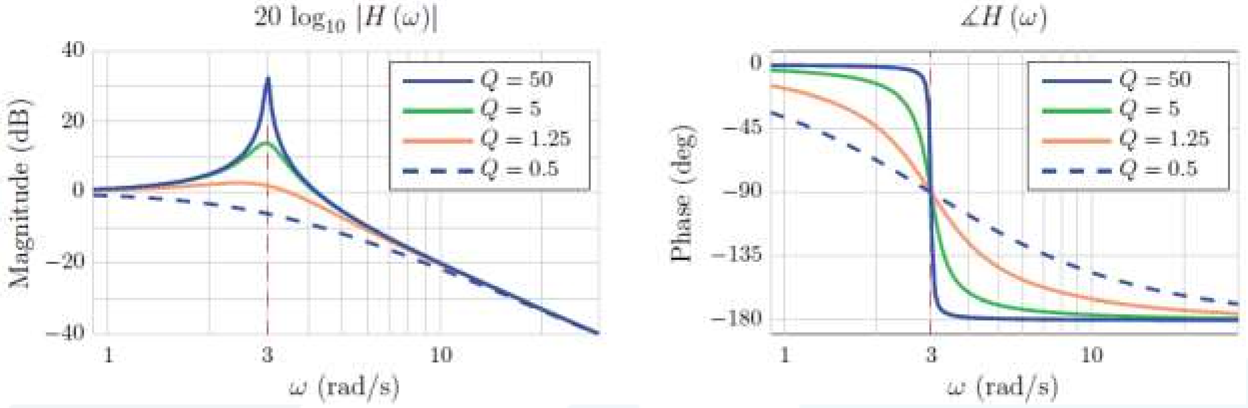
Phase: For $\omega << \omega_0$ the phase is asymptotic to 0°. For $\omega >> \omega_0$ the phase is –180°. At $\omega = \omega_0$ the phase is −90°.



Overdamped: $\zeta > 1 \Rightarrow Q < 0.5$

Critically damped: $\zeta = 1 \Rightarrow Q = 0.5$

Underdamped: $\zeta < 1 \Rightarrow Q > 0.5$



- The response of the second-order system to unit-impulse: $H(s) = \dfrac{k_1}{s - p_1} + \dfrac{k_2}{s - p_2}$

$$k_1 = \frac{p_1 p_2}{p_1 - p_2} = \frac{\omega_0}{2\sqrt{\zeta^2 - 1}}, \qquad k_2 = \frac{p_1 p_2}{p_2 - p_1} = -\frac{\omega_0}{2\sqrt{\zeta^2 - 1}}$$

$$h(t) = k_1 e^{p_1 t} + k_2 e^{p_2 t} = \frac{\omega_0}{2\sqrt{\zeta^2 - 1}} e^{-\zeta\omega_0 t}\left[e^{\omega_0\sqrt{\zeta^2-1}\,t} - e^{-\omega_0\sqrt{\zeta^2-1}\,t}\right]$$

If $\zeta < 1$, $h(t) = \frac{\omega_0}{\sqrt{\zeta^2 - 1}} e^{-\zeta\omega_0 t} \sin\left(\omega_0 \sqrt{\zeta^2 - 1}\, t\right) u(t)$

If $\zeta = 1$, $H(s) = \frac{\omega_0}{(s+\omega_0)^2} \Rightarrow h(t) = \omega_0^2 t e^{-\omega_0 t} u(t)$

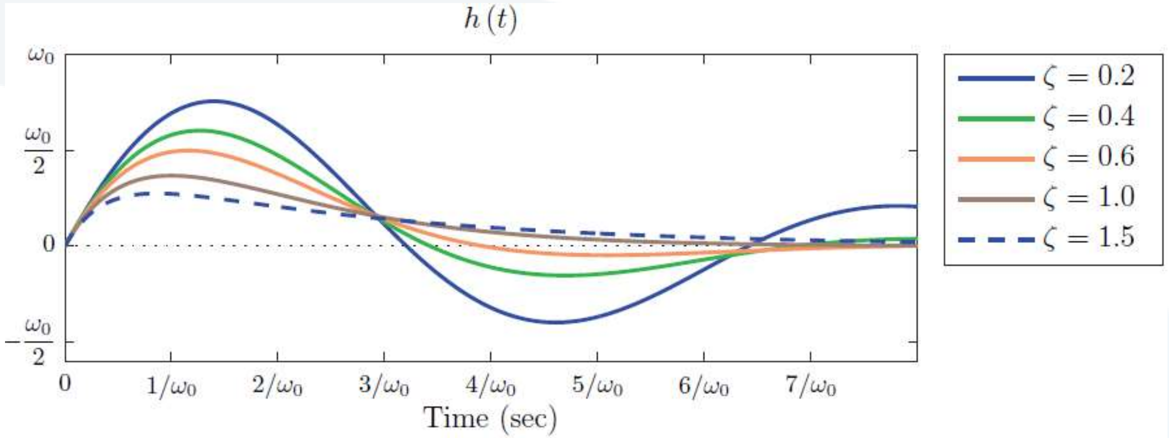
- The response of the second-order system to unit-step:

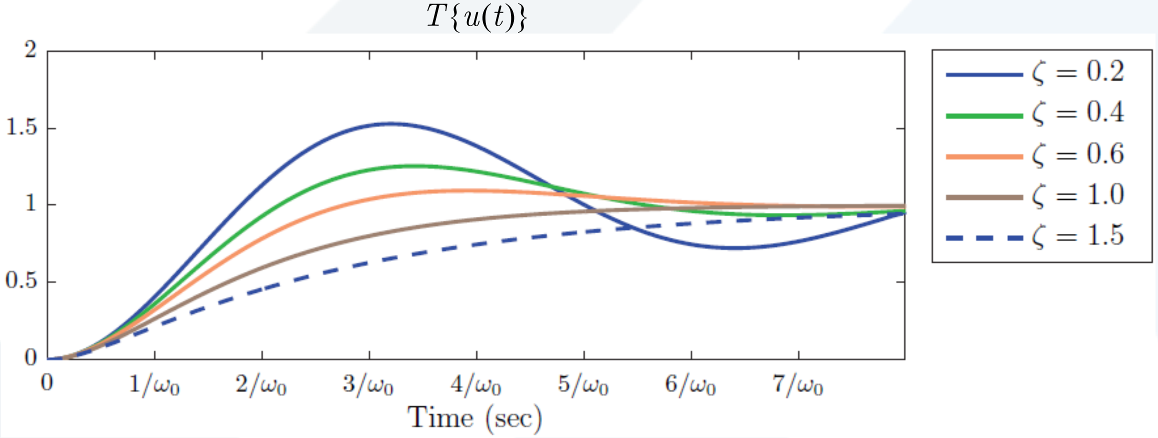
$$T\{u(t)\} = h(t) * u(t) = \int_0^t h(\tau) d\tau, \qquad t \geq 0$$

If $\zeta \neq 1$, $T\{u(t)\} = 1 + \frac{1}{p_1 - p_2}[p_2 e^{p_1 t} - p_1 e^{p_2 t}]u(t)$

$$T\{u(t)\} = 1 + \frac{e^{-\omega_0 \zeta t}}{2\omega_0 \sqrt{1-\zeta^2}} \Big[\left(-\omega_0\zeta - \omega_0\sqrt{1-\zeta^2}\right) e^{\omega_0 \sqrt{1-\zeta^2} t} + \left(\omega_0\zeta - \omega_0\sqrt{1-\zeta^2}\right) e^{-\omega_0 \sqrt{1-\zeta^2} t} \Big] u(t)$$

$$\text{If } \zeta = 1,\ T\{u(t)\} = [1 - e^{-\omega_0 t} - \omega_0 t e^{-\omega_0 t}]u(t)$$





4. Simulation Structures for CTLTI Systems

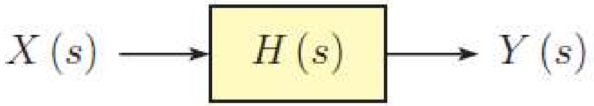
Direct-form implementation

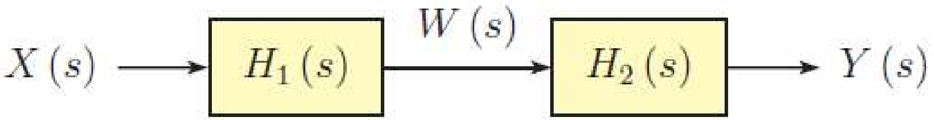
- Consider a third-order CTLTI system described by a TF $H(s)$:

$$H(s) = \frac{Y(s)}{X(s)} = \frac{b_2 s^2 + b_1 s + b_0}{s^3 + a_2 s^2 + a_1 s + a_0}$$

Let us use an intermediate function $W(s)$

$$H(s) = \frac{Y(s)}{W(s)} \frac{W(s)}{X(s)} = \frac{b_2 s^{-1} + b_1 s^{-2} + b_0 s^{-3}}{1 + a_2 s^{-1} + a_1 s^{-2} + a_0 s^{-3}}$$

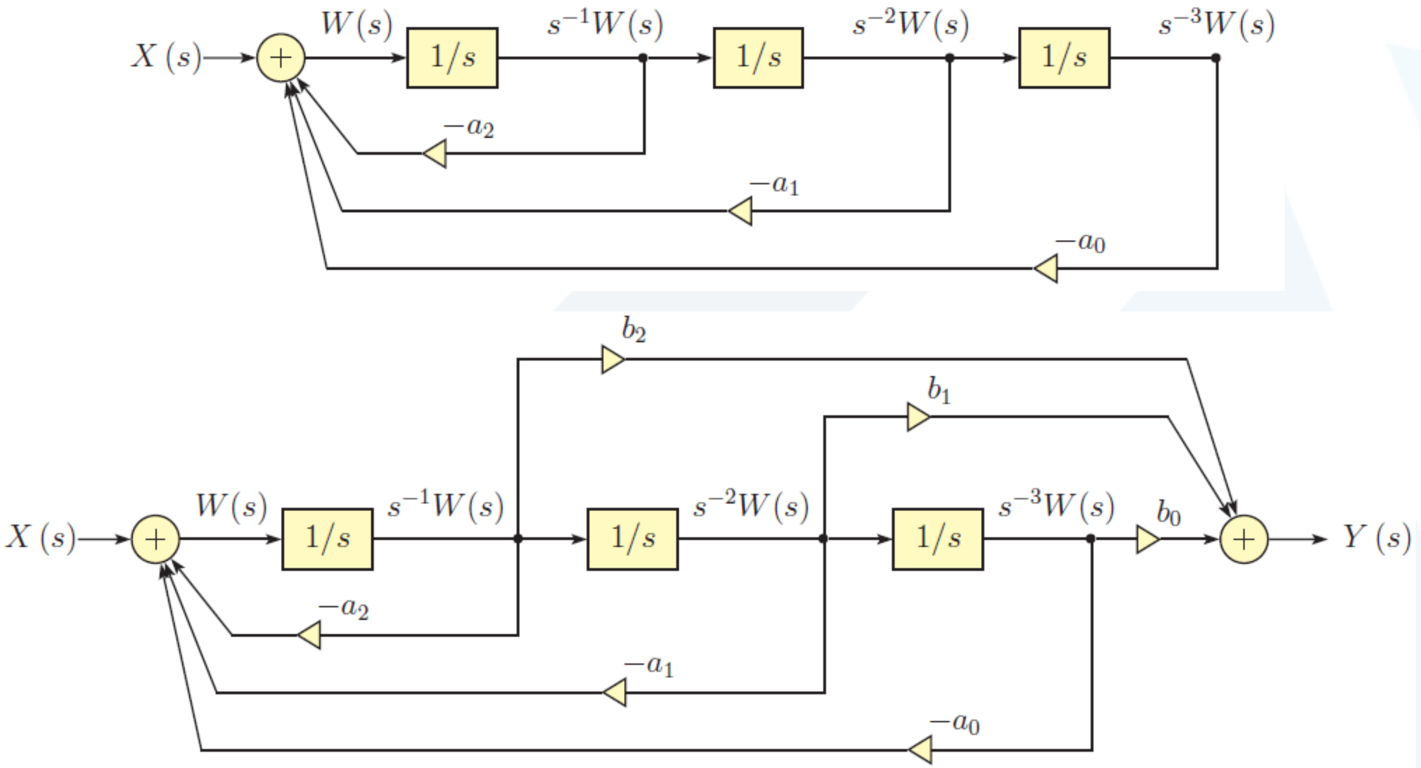




$$H_1(s) = \frac{W(s)}{X(s)} = \frac{1}{1 + a_2 s^{-1} + a_1 s^{-2} + a_0 s^{-3}}, \qquad H_2(s) = \frac{Y(s)}{W(s)} = b_2 s^{-1} + b_1 s^{-2} + b_0 s^{-3}$$

$$W(s) = X(s) - a_2 s^{-1}W(s) - a_1 s^{-2}W(s) - a_0 s^{-3}W(s)$$

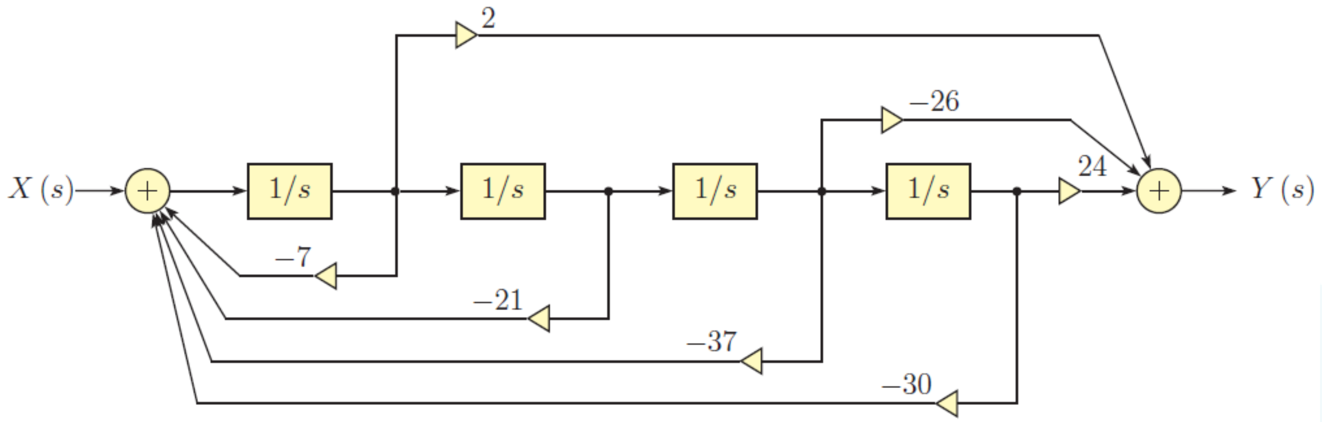
$$Y(s) = b_2 s^{-1}W(s) + b_1 s^{-2}W(s) + b_0 s^{-3}W(s)$$



Completed block diagram for simulating the transfer function $H(s)$

- Example 26: Obtaining a block diagram from transfer function

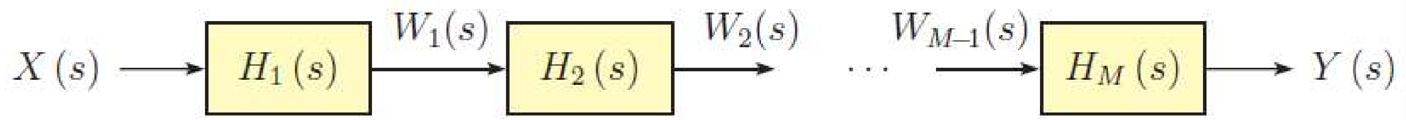
$$H(s) = \frac{2s^3 - 26s + 24}{s^4 + 7s^3 + 21s^2 + 37s + 30}$$



Cascade and parallel forms

Cascade form

$$H(s) = H_1(s)H_2(s)\cdots H_M(s) = \frac{W_1(s)}{X(s)}\frac{W_2(s)}{W_1(s)}\cdots\frac{Y(s)}{W_{M-1}(s)}$$



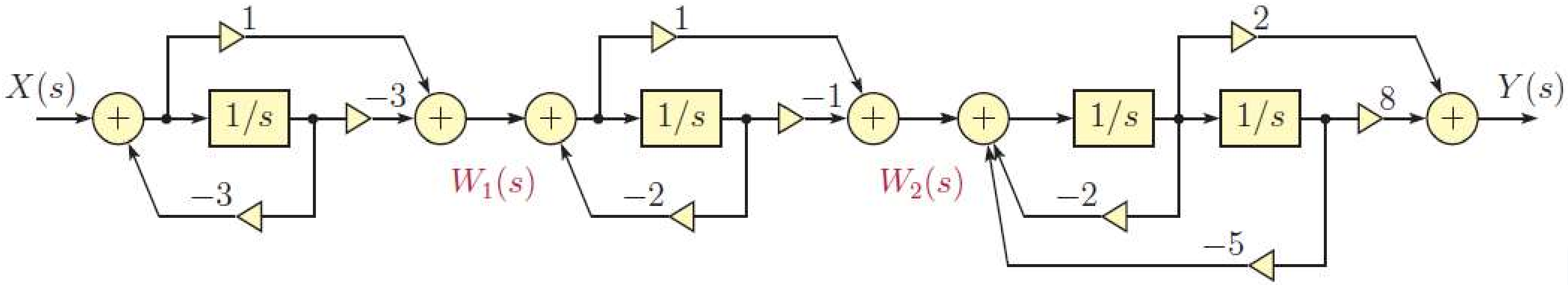
- Example 27: Obtaining a block diagram from transfer function

Develop a cascade form block diagram for the system used in example 26.

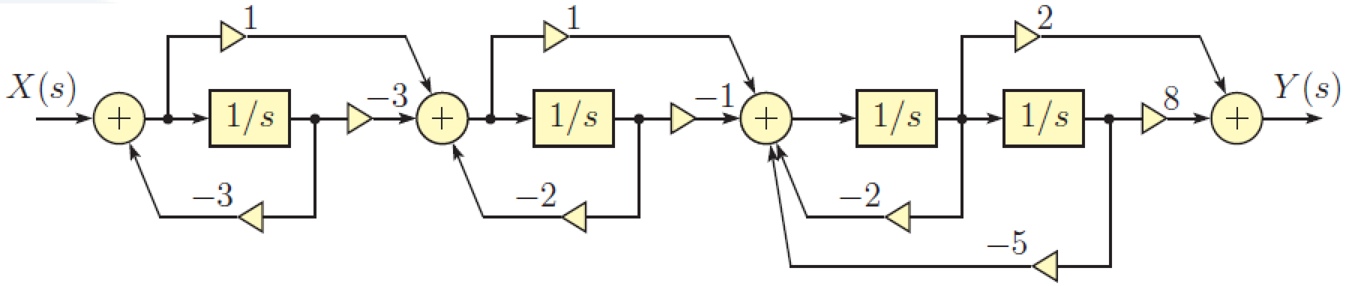
$$H(s)=\frac{Y(s)}{X(s)}=\frac{2(s+4)(s-3)(s-1)}{(s+1-j2)(s+1+j2)(s+3)(s+2)}$$

$$H(s)=H_1(s)H_2(s)H_3(s)$$

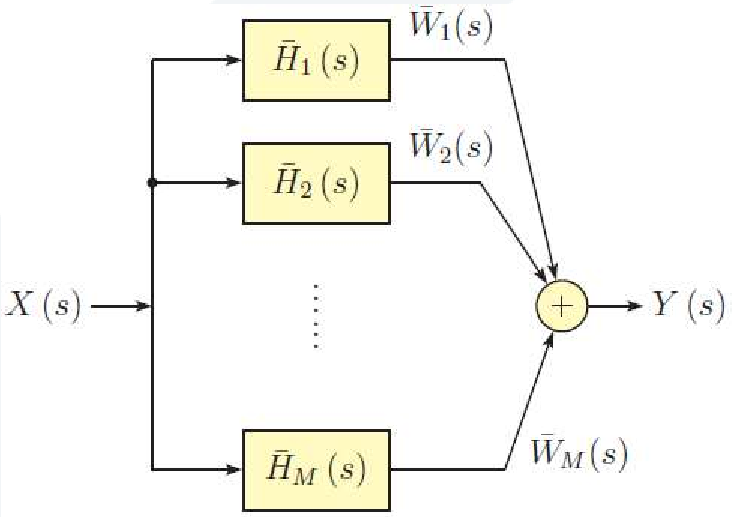
$$H_1(s)=\frac{2(s+4)}{(s+1-j2)(s+1+j2)}=\frac{2s+8}{s^2+2s+5},\qquad H_2(s)=\frac{s-3}{s+3},\qquad H_3(s)=\frac{s-1}{s+2}$$



Further simplified cascade form block diagram:



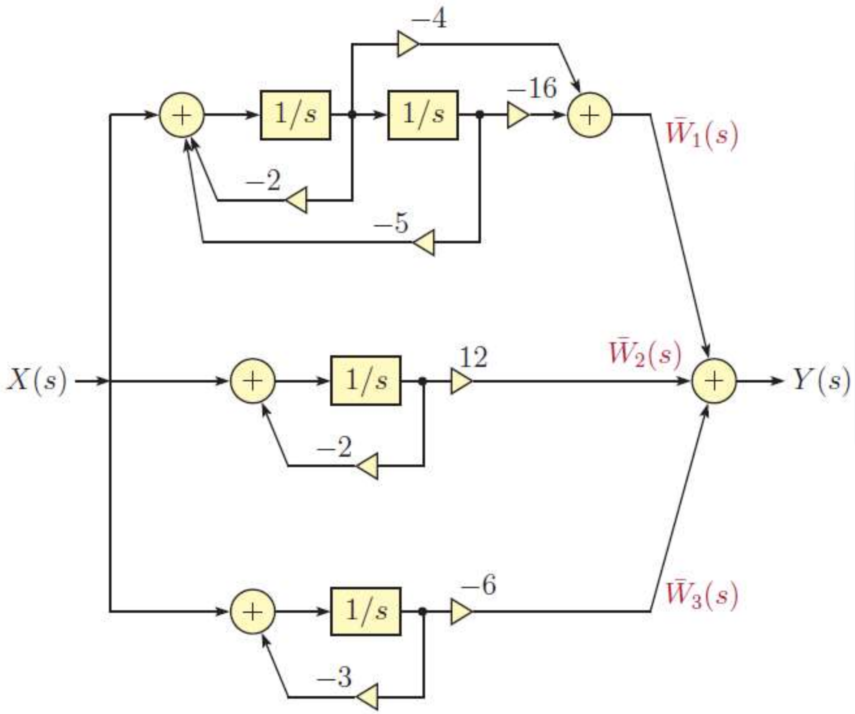
Parallel form $H(s) = \bar{H}_1(s) + \bar{H}_2(s) + \cdots + \bar{H}_M(s) = \dfrac{\bar{W}_1(s)}{X(s)} + \dfrac{\bar{W}_2(s)}{X(s)} + \cdots + \dfrac{\bar{W}_M(s)}{X(s)}$



- Example 28: Obtaining a block diagram from transfer function

  Develop a parallel form BD for simulating the system used in example 26.

$$H(s) = \frac{2s+8}{s^2+2s+5} + \frac{12}{s+2} + \frac{-6}{s+3}$$



5. Unilateral Laplace Transform

The unilateral Laplace transform of the function x is defined as:

$$\mathcal{L}_u\{x(t)\} = X(s) = \int_{0^-}^{\infty} x(t)e^{-st}dt$$

- The unilateral LT is related to the bilateral Laplace transform as follows:

$$\mathcal{L}_u\{x(t)\} = \int_{0^-}^{\infty} x(t)e^{-st}dt = \int_{-\infty}^{\infty} x(t)u(t)e^{-st}dt = \mathcal{L}\{x(t)u(t)\}$$

- With the unilateral LT, the same inverse transform equation is used as in the bilateral case.
- The unilateral LT is only invertible for causal functions.
- For a noncausal function x, we can only recover $x(t)$ for $t \geq 0$.

Unilateral Versus Bilateral Laplace Transform

In the unilateral case:

- The time-domain convolution property has the additional requirement that the functions being convolved must be causal.
- The time/Laplace-domain scaling property has the additional constraint that the scaling factor must be positive.
- The time-domain differentiation property has an extra term in the expression of $\mathcal{L}_u(dx(t)/dt)$, namely $-x(0^-)$.
- The time-domain integration property has a different lower limit in the time-domain integral (0^- instead of $-\infty$);
- The time-domain shifting property does not hold (except in special cases).

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Properties of the Unilateral Laplace Transform

Property	$x(t)$	$X(s)$	ROC
Linearity	$ax_1(t) + bx_2(t)$	$aX_1(s) + bX_2(s)$	$\supset (R_1 \cap R_2)$
Multiply by t	$t\,x(t)$	$-dX(s)/ds$	R
Multiply by $e^{-\alpha t}$	$x(t)e^{-\alpha t}$	$X(s+\alpha)$	Shift R by $-\alpha$
Scaling in t	$x(at),\ a>0$	$\frac{1}{a}X(\frac{s}{a})$	aR
Differentiate in t	$dx(t)/dt$	$sX(s) - x(0^-)$	$\supset R$
Integrate in t	$\int_{0^-}^{t} x(\tau)d\tau$	$\frac{X(s)}{s}$	$\supset (R \cap (\text{Re}(s) > 0))$
Convolve in t	$x_1(t) * x_2(t)$	$X_1(s)\ X_2(s)$	$\supset (R_1 \cap R_2)$

Unilateral Laplace Transform Pairs

Pair	$x(t);\ t \geq 0$	$X(s)$
1	$\delta(t)$	1
2	1	$\frac{1}{s}$
3	t^n	$\frac{n!}{s^{n+1}}$
4	e^{-at}	$\frac{1}{s+a}$
5	$t^n e^{-at}$	$\frac{n!}{(s+a)^{n+1}}$

Pair	$x(t);\ t \geq 0$	$X(s)$
6	$\cos\omega_0 t$	$\frac{s}{s^2+\omega_0^2}$
7	$\sin\omega_0 t$	$\frac{\omega_0}{s^2+\omega_0^2}$
8	$e^{-at}\cos\omega_0 t$	$\frac{s+a}{(s+a)^2+\omega_0^2}$
9	$e^{-at}\sin\omega_0 t$	$\frac{\omega_0}{(s+a)^2+\omega_0^2}$

Analysis of LTI Systems Represented by Differential Equations

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}, \qquad \left.\frac{d^k y(t)}{dt^k}\right|_{t=0}, \; 0 \le k \le N-1$$

$$\frac{d^N y(t)}{dt^N} + \sum_{k=0}^{N-1} \alpha_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} \beta_k \frac{d^k x(t)}{dt^k}, \qquad \alpha_k = \frac{a_k}{a_N}, \; \beta_k = \frac{b_k}{a_N}$$

$$Y(s) = \frac{B(s)}{A(s)} X(s) + \frac{1}{A(s)} I(s)$$

$$A(s) = \sum_{k=0}^{N} \alpha_k s^k, \qquad \alpha_N = 1 \qquad\qquad B(s) = \sum_{k=0}^{M} \beta_k s^k$$

$$I(s) = \sum_{k=1}^{N} \alpha_k \left(\sum_{m=0}^{k-1} s^{k-m-1} \left.\frac{d^m y(t)}{dt^m}\right|_{t=0} \right), \qquad \alpha_N = 1$$

$$H(s) = \frac{B(s)}{A(s)}, \quad H_1(s) = \frac{1}{A(s)}$$

$$\Rightarrow Y(s) = H(s)X(s) + H_1(s)I(s)$$

the complete response: $y(t) = y_{zs}(t) + y_{zi}(t)$, where

$y_{zs}(t) = \mathcal{L}^{-1}[H(s)X(s)]$ is the system's zero-state response,

$y_{zi}(t) = \mathcal{L}^{-1}[H_1(s)I(s)]$ is the system's zero-input response.

- Example 29: Response of a second-order system (RLC circuit)

  A voltage $x(t) = 10e^{-3t}u(t)$ is applied at the input of the RLC circuit. Find the output current $i_L(t) = y(t)$ for $t \geq 0$ if the initial inductor current is $i_L(0^-) = 0$, and the initial capacitor voltage $v_C(0^-) = 5$ V. Use $R = 3\ \Omega$, $L = 1$ H and $C = 1/2$ F.

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = \frac{dx(t)}{dt}, \qquad y(0^-) = 0, \left.\frac{dy(t)}{dt}\right|_{t=0^-} = -5$$

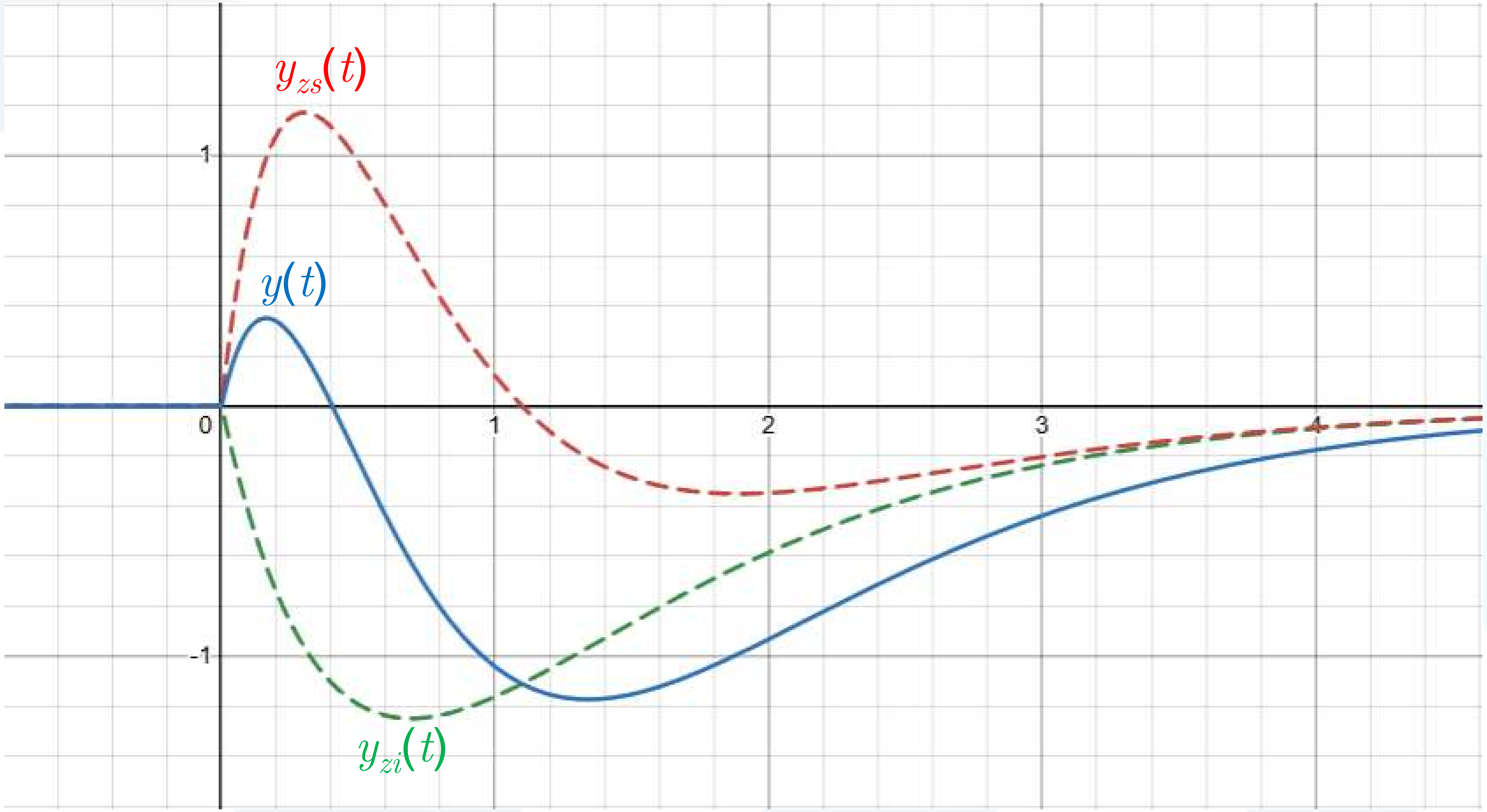
$$s^2Y(s) - 0s - (-5) + 3(sY(s) - 0) + 2Y(s) = sX(s) = \frac{10s}{s+3}$$

$$(s^2 + 3s + 2)\,Y(s) = -5 + \frac{10s}{s+3}$$

$$Y(s) = \frac{-5}{s^2 + 3s + 2} + \frac{10s}{(s+3)(s^2 + 3s + 2)}$$

$$y(t) = \underbrace{-5e^{-t} + 5e^{-2t}}_{y_{zi}(t)} + \underbrace{-5e^{-t} + 20e^{-2t} - 15e^{-3t}}_{y_{zs}(t)},\ t \geq 0$$

$$y(t) = -10e^{-t} + 25e^{-2t} - 15e^{-3t},\ t \geq 0$$


$y_{zs}(t)$
$y(t)$
$y_{zi}(t)$
1
-1
0
1
2
3
4

Transient and Steady State Response

- The response a CTLTI system to an everlasting exponential input $x(t) = e^{st}$ is also an everlasting exponential $H(s)e^{st}$ $\quad e^{st} \Rightarrow e^{st}H(s)$

  for BIBO-stable systems ($s = j\omega$) $\quad e^{j\omega t} \Rightarrow e^{j\omega t}H(\omega)$

  $\cos \omega t = Re(e^{j\omega t}) \qquad \cos \omega t \Rightarrow Re[H(\omega)e^{j\omega t}] = |H(\omega)|\cos[\omega t + \angle H(\omega)]$

  The system response $y(t)$ to a sinusoidal input $\cos \omega t$ is given by:

$$y(t) = |H(\omega)|\cos[\omega t + \angle H(\omega)]$$

Steady-State Response to Causal Sinusoidal Inputs

- Consider the input $e^{j\omega t}u(t)$, which starts at $t = 0$ rather than at $t = -\infty$. $X(s) = 1/(s - j\omega)$.

$$Y(s) = H(s)X(s) = \frac{N(s)}{(s-p_1)(s-p_2)\cdots(s-p_N)(s-j\omega)}$$

$$Y(s) = \sum_{i=1}^{N} \frac{k_i}{s - p_i} + \frac{H(j\omega)}{s - j\omega}$$

$$y(t) = \underbrace{\sum_{i=1}^{N} k_i e^{p_i t} u(t)}_{transient\ component\ y_{tr}(t)} + \underbrace{H(\omega) e^{j\omega t} u(t)}_{steady-state\ component\ y_{ss}(t)}$$

The terms $e^{p_i t}$ decay with time, and, therefore, constitute the so-called transient component of the response. The last term $H(\omega)e^{j\omega t}$ persists forever, and is the steady-state component of the response given by: $y_{ss}(t) = H(\omega)e^{j\omega t}u(t)$.

For BIBO systems at any finite time, the total response appears to be $H(\omega)e^{j\omega t}$.

For a causal sinusoidal input $\cos\omega t$, the steady-state response $y_{ss}(t)$ is given by: $y_{ss}(t) = |H(\omega)| \cos[\omega t + \angle H(\omega)] u(t)$

- Note: $|H(\omega)| \cos[\omega t + \angle H(\omega)]$ is the total response to everlasting sinusoid $\cos\omega t$. In contrast, it is the steady-state response to the same input applied at $t = 0$.