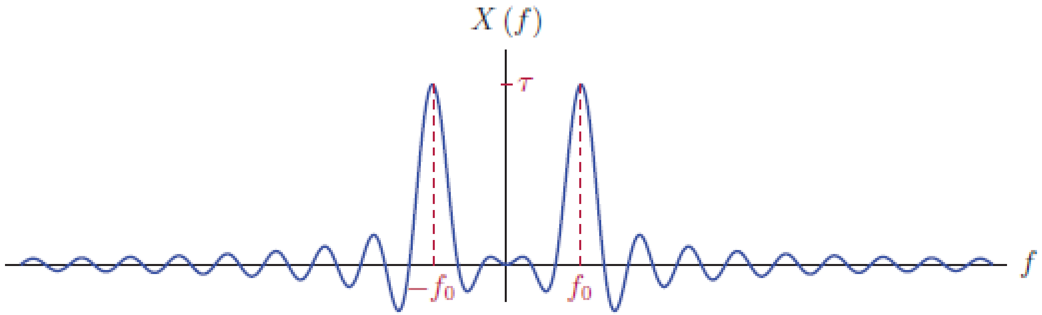
CEDC403: Signals and Systems

Lecture Notes 11: Z-Transform for Discrete-Time Signals and Systems



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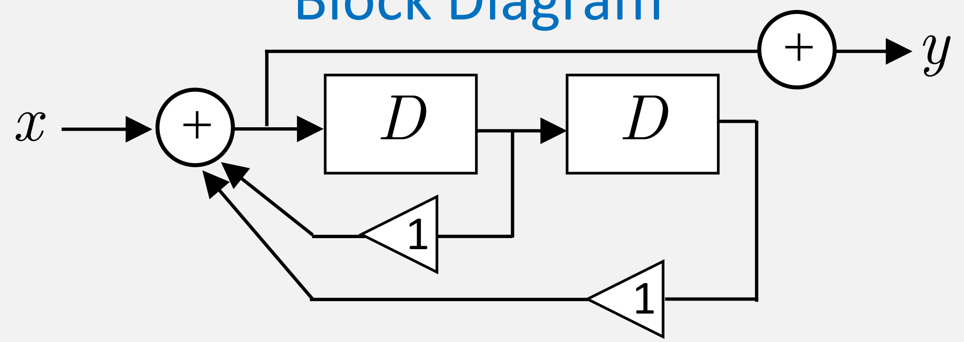
Chapter 8

Z-Transform for Discrete-Time Signals and Systems

Concept Map for DT Systems

Block Diagram



Impulse Response

$h[n] = \{1, 1, 2, 3, 5, 8, 13, \ldots\}$

Difference Equation

$$y[n] = x[n] + y[n-1] + y[n-2]$$

Transfer Function

$$\frac{Y(z)}{X(z)} = \frac{z^2}{z^2 - z - 1}$$

Introduction

- The z-transform (ZT) can be viewed as a generalization of the discrete time Fourier transform.
- The ZT representation exists for some sequences that do not have a discrete Fourier transform representation. So, we can handle some sequences with the ZT that cannot be handled with the DTFT ($x[n] = nu[n]$).

1. Z-Transform

- The z-transform of a discrete-time signal $x[n]$ is defined as:

$$Z\{x[n]\} = X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

where z, the independent variable of the transform is a complex number.

- The z-transform defined is sometimes referred to as the bilateral (two sided) z-transform. A simplified variant of the transform termed the unilateral (one-sided) z-transform is introduced as an alternative analysis tool.

Relationship Between ZT and Discrete-Time FT

$$X(r,\Omega) = X(z)\Big|_{z=re^{j\Omega}} = \sum_{n=-\infty}^{\infty} x[n](re^{j\Omega})^{-n} = \sum_{n=-\infty}^{\infty} x[n]r^{-n}e^{-j\Omega n} = \mathcal{F}\{r^{-n}x[n]\}$$

$$X(z)\Big|_{z=e^{j\Omega}} = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \mathcal{F}\{x[n]\}$$

- Example 1: A simple z-transform example

$$x[n] = \{\underset{\uparrow}{3.7}, 1.3, -1.5, 3.4, 5.2\}$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = 3.7 + 1.3z^{-1} - 1.5z^{-2} + 3.4z^{-3} + 5.2z^{-4}$$

The transform converges at all points in the complex z-plane except of $z = 0$.

- Example 2: z-transform of a non-causal signal $x[n] = \{3.7, 1.3, -\underset{\uparrow}{1.5}, 3.4, 5.2\}$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = 3.7z^{2} + 1.3z^{1} - 1.5 + 3.4z^{-1} + 5.2z^{-2}$$

It converges at every point in the z-plane except, the origin and infinity.

- Example 3: z-Transform of the unit-impulse

$\mathcal{Z}\{\delta[n]\} = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = x[0]z^{0} = 1$ It converges at every point in the z-plane

- Example 4: z-Transform of a time shifted the unit-impulse

$$X(z) = \mathcal{Z}\{\delta[n-k]\} = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = z^{-k}$$

1. If $k > 0$ then the transform does not converge at the origin $z = 0$.
2. If $k < 0$ then the transform does not converge at infinity.

Regions of Convergence (ROC)

- For the z-transform $X(z)$ of $x[n]$ to exist we need that:

  $$\left|X(z)\right| = \left|\sum_{n=-\infty}^{\infty} x[n]z^{-n}\right| \le \sum_{n=-\infty}^{\infty} \left|x[n]\right|\left|r^{-n}e^{-j\Omega n}\right| = \sum_{n=-\infty}^{\infty} \left|x[n]\right| r^{-n} < \infty$$

  Thus, the ROC depends only on r and not on Ω.

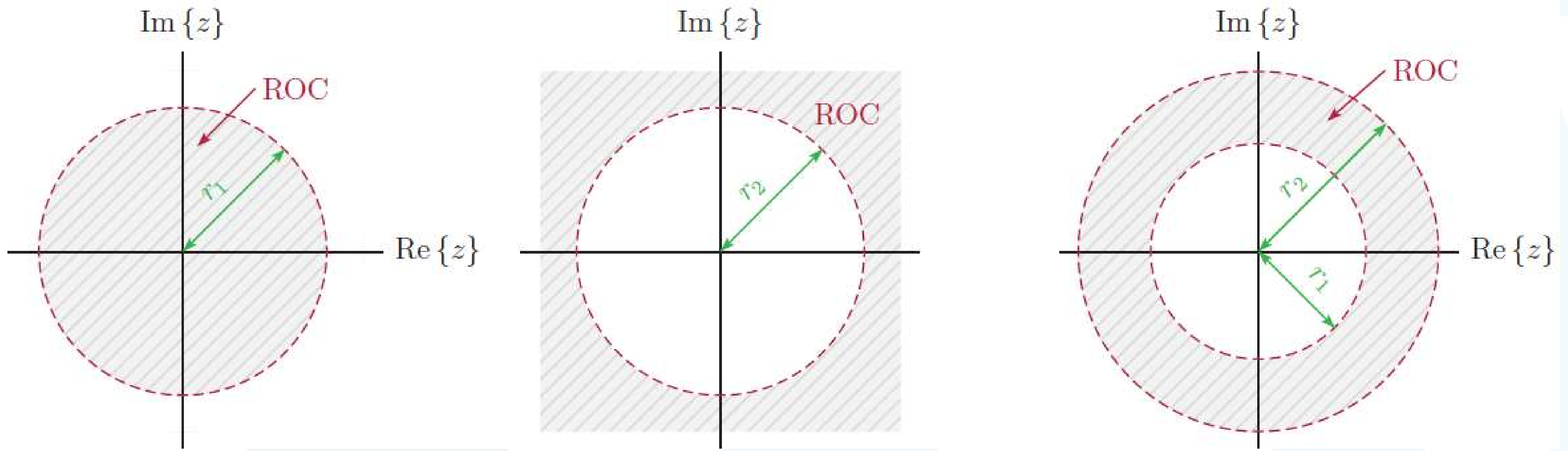
- The region of convergence (ROC) of the z-transform of a signal $x[n]$ of finite support $[N_0, N_1]$, where $-\infty < N_0 \le n \le N_1 < \infty$, is the whole z-plane, excluding the origin $z = 0$ and/or $z = \pm\infty$ depending on N_0 and N_1. $X(z) = \sum_{n=N_0}^{N_1} x[n]z^{-n}$

- ROC of Infinite-Support Signals:

1. Causal signal $x[n]$ has a ROC $|z| > r_1$, where r_1 is the largest radius of the poles of $X(z)$,

2. anticausal signal $x[n]$ has as region of convergence the inside of the circle defined by the smallest radius r_2 of the poles of $X(z)$, or $|z| < r_2$,

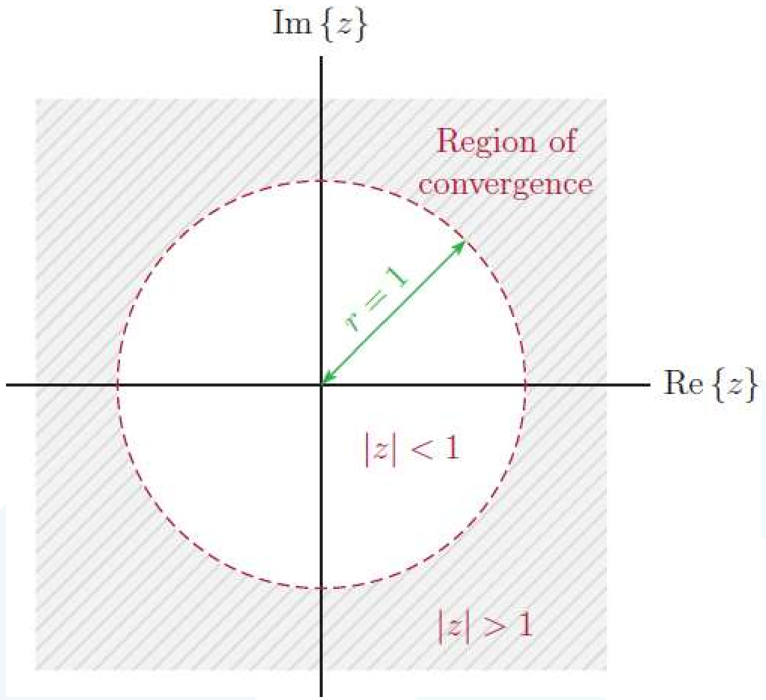
3. noncausal signal $x[n]$ has as ROC $r_1 < |z| < r_2$, where r_1 and r_2 corresponds to the maximum and minimum radii of the poles of $X_c(z)$ and $X_{ac}(z)$: the z-transforms of the causal and anticausal components of $x[n] = x_c[n] + x_{ac}[n]$.



- **Example 5:** z-Transform of the unit-step signal

$$X(z) = Z\{u[n]\} = \sum_{n=0}^{\infty} z^{-n} = \frac{1}{1 - z^{-1}} = \frac{z}{z - 1}$$

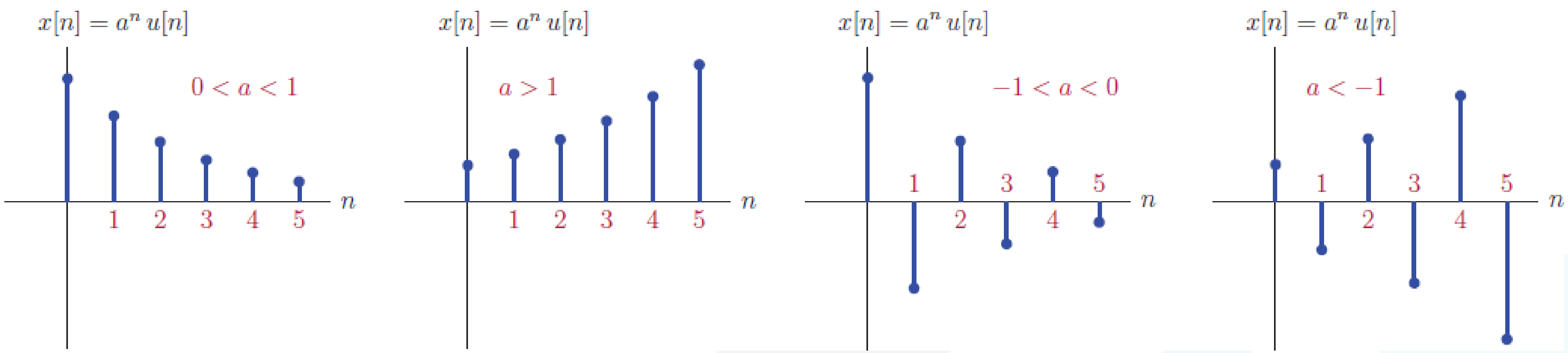
converge if: $|z^{-1}| < 1 \Rightarrow |z| > 1$

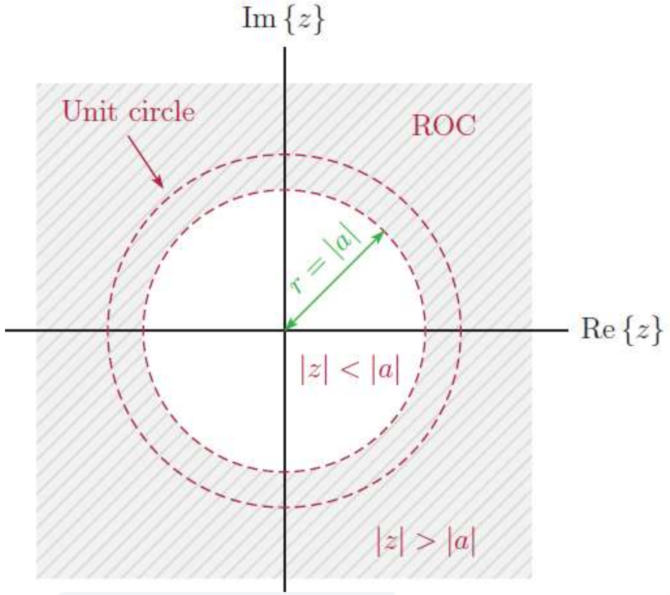


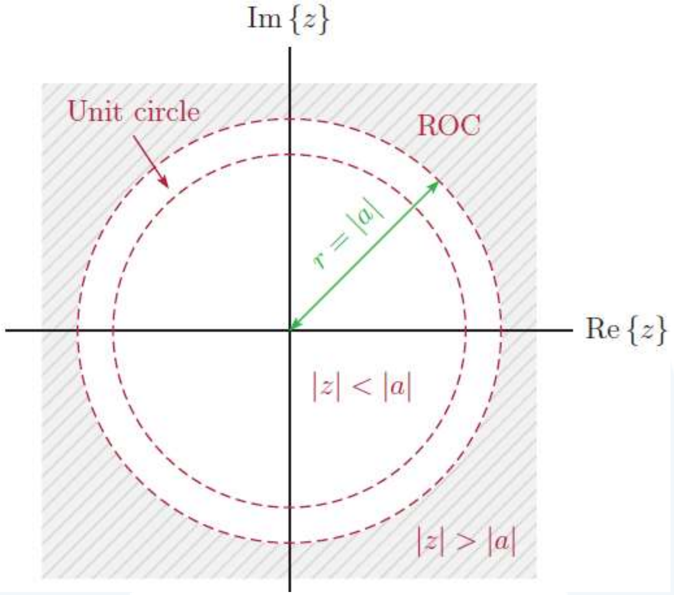
- **Example 6:** z-Transform of a causal exponential signal $x[n] = a^n u[n]$

$$X(z) = \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}$$

converge if: $|az^{-1}| < 1 \Rightarrow |z| > |a|$


$x[n] = a^n u[n]$
$0 < a < 1$
$x[n] = a^n u[n]$
$a > 1$
$x[n] = a^n u[n]$
$-1 < a < 0$
$x[n] = a^n u[n]$
$a < -1$
n
1 2 3 4 5

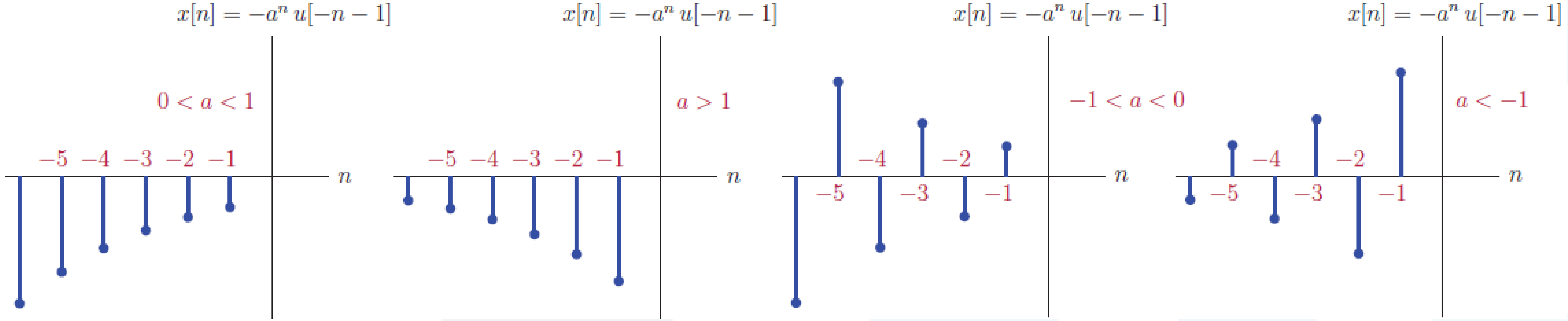

Im $\{z\}$
Unit circle
ROC
$r = |a|$
Re $\{z\}$
$|z| < |a|$
$|z| > |a|$


Im $\{z\}$
Unit circle
ROC
$r = |a|$
Re $\{z\}$
$|z| < |a|$
$|z| > |a|$

- Example 7: z-Transform of an anti-causal exponential signal $x[n] = -a^n u[-n-1]$

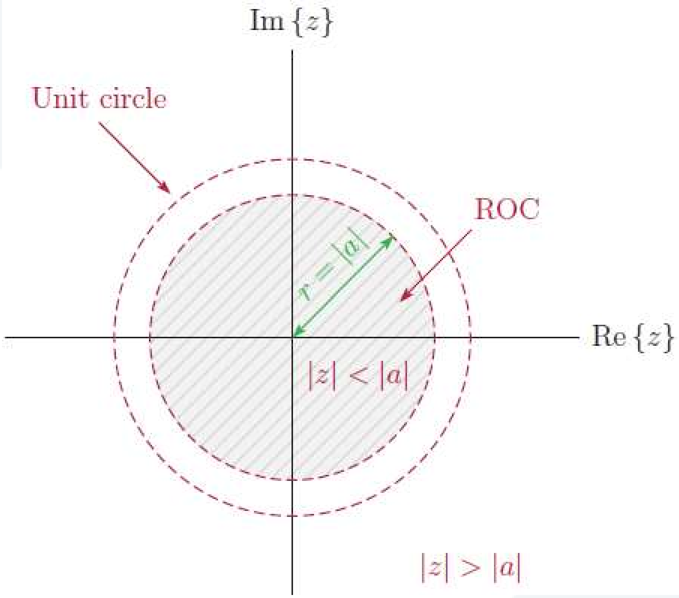
$$X(z) = \sum_{n=-\infty}^{\infty} -a^n u[-n-1] z^{-n} = -\sum_{n=-\infty}^{-1} a^n z^{-n} = -\sum_{m=1}^{\infty} a^{-m} z^m = -\sum_{m=1}^{\infty} (a^{-1}z)^m$$

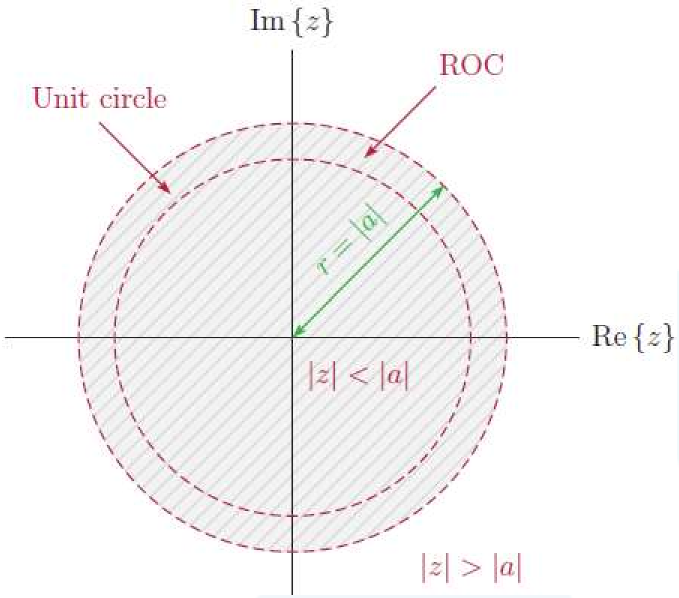
$$= -a^{-1}z \frac{1}{1-a^{-1}z} = \frac{z}{z-a}$$ converge if: $|a^{-1}z| < 1 \Rightarrow |z| < |a|$



$$\mathcal{Z}\{a^n u[n]\} = \frac{z}{z-a}, \text{ ROC: } |z| > |a|; \quad \mathcal{Z}\{-a^n u[-n-1]\} = \frac{z}{z-a}, \text{ ROC: } |z| < |a|$$

- Note: two different signals have the same $X(z) \Rightarrow$ the ROC must be specified.



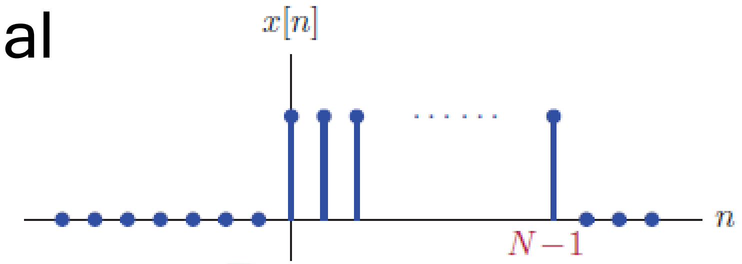


- In the general case, a rational transform $X(z)$ is expressed in the form

$$X(z) = K\,\frac{B(z)}{A(z)} = K\,\frac{(z - z_1)(z - z_2)\cdots(z - z_M)}{(p - p_1)(p - p_2)\cdots(p - p_N)}$$

Max(M, N) is the order of the transform $X(z)$

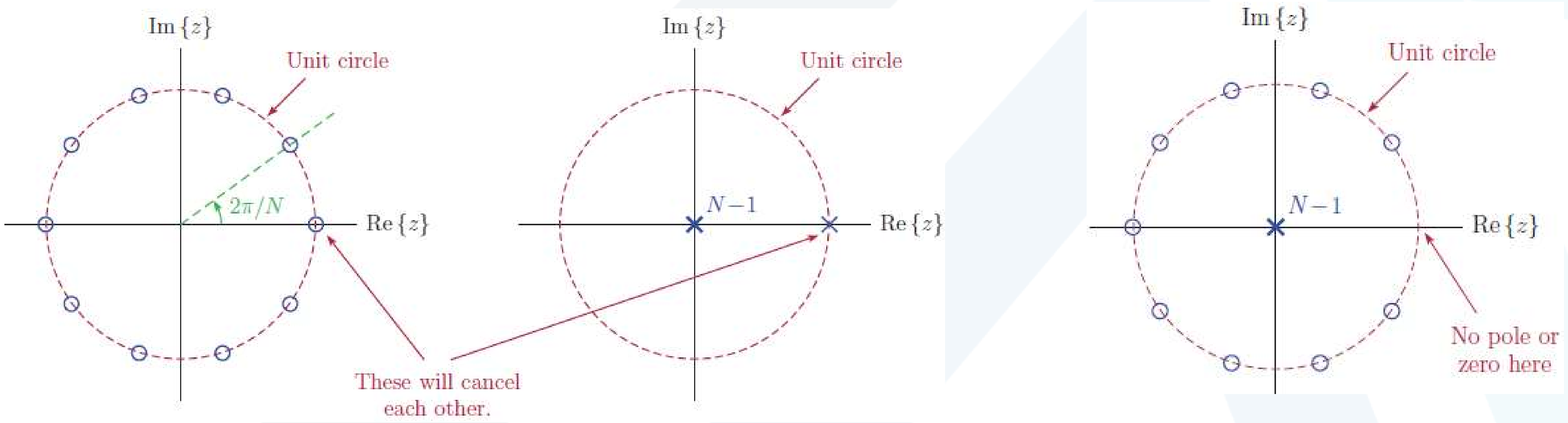
- Example 8: z-Transform of a discrete-time pulse signal



$$X(z)=\sum_{n=0}^{N-1}(1)z^{-n}=\frac{1-z^{-N}}{1-z^{-1}}=\frac{z^N-1}{z^{N-1}(z-1)},\ \text{ROC: } |z|>0$$

It seems as though $X(z)$ might have a pole at $z=1$

Zeros: $z_k=e^{j2\pi k/N},\ k=1,\cdots,N-1$ Poles: $z=1$ and $p_k=0,\ k=1,\cdots,N-1$



Properties of z-Transform

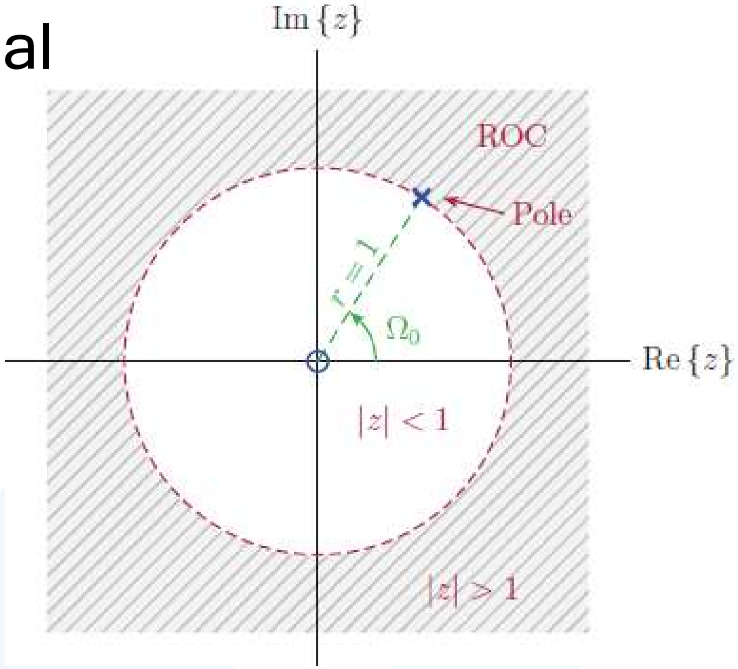
Property	$x[n]$	$X(z)$	ROC
Linearity	$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	$\supset (R_1 \cap R_2)$
Time shifting	$x[n - k]$	$X(z)z^{-k}$	$R \pm \{0 \text{ or } \infty\}$
Time reversal	$x[-n]$	$X(z^{-1})$	R^{-1}
Multiply by exp.	$x[n]a^n$	$X(z/a)$	$\|a\|R$
Differentiate in z	$nx[n]$	$-z\,dX(z)/dz$	R
Convolution	$x_1[n] * x_2[n]$	$X_1(z)\,X_2(z)$	$\supset (R_1 \cap R_2)$
Summation	$\sum_{k=-\infty}^{n} x[k]$	$\frac{z}{z-1}X(z)$	$\supset (R \cap (z > 1))$

- Example 9: z-Transform of complex exponential signal

$$x[n] = e^{j\Omega_0 n}u[n]$$

$$X(z) = \sum_{n=0}^{\infty} e^{j\Omega_0 n} z^{-n} = \sum_{n=0}^{\infty} \left(e^{j\Omega_0} z^{-1}\right)^n = \frac{1}{1 - e^{j\Omega_0} z^{-1}}$$

$$X(z) = \frac{z}{z - e^{j\Omega_0}} \qquad \text{ROC: } \left|e^{j\Omega_0} z^{-1}\right| < 1 \Rightarrow |z| > 1$$



- Example 10: z-Transform of a cosine and sine signals

$$x[n] = \cos(\Omega_0 n)u[n] \qquad x[n] = \sin(\Omega_0 n)u[n]$$

$$\cos(\Omega_0 n)u[n] = \tfrac{1}{2} e^{j\Omega_0 n}u[n] + \tfrac{1}{2} e^{-j\Omega_0 n}u[n] \qquad \sin(\Omega_0 n)u[n] = \tfrac{1}{2j} e^{j\Omega_0 n}u[n] - \tfrac{1}{2j} e^{-j\Omega_0 n}u[n]$$

$$\mathcal{Z}\{\cos(\Omega_0 n)u[n]\} = \tfrac{1}{2}\mathcal{Z}\{e^{j\Omega_0 n}u[n]\} + \tfrac{1}{2}\mathcal{Z}\{e^{-j\Omega_0 n}u[n]\}$$

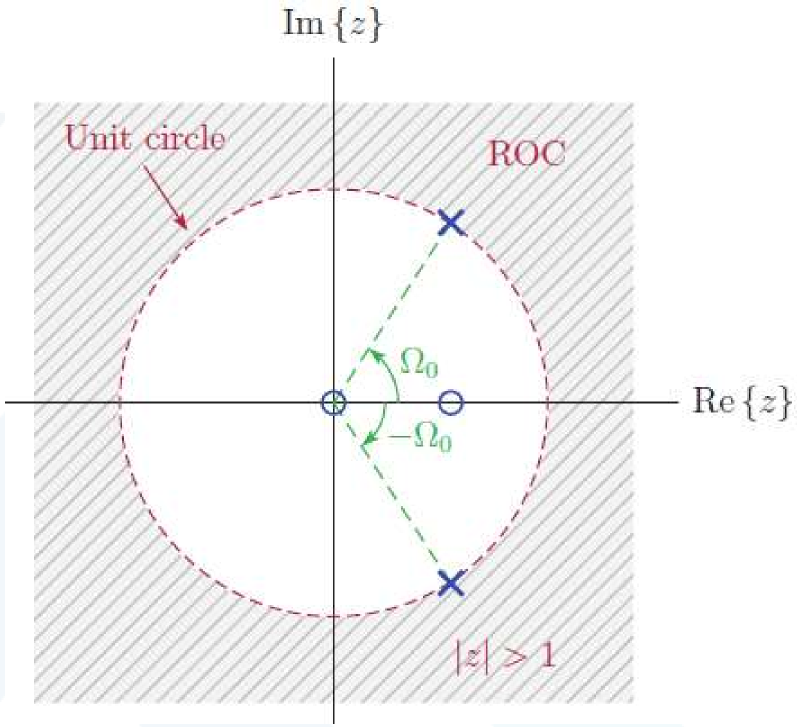
$$= \frac{1/2}{1-e^{j\Omega_0}z^{-1}} + \frac{1/2}{1-e^{-j\Omega_0}z^{-1}} = \frac{1-\cos(\Omega_0)z^{-1}}{1-2\cos(\Omega_0)z^{-1}+z^{-2}}$$

$$= \frac{z[z-\cos(\Omega_0)]}{z^2-2\cos(\Omega_0)z+1}$$ ROC is $|z|>1$

$$\mathcal{Z}\{\sin(\Omega_0 t)u[n]\} = \tfrac{1}{2j}\mathcal{Z}\{e^{j\Omega_0 n}u[n]\} - \tfrac{1}{2j}\mathcal{Z}\{e^{-j\Omega_0 n}u[n]\}$$

$$= \frac{1/2j}{1-e^{j\Omega_0}z^{-1}} - \frac{1/2j}{1-e^{-j\Omega_0}z^{-1}} = \frac{\sin(\Omega_0)z^{-1}}{1-2\cos(\Omega_0)z^{-1}+z^{-2}}$$

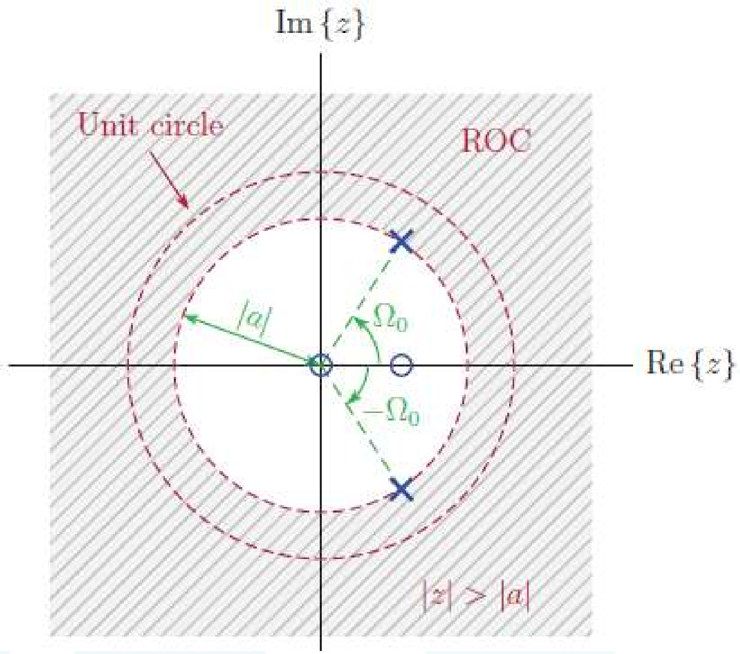
$$= \frac{\sin(\Omega_0)z}{z^2-2\cos(\Omega_0)z+1}$$ ROC is $|z|>1$



- Example 11: Multiplication by an exponential signal: $x[n] = a^n \cos(\Omega_0 n) u[n]$

$$x[n] = a^n x_1[n], \quad X_1(z) = \frac{z[1 - \cos(\Omega_0)]}{z^2 - 2\cos(\Omega_0)z + 1}$$

$$X_1(z) = X_1(z/a) = \frac{z[z - a\cos(\Omega_0)]}{z^2 - 2a\cos(\Omega_0)z + a^2}$$



The transform $X(z)$ has two poles at:

$$z = ae^{\pm j\Omega_0} \Rightarrow \text{ROC: } |z| > |a|$$

- Example 12: Using the differentiation property: $x[n] = na^n u[n]$

$$\mathcal{Z}\{a^n u[n]\} = \frac{z}{z - a}, \quad \text{ROC: } |z| > |a|$$

$$X(z) = (-z)\frac{d}{dz}\frac{z}{z-a} = \frac{az}{(z-a)^2}, \quad \text{ROC: } |z| > |a|$$

z-Transform of a unit-ramp signal $x[n] = nu[n]$

Setting $a = 1 \Rightarrow$ $X(z) = \left.\frac{az}{(z-a)^2}\right|_{a=1} = \frac{z}{(z-1)^2}, \quad \text{ROC: } |z| > 1$

- Example 13: Using the convolution property $x_1[n] = \{\underset{\uparrow}{4}, 3, 2, 1\}, \quad x_2[n] = \{\underset{\uparrow}{3}, 7, 4\}$

  Determine $x[n] = x_1[n] * x_2[n]$ using z-transform techniques.

$$X_1(z) = 4 + 3z^{-1} + 2z^{-2} + z^{-3}, \quad X_2(z) = 3 + 7z^{-1} + 4z^{-2}$$

$$X(z) = X_1(z)X_2(z) = 12 + 37z^{-1} + 43z^{-2} + 29z^{-3} + 15z^{-4} + 4z^{-5}$$

$$x[n] = \{\underset{\uparrow}{12}, 37, 43, 29, 15, 4\}$$

- Example 14: Determine $x[n] = x_1[n] * x_2[n]$ using z-transform techniques

$$x_1[n] = \{\underset{\uparrow}{4}, 3, 2, 1\}, \quad x_2[n] = \{\underset{\uparrow}{3}, 7, 4\}$$

$$X_1(z) = 4 + 3z^{-1} + 2z^{-2} + z^{-3}, \quad X_2(z) = 3 + 7z^{-1} + 4z^{-2}$$

$$X(z) = X_1(z)X_2(z) = 12 + 37z^{-1} + 43z^{-2} + 29z^{-3} + 15z^{-4} + 4z^{-5}$$

$$x[n] = \{\underset{\uparrow}{12}, 37, 43, 29, 15, 4\}$$

- Example 15: Finding the output signal of a DTLTI system using inverse z-transform: $h[n] = (0.9)^n u[n]$, $x[n] = u[n] - u[n-7]$

$$H(z) = \mathcal{Z}\{h[n]\} = \frac{z}{z-0.9}, \quad \text{ROC: } |z| > 0.9$$

$$X(z) = \sum_{n=0}^{6} z^{-n} = 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5} + z^{-6} = \frac{z^7 - 1}{z^6(z-1)}, \quad \text{ROC: } |z| > 0$$

$$Y(z) = X(z)H(z)$$
$$= H(z) + z^{-1}H(z) + z^{-2}H(z) + z^{-3}H(z) + z^{-4}H(z) + z^{-5}H(z) + z^{-6}H(z)$$
$$y[n] = h[n] + h[n-1] + h[n-2] + h[n-3] + h[n-4] + h[n-5] + h[n-6]$$
$$y[n] = (0.9)^{n}u[n] + (0.9)^{n-1}u[n-1] + (0.9)^{n-2}u[n-2] + (0.9)^{n-3}u[n-3]$$
$$+ (0.9)^{n-4}u[n-4] + (0.9)^{n-5}u[n-5] + (0.9)^{n-6}u[n-6]$$

Initial and final value Theorems

Initial and final value properties of the z-transform applies to causal signals only.

Initial value: $x[0] = \lim_{z\to\infty} X(z)$ Final value: $\lim_{n\to\infty} x[n] = \lim_{z\to 1}(z-1)X(z)$

- Example 16: Determine the initial value $x[0]$ of the signal $X(z) = \dfrac{3z^3 + 2z + 5}{2z^3 - 7z^2 + z - 4}$

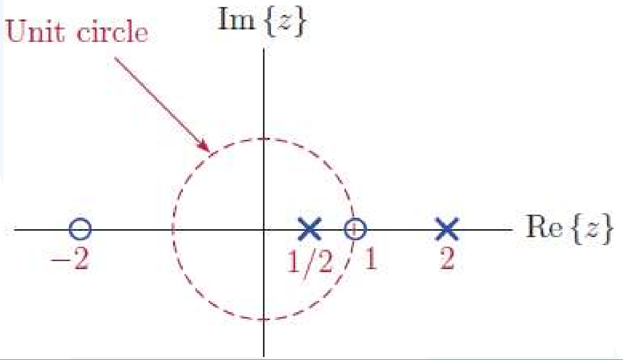
$$x[0] = \lim_{z\to\infty} \frac{3z^3 + 2z + 5}{2z^3 - 7z^2 + z - 4} = \frac{3}{2}$$

2. Inverse Z-Transform

- Recall that the inverse z-transform x of X is given by: $x[n] = \frac{1}{2\pi j}\oint X(z)z^{n-1}dz$ where Γ is a counterclockwise closed circular contour centered at the origin and with radius r such that Γ is in the ROC of X.
- For rational functions, the inverse z-transform can be more easily computed using partial fraction expansions (PFE).
- Example 17: Finding the inverse z-transform using PFE $X(z) = \dfrac{(z-1)(z+2)}{(z-1/2)(z-2)}$

$$\frac{X(z)}{z} = \frac{(z-1)(z+2)}{z(z-1/2)(z-2)} = \frac{-2}{z} + \frac{\frac{5}{3}}{(z-\frac{1}{2})} + \frac{\frac{4}{3}}{(z-2)}$$

$$X(z) = -2 + \frac{\frac{5}{3}z}{(z-\frac{1}{2})} + \frac{\frac{4}{3}z}{(z-2)} = X_1(z) + X_2(z) + X_3(z)$$



$X_1(z)$, is a constant, and its ROC is the entire z-plane. $x_1[n] = \mathcal{Z}^{-1}\{-2\} = -2\delta[n]$

Possibility 1: ROC: $|z| < ½$

$X_2(z)$ and $X_3(z)$ must correspond to anti-causal signals:

ROC for $X_2(z)$: $|z| < ½$

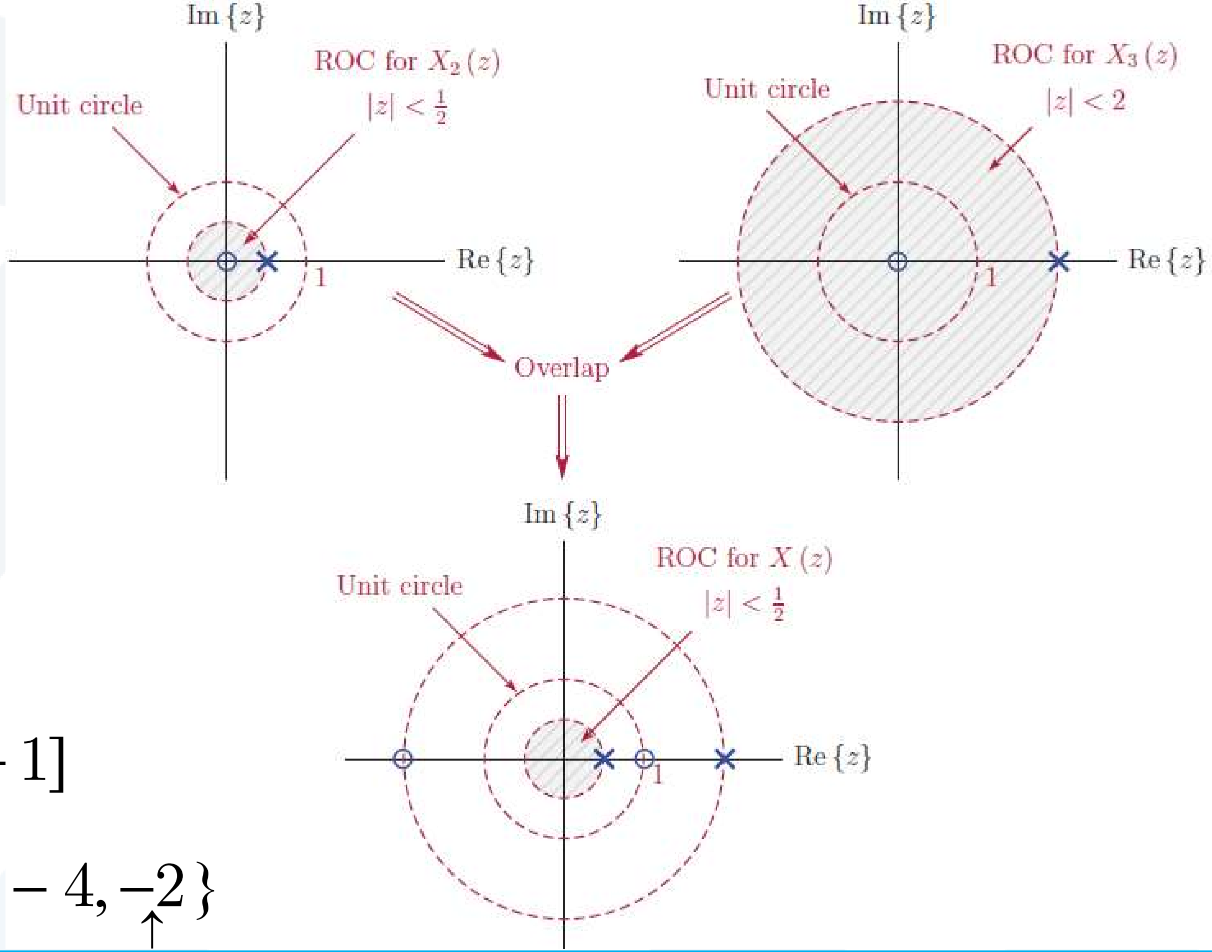
ROC for $X_3(z)$: $|z| < 2$

$$x_2[n] = -\frac{5}{3}\left(\frac{1}{2}\right)^n u[-n-1]$$

$$x_3[n] = -\frac{4}{3}\left(2\right)^n u[-n-1]$$

$$x[n] = -2\delta[n] - \left[\frac{5}{3}\left(\frac{1}{2}\right)^n + \frac{4}{3}\left(2\right)^n\right]u[-n-1]$$

$$x[n] = \{\cdots, -53.375, -26.75, -13.5, -7, -4, \underset{\uparrow}{-2}\}$$



Possibility 2: ROC: $|z| > 2$

$X_2(z)$ and $X_3(z)$ must correspond to causal signals. We need:

ROC for $X_2(z)$: $|z| > ½$

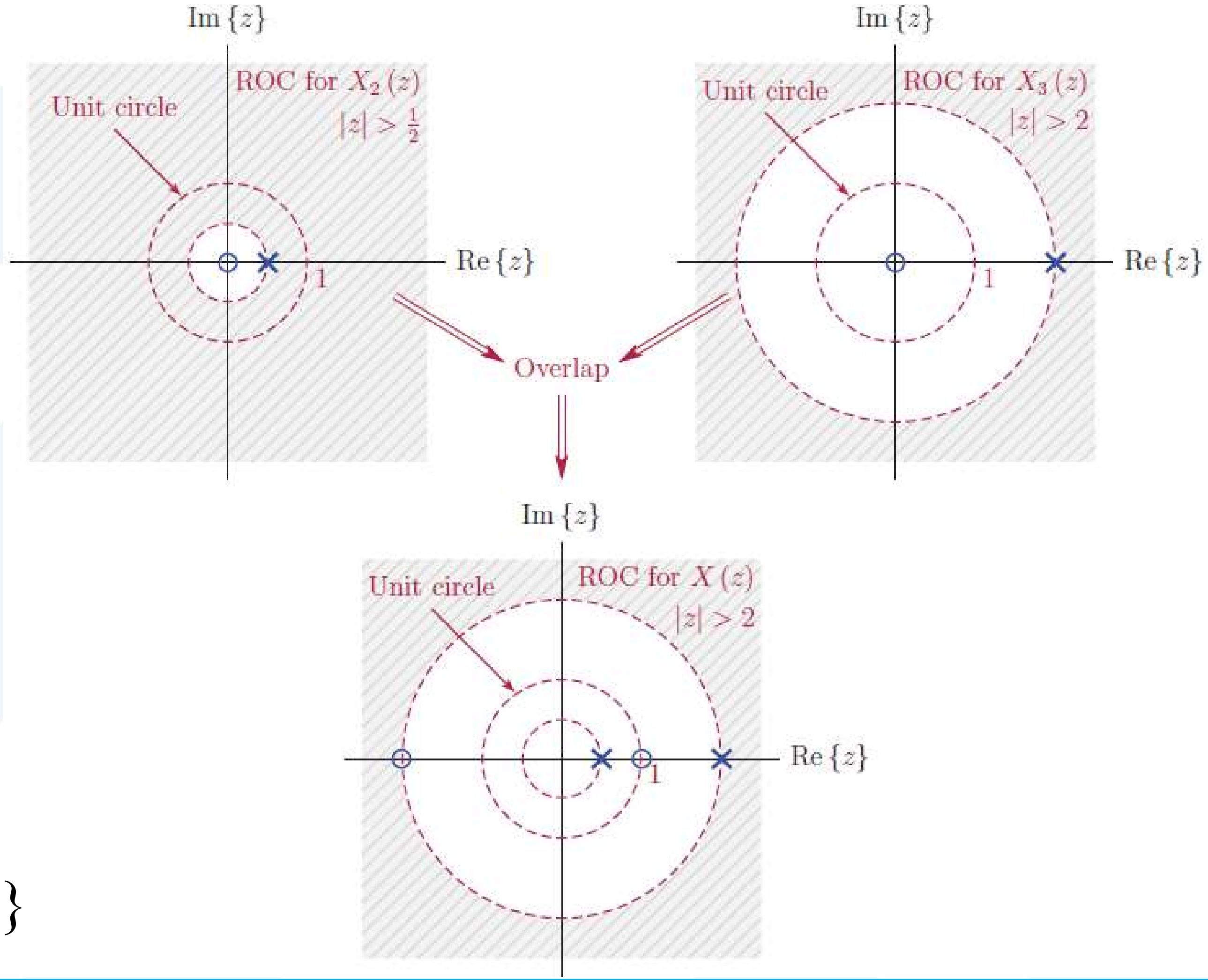
ROC for $X_3(z)$: $|z| > 2$

$$x_2[n] = \frac{5}{3}\left(\frac{1}{2}\right)^n u[n]$$

$$x_3[n] = \frac{4}{3}\left(2\right)^n u[n]$$

$$x[n] = -2\delta[n] + \left[\frac{5}{3}\left(\frac{1}{2}\right)^n + \frac{4}{3}\left(2\right)^n\right]u[n]$$

$$x[n] = \{\underset{\uparrow}{1}, 3.5, 5.75, 8.208, 13.385, \cdots\}$$



Possibility 3: ROC: ½ < $|z|$ < 2

$X_2(z)$ and $X_3(z)$ must correspond to noncausal signals. We need:

ROC for $X_2(z)$: $|z| > ½$

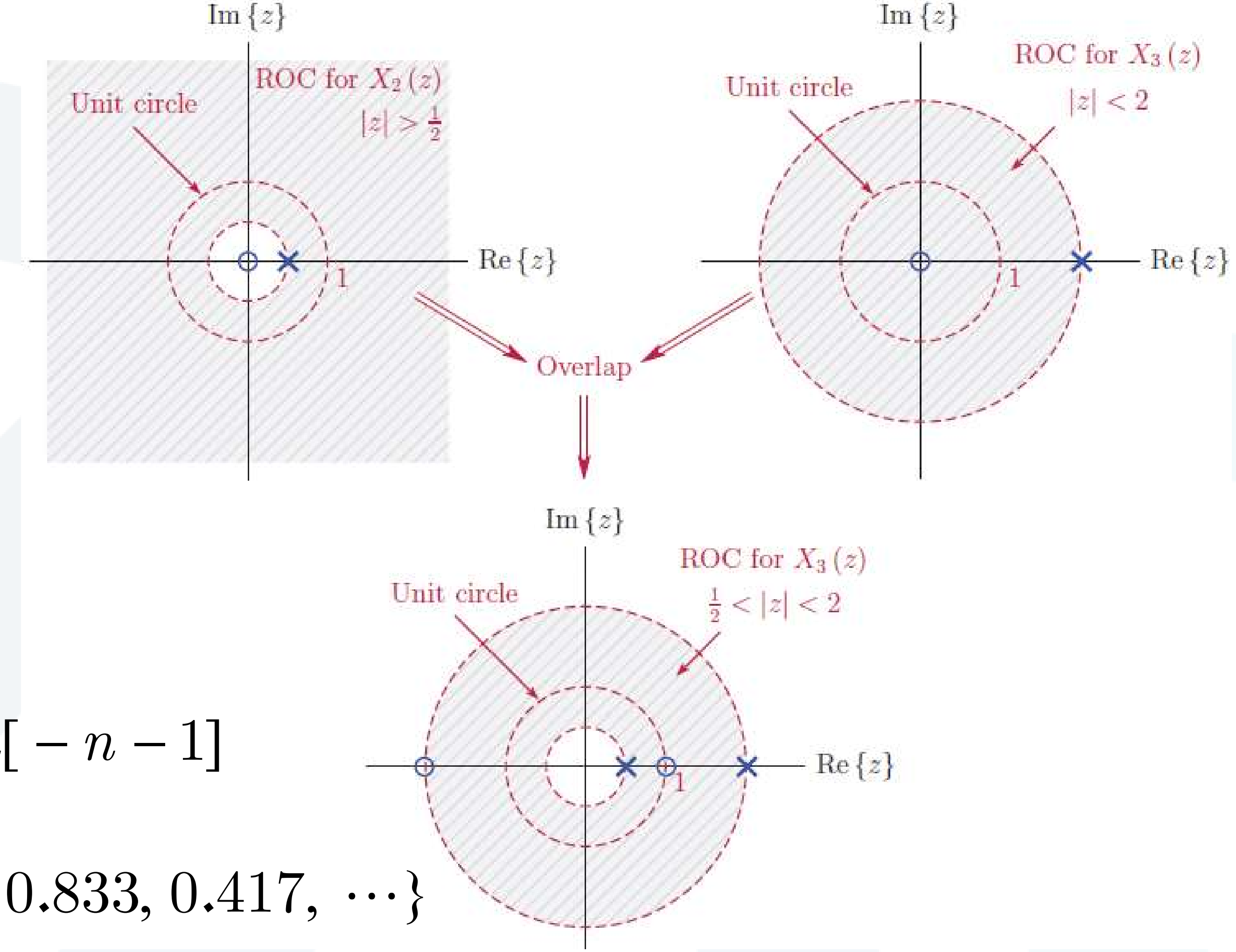
ROC for $X_3(z)$: $|z| < 2$

$$x_2[n] = \frac{5}{3}\left(\frac{1}{2}\right)^n u[n]$$

$$x_3[n] = -\frac{4}{3}(2)^n u[-n-1]$$

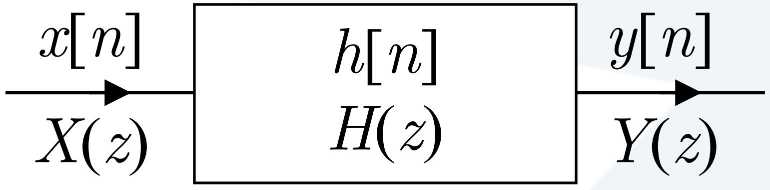
$$x[n] = -2\delta[n] + \frac{5}{3}\left(\frac{1}{2}\right)^n u[n] - \frac{4}{3}(2)^n u[-n-1]$$

$$x[n] = \{\cdots, -0.333, -0.667, \underset{\uparrow}{-0.333}, 0.833, 0.417, \cdots\}$$



3. Using the Z-Transform with DTLTI Systems

Transfer Function and LTI Systems



Block Diagram Representation

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

$$Y(z) = X(z)H(z)$$

- Since $y[n] = x[n] * h[n]$ (zero-state response), the system is characterized in the z-domain by $Y(z) = X(z)H(z)$.
- $H(z)$ is the transfer function (or system function) of the system (i.e., the transfer function is the z-Transform of the impulse response).
- A LTI system is completely characterized by its transfer function H.

Relating the transfer function to the difference equation

- Many DTLTI systems of practical interest can be represented using an Nth-order linear difference equation with constant coefficients.
- Consider a system with input x and output y that is characterized by the DE:

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

where the a_k and b_k are complex constants and

$$\mathcal{Z}\left\{\sum_{k=0}^{N} a_k y[n-k]\right\} = \mathcal{Z}\left\{\sum_{k=0}^{M} b_k x[n-k]\right\} \Rightarrow \sum_{k=0}^{N} \mathcal{Z}\{a_k y[n-k]\} = \sum_{k=0}^{M} \mathcal{Z}\{b_k x[n-k]\}$$

$$\sum_{k=0}^{N} a_k \mathcal{Z}\{y[n-k]\} = \sum_{k=0}^{M} b_k \mathcal{Z}\{x[n-k]\}$$

$$\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z) \Rightarrow H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

- The impulse response of the system $h[n] = \mathcal{Z}^{-1}\{H(z)\}$.
- The transfer function concept is meaningful only for LTI systems that are both linear and time invariant.
- In determining the transfer function from the difference equation, all initial conditions must be assumed to be zero.
- Example 18: A DTLTI system is defined by means of the difference equation:

$$y[n] - 0.4y[n-1] + 0.89y[n-2] = x[n] - x[n-1]$$

Find the transfer function $H(z)$ for this system.

$$Y(z) - 0.4z^{-1}Y(z) + 0.89z^{-2}Y(z) = X(z) - z^{-1}X(s)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 - z^{-1}}{1 - 0.4z^{-1} + 0.89z^{-2}} = \frac{z(z-1)}{z^2 - 0.4z + 0.89}$$

Response of a DTLTI system to a complex exponential signal $x[n] = z_0^n$

where z_0 represents a point in the z-plane within the ROC of the TF.

$$y[n] = h[n] * x[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] = \sum_{k=-\infty}^{\infty} h[k]z_0^{n-k} = z_0^n H(z_0)$$

- Complex geometric sequences are eigenfunctions of DTLTI systems.

Transfer function and causality

- For a DTLTI system to be causal, its impulse response $h[n]$ needs to be equal to zero for $n < 0$. $H(z) = \sum_{k=-\infty}^{\infty} h[n]z^{-n} = \sum_{k=0}^{\infty} h[n]z^{-n}$
- The ROC for the transfer function of a causal system is the outside of a circle in the z-plane. Consequently, the transfer function must also converge at $|z| \to \infty$. Consider a transfer function in the form:

$$H(z)=\frac{Y(z)}{X(z)}=\frac{b_M z^M+b_{M-1}z^{M-1}+\cdots+b_1 z+b_0}{a_N z^N+a_{N-1}z^{N-1}+\cdots+a_1 z+a_0}$$

For $H(z)$ to be causal: $\lim_{z\to\infty} H(z)=\lim_{z\to\infty}\frac{b_M}{a_N}z^{M-N}<\infty \Leftrightarrow M-N\le 0 \Rightarrow M\le N$

- Note: this condition is necessary for a system to be causal, but it is not sufficient. It is also possible for a non-causal system to have a TF with $M \le N$.

Transfer function and stability:

- For a DTLTI system to be stable $\sum_{k=-\infty}^{\infty}\left|h[n]\right|z^{-n}<\infty$.
- The FT of a signal exists if the signal is absolute summable. But the FT of the impulse response is equal to the z-domain transfer function evaluated on the unit circle of the z-plane, that is,

$H(\Omega) = H(z)\Big|_{z=e^{j\Omega}}$ provided that the unit circle of the z-plane is within the ROC.

Stability condition:

- For a DTLTI system to be stable, the ROC of its z-domain transfer function must include the unit circle.
- For a causal system to be stable, the transfer function must not have any poles on or outside the unit circle of the z-plane.
- For an anticausal system to be stable, the transfer function must not have any poles on or inside the unit circle of the z-plane.
- For a noncausal system the ROC for the transfer function, if it exists, is the region between two circles with radii r_1 and r_2, $r_1 < |z| < r_2$. The poles of the transfer function may be either:

a. On or inside the circle with radius r_1

b. On or outside the circle with radius r_2

and the ROC must include the unit circle.

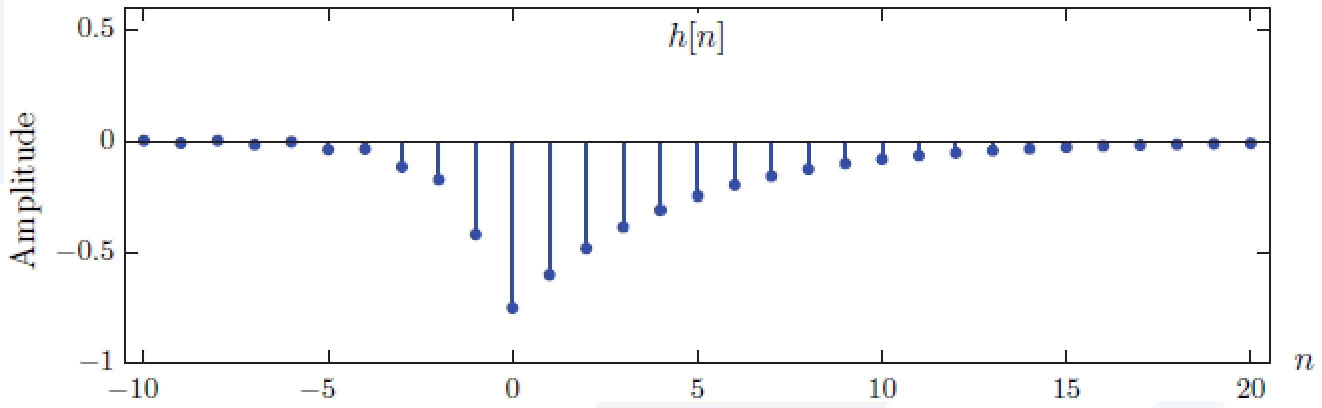
- Example 19: Determine the impulse response of a stable system:

$$H(z)=\frac{z(z+1)}{(z-0.8)(z+1.2)(z-2)}$$

The poles are at $p=-1.2$, 0.8, 2. Since the system is stable, its ROC must include the unit circle. The only possible choice is $0.8<|z|<1.2$.

$$H(z)=\frac{z(z+1)}{(z-0.8)(z+1.2)(z-2)}=-\frac{0.75z}{z-0.8}-\frac{0.0312z}{z+1.2}+\frac{0.7813z}{z-2}$$

$$h[n]=-0.75(0.8)^n u[n]+0.0312(-1.2)^n u[-n-1]-0.7813(2)^n u[-n-1]$$



- Example 20: A DTLTI system is characterized by the difference equation:

$$y[n] = x[n-1] + 3x[n-2] + 2x[n-3] + 2.3y[n-1] - 2y[n-2] + 1.2y[n-3]$$

Comment on the stability of this system.

$$y[n] - 2.3y[n-1] + 2y[n-2] - 1.2y[n-3] = x[n-1] + 3x[n-2] + 2x[n-3]$$

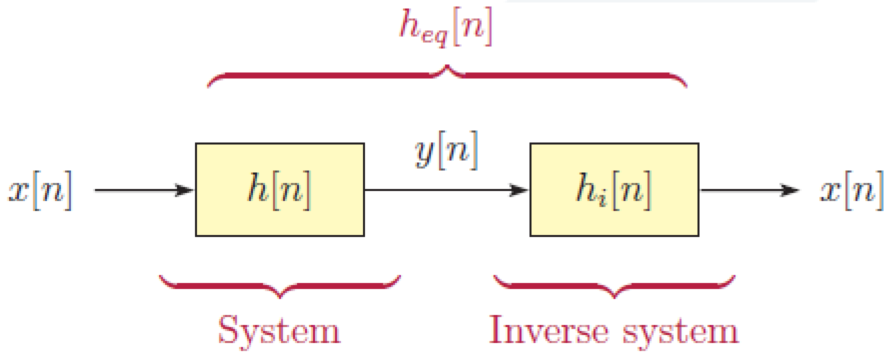
$$H(z) = \frac{z^2 + 3z + 2}{z^3 - 2.3z^2 + 2z - 1.2} = \frac{(z+1)(z+2)}{(z - 0.4 + j0.8)(z - 0.4 - j0.8)(z - 1.5)}$$

Since the system is known to be causal (form of the difference equation), its ROC must be the outside of a circle. ROC: $|z| > 1.5$

Since the ROC does not include the unit circle, the system is unstable.

Inverse systems

- The inverse of a system is another system which, when connected in cascade with the original system, forms an identity system.
- Let the original system $h[n]$ and its inverse $h_i[n]$ be both DTLTI systems.



$$h_{eq}[n] = h[n] * h_i[n] = \delta[n]$$

$$H_{eq}(z) = H(z)H_i(z) \Rightarrow H_i(z) = \frac{1}{H(z)}$$

- Two important characteristics of the inverse system are causality and stability.

$$H(z) = \frac{B(z)}{A(z)} = \frac{b_M z^M + b_{M-1} z^{M-1} + \cdots + b_1 z + b_0}{a_N z^N + a_{N-1} z^{N-1} + \cdots + a_1 z + a_0}$$

$$H_i(z) = \frac{A(z)}{B(z)} = \frac{a_N z^N + a_{N-1} z^{N-1} + \cdots + a_1 z + a_0}{b_M z^M + b_{M-1} z^{M-1} + \cdots + b_1 z + b_0}$$

- If the original system $H(z)$ is causal then $M \le N$. Causality of the inverse system $H_i(z)$ requires $N \le M$. Hence, we need $N = M$ if both the original system and its inverse are required to be causal.
- To analyze the stability of the inverse system, using $N = M$ we have:

$$H(z) = \frac{b_N (z - z_1)(z - z_2) \cdots (z - z_N)}{a_N (z - p_1)(z - p_2) \cdots (z - p_N)}$$ $H(s)$ in pole-zero form

- If the original system is both causal and stable, $|p_k| < 1$, $k = 1, \ldots, N$.

$$H_i(z) = \frac{a_N(z-p_1)(z-p_2)\cdots(z-p_N)}{b_N(z-z_1)(z-z_2)\cdots(z-z_N)}$$

- For the inverse system to be stable, both zeros and poles of the original system must be inside the unit circle: $|p_k| < 1$, and $|z_k| < 1$, $k = 1, \ldots, N$.
- A causal DTLTI system that has all of its zeros and poles inside the unit circle is referred to as a minimum-phase system. A minimum-phase system and its inverse are both causal and stable.

- Example 21: A DTLTI system is described by a difference equation

$$y[n] = 0.1y[n-1] + 0.72y[n-2] + x[n] + 0.5x[n-1]$$

Determine if a causal and stable inverse can be found for this system.

$$(1-0.1z^{-1}-0.72z^{-2})Y(z)=(1+0.5z^{-1})X(z) \Rightarrow H(z)=\frac{1+0.5z^{-1}}{1-0.1z^{-1}-0.72z^{-2}}$$

$$H(z)=\frac{z(z+0.5)}{(z+0.8)(z-0.9)}$$

Since all poles and zeros are inside the unit circle, $H(z)$ is a minimum phase system.

$$H_i(z)=\frac{(z+0.8)(z-0.9)}{z(z+0.5)}=\frac{1-0.1z^{-1}-0.72z^{-2}}{1+0.5z^{-1}}$$

causal and stable. Its difference equation is:

$$y[n]=-0.5y[n-1]+x[n]-0.1x[n-1]-0.72x[n-2]$$

4. Simulation Structures for DTLTI Systems

Direct-form implementation

- The general form of the z-domain transfer function for a DTLTI system is:

$$H(z)=\frac{Y(z)}{X(z)}=\frac{b_0 z^{M}+b_1 z^{-1}+b_2 z^{-2}+\cdots+b_M z^{-M}}{1+a_1 z^{-1}+a_2 z^{-2}+\cdots+a_N z^{-N}}$$

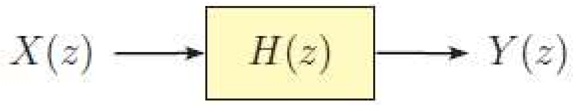
- The method of obtaining a block diagram from an z-domain TF will be derived using a third-order system, but its generalization to higher-order TF is quite straightforward. Consider a DTLTI system described by a TF $H(z)$:

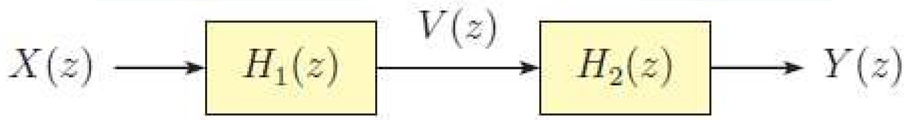
$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3}}{1 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3}}$$

Let us use an intermediate function $V(z)$

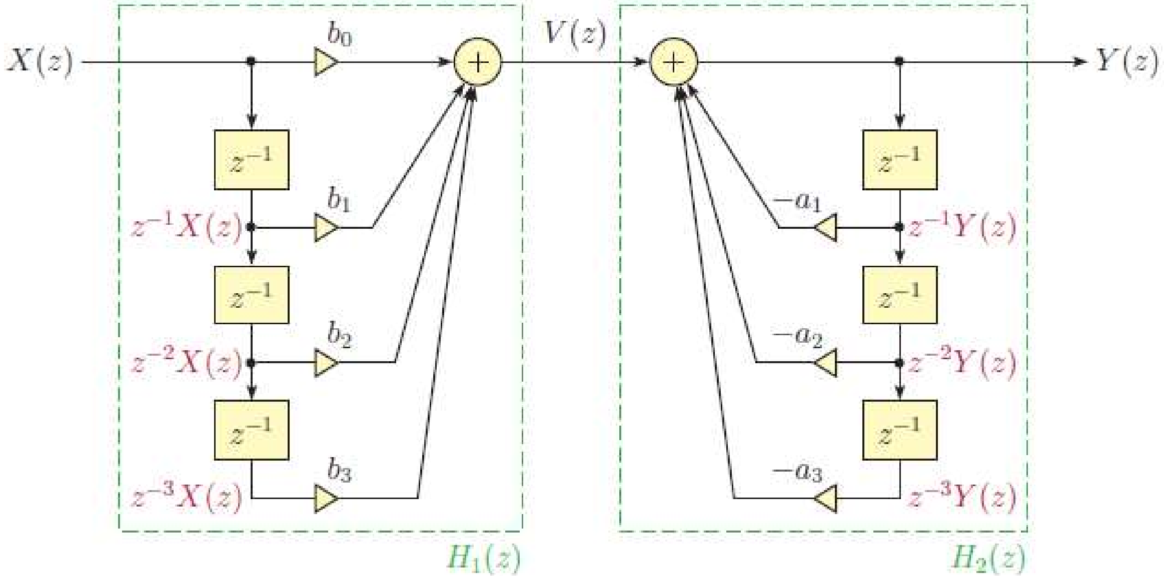
$$H(z) = H_1(z) H_2(z) = \frac{Y(z)}{V(z)} \frac{V(z)}{X(z)}$$

$$H_1(z) = \frac{V(z)}{X(z)} = b_0 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3}, \quad H_2(z) = \frac{Y(z)}{V(z)} = \frac{1}{1 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3}}$$

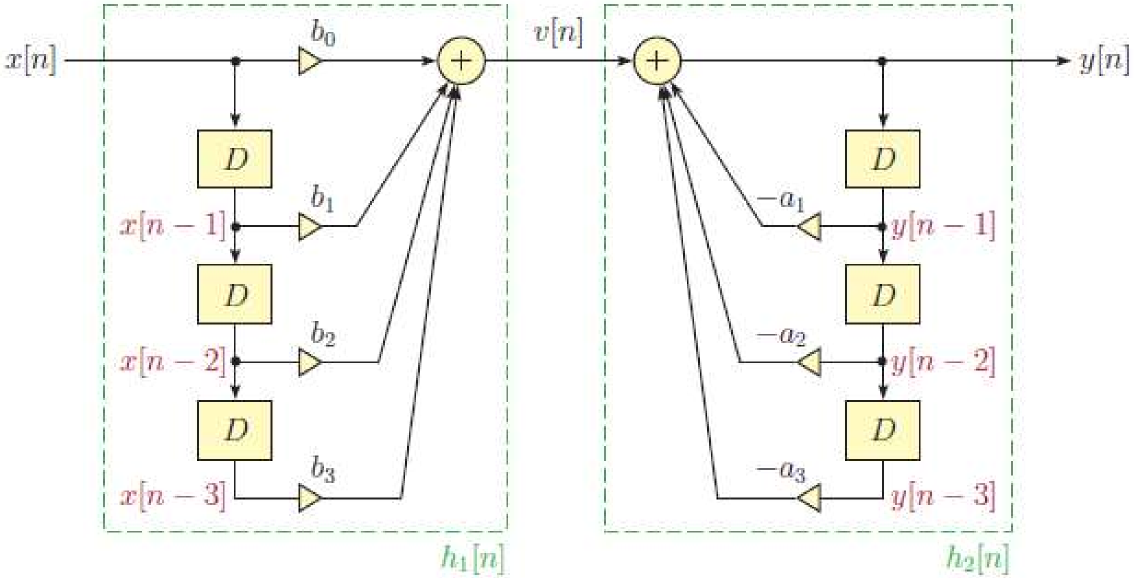




$$V(z) = b_0X(z) + b_1z^{-1}X(z) + b_2z^{-2}X(z) + b_3z^{-3}X(z)$$
$$v[n] = b_0x[n] + b_1x[n-1] + b_2x[n-2] + b_3x[n-3]$$
$$Y(z) = V(z) - a_1z^{-1}Y(z) - a_2z^{-2}Y(z) - a_3z^{-3}Y(z)$$
$$y[n] = v[n] - a_1y[n-1] - a_2y[n-2] - a_3y[n-3]$$



Direct-form I realization of $H(z)$



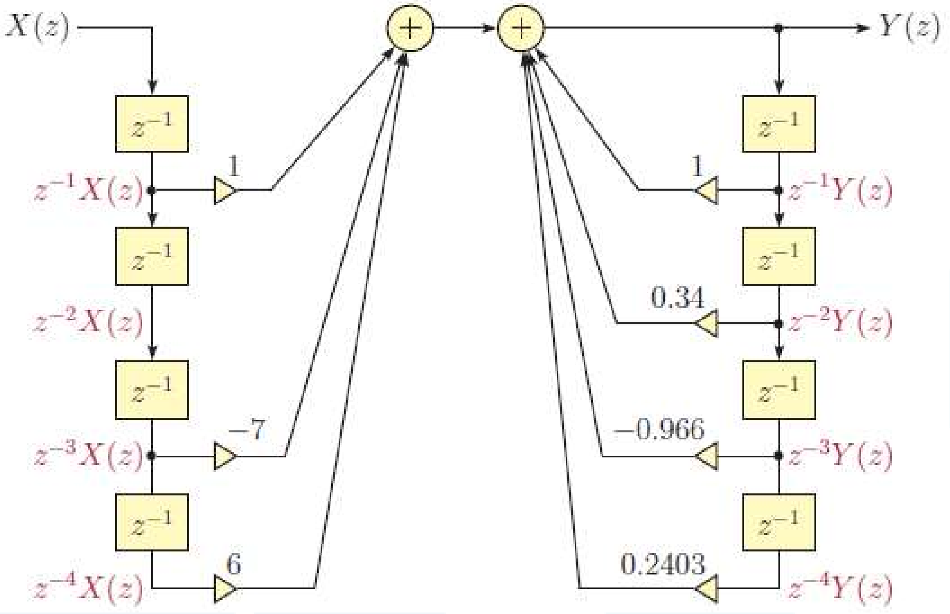
Direct-form I realization of $H(z)$ using time-domain quantities

- Example 22: A causal DTLTI system is described by the transfer function

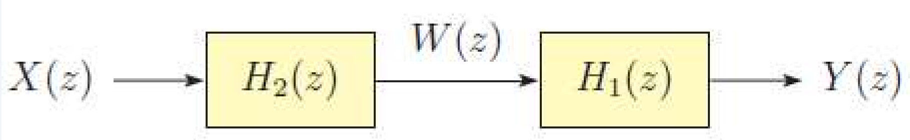
$$H(z)=\frac{z^3-7z+6}{z^4-z^3-0.34z^2+0.966z-0.2403}$$

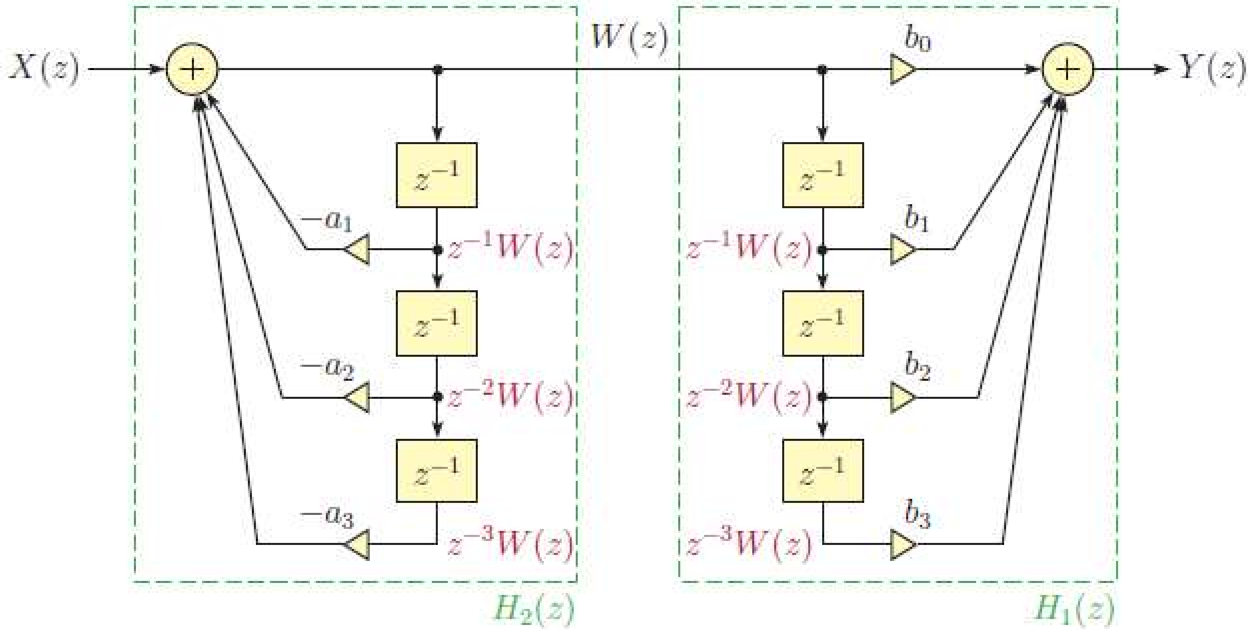
Draw a direct-form type-I block diagram for implementing this system.

$$H(z)=\frac{z^{-1}-7z^{-3}+6z^{-4}}{1-z^{-1}-0.34z^{-2}+0.966z^{-3}-0.2403z^{-4}}$$

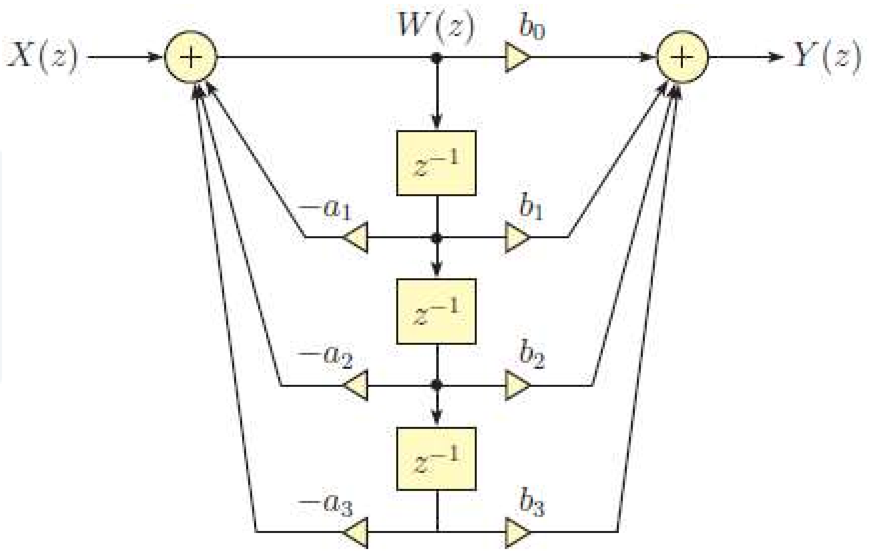


Since each subsystem, $H_1(z)$ and $H_2(z)$, is linear, it does not matter which one comes first in a cascade connection.





Block diagram obtained by swapping the order of two subsystems

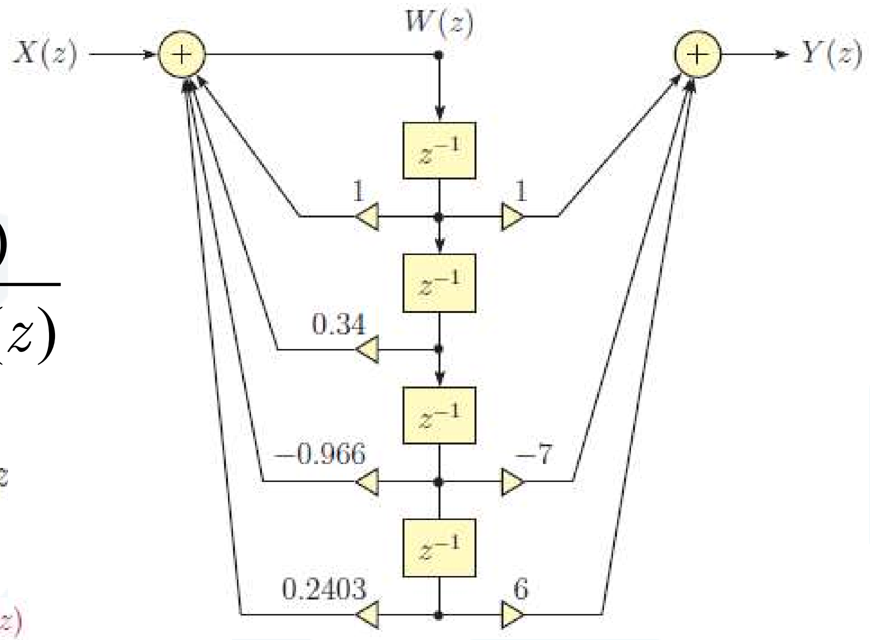


Direct-form II realization of $H(z)$

- Example 23: Direct-form type-II implementation of a system

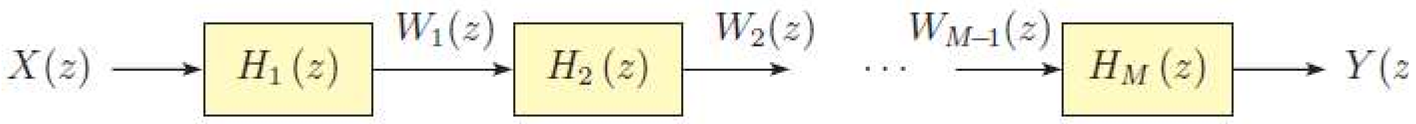
  Draw a direct-form II block diagram for implementing the system used in Example 22.

Cascade and parallel forms



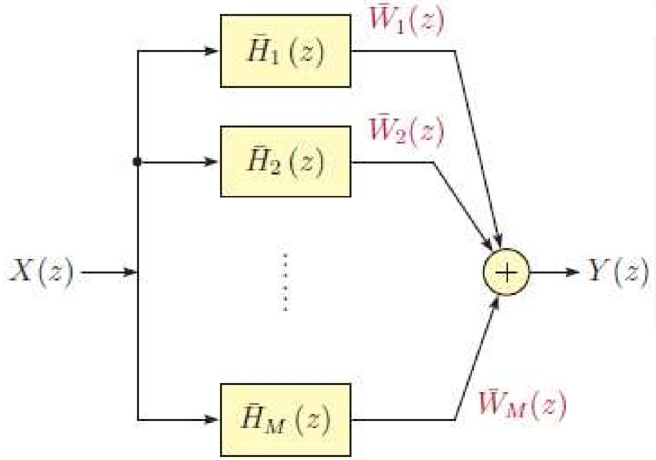
Cascade form

$$H(z) = H_1(z)H_2(z)\cdots H_M(z) = \frac{W_1(z)}{X(z)}\frac{W_2(z)}{W_1(z)}\cdots\frac{Y(z)}{W_{M-1}(z)}$$



Parallel form

$$H(z) = \bar{H}_1(z) + \bar{H}_2(z) + \cdots + \bar{H}_M(z)$$
$$= \frac{\bar{W}_1(z)}{X(z)} + \frac{\bar{W}_2(z)}{X(z)} + \cdots + \frac{\bar{W}_M(z)}{X(z)}$$



- Example 24: Cascade form block diagram

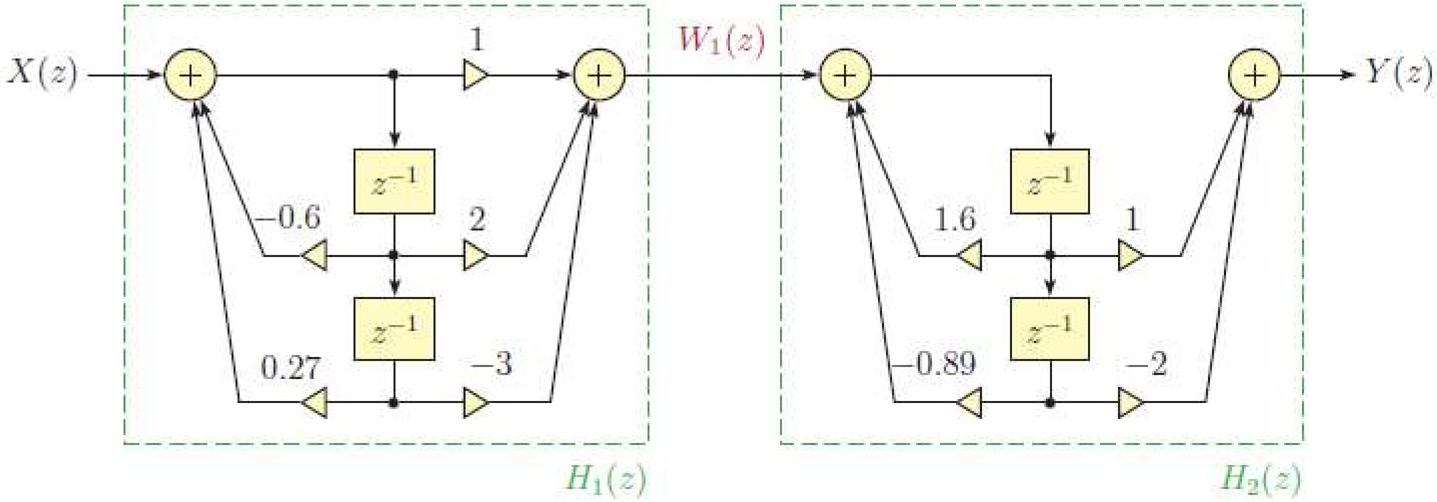
  Develop a cascade form block diagram for the DTLTI system used in Example 22

$$H(z) = \frac{(z+3)(z-1)(z-2)}{(z+0.9)(z-0.3)(z-0.8-j0.5)(z-0.8+j0.5)}$$

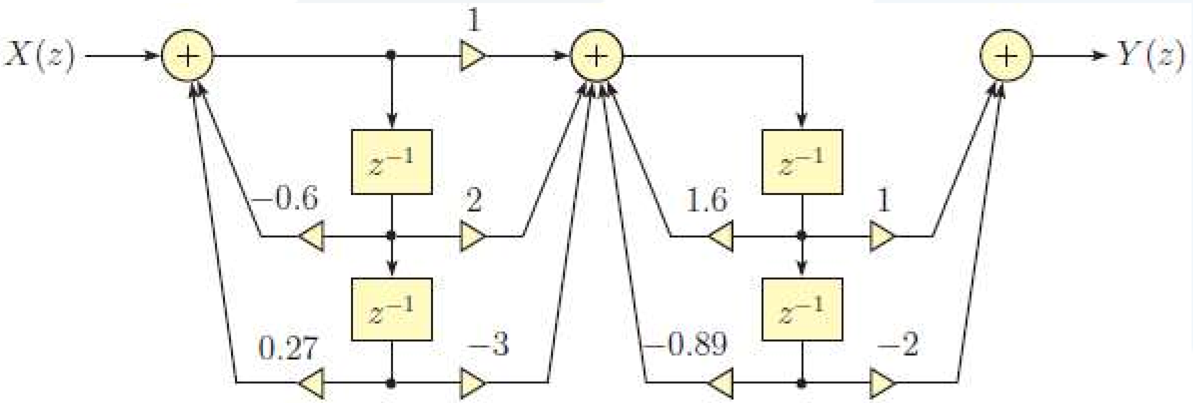
The system function can be broken down into cascade sections in a number of ways. Let us choose to have two second-order sections

$$H_1(z) = \frac{(z+3)(z-1)}{(z+0.9)(z-0.3)} = \frac{z^2+2z-3}{z^2+0.6z-0.27}$$

$$H_2(z) = \frac{(z-2)}{(z-0.8-j0.5)(z-0.8+j0.5)} = \frac{z-2}{z^2-1.6z+0.89}$$



It is also possible to consolidate the neighboring adders in the middle



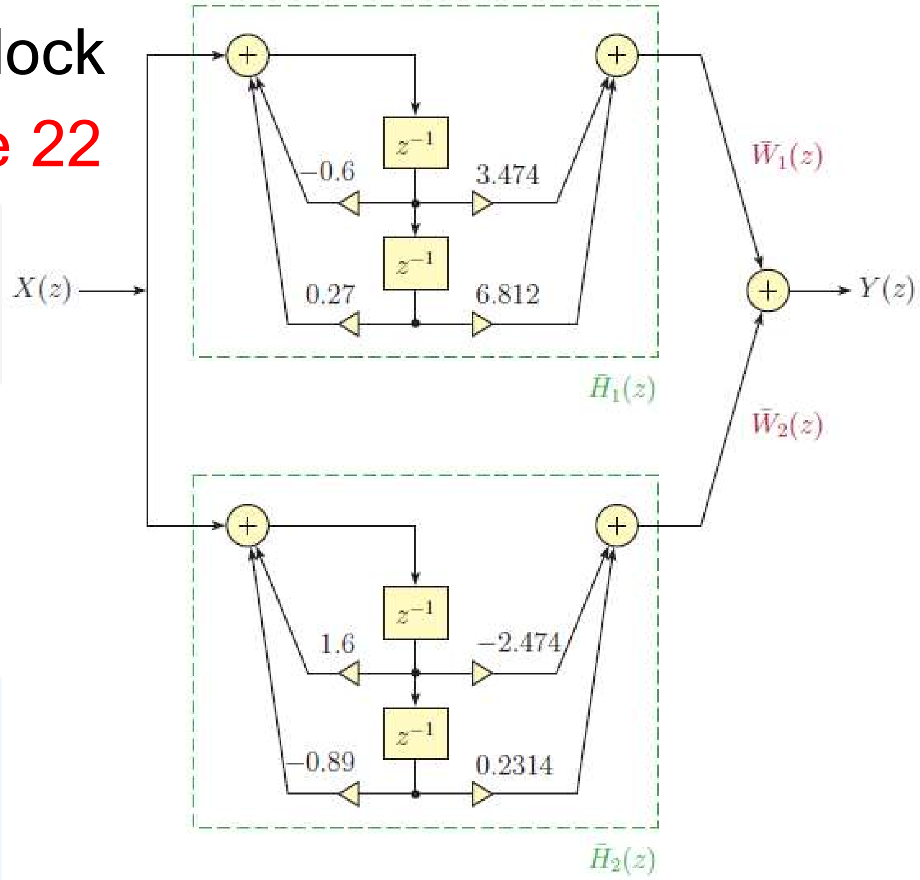
- Example 25: Develop a parallel form block diagram for the DTLTI system used in Example 22

$$H(z)=\frac{-3.0709}{z+0.9}+\frac{6.5450}{z-0.3}+\frac{-1.2371+j1.7480}{z-0.8-j0.5}+\frac{-1.2371-j1.7480}{z-0.8+j0.5}$$

Let us use two second-order sections

$$\bar{H}_1(z)=\frac{-3.0709}{z+0.9}+\frac{6.5450}{z-0.3}=\frac{3.474z+6.812}{z^2+0.6z-0.27}$$

$$\bar{H}_2(z)=\frac{-1.2371+j1.7480}{z-0.8-j0.5}+\frac{-1.2371-j1.7480}{z-0.8+j0.5}=\frac{-2.474z+0.2314}{z^2-1.6z+0.89}$$



5. Unilateral Z-Transform

The unilateral z-transform of $x[n]$ is defined as: $X_u(z) = \mathcal{Z}_u\{x[n]\} = \sum_{n=0}^{\infty} x[n]z^{-n}$

- If $x[n]$ is a causal signal, then $X_u(z)$ becomes identical to $X(z)$.
- The unilateral ZT is related to the bilateral z-transform as follows:

$$\mathcal{Z}_u\{x[n]\} = \mathcal{Z}\{x[n]u[n]\} = \sum_{n=-\infty}^{\infty} x[n]u[n]z^{-n}$$

- One property of the unilateral z-transform that differs from its counterpart for the bilateral z-transform is the time-shifting property.

$$\mathcal{Z}\{x[n-1]\} = z^{-1}\mathcal{Z}\{x[n]\} = z^{-1}X(z)$$

$$\mathcal{Z}_u\{x[n-1]\} = \sum_{n=0}^{\infty} x[n-1]z^{-n} = x[-1] + \sum_{n=1}^{\infty} x[n-1]z^{-n} = x[-1] + z^{-1}\sum_{n=0}^{\infty} x[n]z^{-n}$$

$$\mathcal{Z}_u\{x[n-1]\} = x[-1] + z^{-1}X_u(z)$$

$$\mathcal{Z}_u\{x[n-k]\} = \sum_{n=-k}^{-1} x[n]z^{-n-k} + z^{-k}X_u(z), \quad k > 0$$

$$\mathcal{Z}_u\{x[n+k]\} = z^k X_u(z) - \sum_{n=0}^{k-1} x[n]z^{k-n}, \quad k > 0$$

- Note: The unilateral z-transform is useful in the use of z-transform techniques for solving difference equations with specified initial conditions.
- Example 26: Using z-transform techniques, determine the natural response of the system for the initial conditions: $y[-1] = 19$, $y[-2] = 53$.

$$y[n] - \frac{5}{6}y[n-1] + \frac{1}{6}y[n-2] = 0$$

$$\mathcal{Z}_u\{y[n-1]\} = y[-1] + z^{-1}Y_u(z) = 19 + z^{-1}Y_u(z)$$

$$\mathcal{Z}_u\{y[n-2]\} = y[-2] + y[-1]z^{-1} + z^{-2}Y_u(z) = 53 + 19z^{-1} + z^{-2}Y_u(z)$$

$$Y_u(z) - \tfrac{5}{6}[19 + z^{-1}Y_u(z)] + \tfrac{1}{6}[53 + 19z^{-1} + z^{-2}Y_u(z)] = 0$$

$$Y_u(z) = \frac{z(7z - \frac{19}{6})}{z^2 - \frac{5}{6}z + \frac{1}{6}} = \frac{z(7z - \frac{19}{6})}{(z - \frac{1}{2})(z - \frac{1}{3})} = \frac{2z}{z - \frac{1}{2}} + \frac{5z}{z - \frac{1}{3}}$$

$$y_h[n] = 2(\tfrac{1}{2})^n u[n] + 5(\tfrac{1}{3})^n u[n]$$

- Example 27: Determine the response of this system for the input signal $x[n] = 20\cos(0.2\pi n)$ if the initial value of the output is $y[-1] = 2.5$.

$$y[n] = 0.9y[n-1] + 0.1x[n]$$

$$\mathcal{Z}\{\cos(\Omega_0 n)u[n]\} = \frac{z[z - \cos(\Omega_0)]}{z^2 - 2\cos(\Omega_0)z + 1} \Rightarrow \mathcal{Z}_u\{20\cos(0.2\pi n)\} = \frac{20z[z - \cos(0.2\pi)]}{z^2 - 2\cos(0.2\pi)z + 1}$$

$$\mathcal{Z}_u\{y[n-1]\} = y[-1] + z^{-1}Y_u(z) = 2.5 + z^{-1}Y_u(z)$$

$$\mathcal{Z}_u\{y[n]\} = 0.9\mathcal{Z}_u\{y[n-1]\} + 0.1\mathcal{Z}_u\{x[n]\}$$

$$Y_u(z) = 0.9[2.5 + z^{-1}Y_u(z)] + 0.1\,\frac{20z[z - \cos(0.2\pi)]}{z^2 - 2\cos(0.2\pi)z + 1}$$

$$Y_u(z) = 0.9z^{-1}Y_u(z) + 2.25 + \frac{2z[z - \cos(0.2\pi)]}{z^2 - 2\cos(0.2\pi)z + 1}$$

$$Y_u(z) = \frac{2z^2[z - \cos(0.2\pi)] + 2.25(z - e^{j0.2\pi})(z - e^{-j0.2\pi})}{(z - 0.9)(z - e^{j0.2\pi})(z - e^{-j0.2\pi})}$$

$$Y_u(z) = \frac{2.7129}{z - 0.9} + \frac{0.7685 - j1.4953}{z - e^{j0.2\pi}} + \frac{0.7685 + j1.4953}{z - e^{-j0.2\pi}}$$

The forced response of the system is:

$$y[n] = 2.7129\ (0.9)^n\, u[n] + 1.5371\cos(0.2\pi n)u[n] + 2.9907\sin(0.2\pi n)u[n]$$

Analysis of LTI Systems Represented by Difference Equations

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k],\ y[-k] \text{ for } k = 1, \dots, N$$

$$y[n-N] + \sum_{k=0}^{N-1} \alpha_k y[n-k] = \sum_{k=0}^{M} \beta_k x[n-k], \qquad \alpha_k = \frac{a_k}{a_N}, \beta_k = \frac{b_k}{a_N}$$

$$Y(z) = \frac{B(z)}{A(z)} X(z) + \frac{1}{A(z)} I(z)$$

$$A(z) = \sum_{k=0}^{N} \alpha_k z^{-k}, \qquad \alpha_N = 1 \qquad\qquad B(z) = \sum_{k=0}^{M} \beta_k z^{-k}$$

$$I(z) = \sum_{k=1}^{N} \alpha_k \left(\sum_{n=-k}^{-1} x[n] z^{-n-k} \right), \qquad \alpha_N = 1$$

$$H(z) = \frac{B(z)}{A(z)},\ H_1(z) = \frac{1}{A(z)} \Rightarrow Y(z) = H(z)X(z) + H_1(z)I(z)$$

the complete response: $y[n] = y_{zs}[n] + y_{zi}[n]$, where

$y_{zs}[n] = \mathcal{Z}_u^{-1}\{H(z)X(z)\}$ is the system's zero-state response,

$y_{zi}[n] = \mathcal{Z}_u^{-1}\{H_1(z)I(z)\}$ is the system's zero-input response.

- Example 28: DLTI system described by the DE: $y[n]+3y[n-1]+2y[n-2]=x[n]$ has input $x[n]=2^n u[n]$ and initial conditions $y[-1]=0$ and $y[-2]=1/2$. Determine the zero-input response and the zero-state response of this system.

$$Y(z)+3(y[-1]+z^{-1}Y(z))+2(y[-2]+y[-1]z^{-1}+z^{-2}Y(z)] = X(z)$$

$$Y(z)+3(z^{-1}Y(z))+2(0.5+z^{-2}Y(z)] = \frac{z}{z-2} \Rightarrow Y(z)(1+3z^{-1}+2z^{-2}) = \frac{z}{z-2}-1$$

$$Y(z) = \frac{1}{(1+3z^{-1}+2z^{-2})}\frac{z}{(z-2)} + \frac{-1}{(1+3z^{-1}+2z^{-2})}$$

$$Y(z) = \frac{z^3}{(z-2)(z^2+3z^1+2)} + \frac{-z^2}{(z^2+3z^1+2)}$$

$$Y(z)=\underbrace{\frac{z^3}{(z-2)(z+2)(z+1)}}_{zero-state\ response}+\underbrace{\frac{-z^2}{(z+2)(z+1)}}_{zero-input\ response}$$

$$\frac{Y(z)}{z}=\frac{z^2}{(z-2)(z+2)(z+1)}+\frac{-z}{(z+2)(z+1)}=\frac{-1/3}{z+1}+\frac{1}{z+2}+\frac{1/3}{z-2}+\frac{1}{z+1}+\frac{-2}{z+2}$$

$$Y(z)=\underbrace{\frac{-1/3z}{(z+1)}+\frac{z}{(z+2)}+\frac{1/3z}{(z-2)}}_{zero-state\ response}+\underbrace{\frac{z}{(z+1)}+\frac{-2z}{(z+2)}}_{zero-input\ response}$$

$$y_{zs}[n]=-\tfrac{1}{3}(-1)^n+(-2)^n+\tfrac{1}{3}(2)^n,\quad n\geq 0$$

$$y_{zi}[n]=(-1)^n-2(-2)^n,\quad n\geq 0$$

$$y[n]=\tfrac{2}{3}(-1)^n-(-2)^n+\tfrac{1}{3}(2)^n,\quad n\geq 0$$

Concept Map: relations between CT and DT representations