Undergraduate Course

FINITE ELEMENT METHODS

Spring Element

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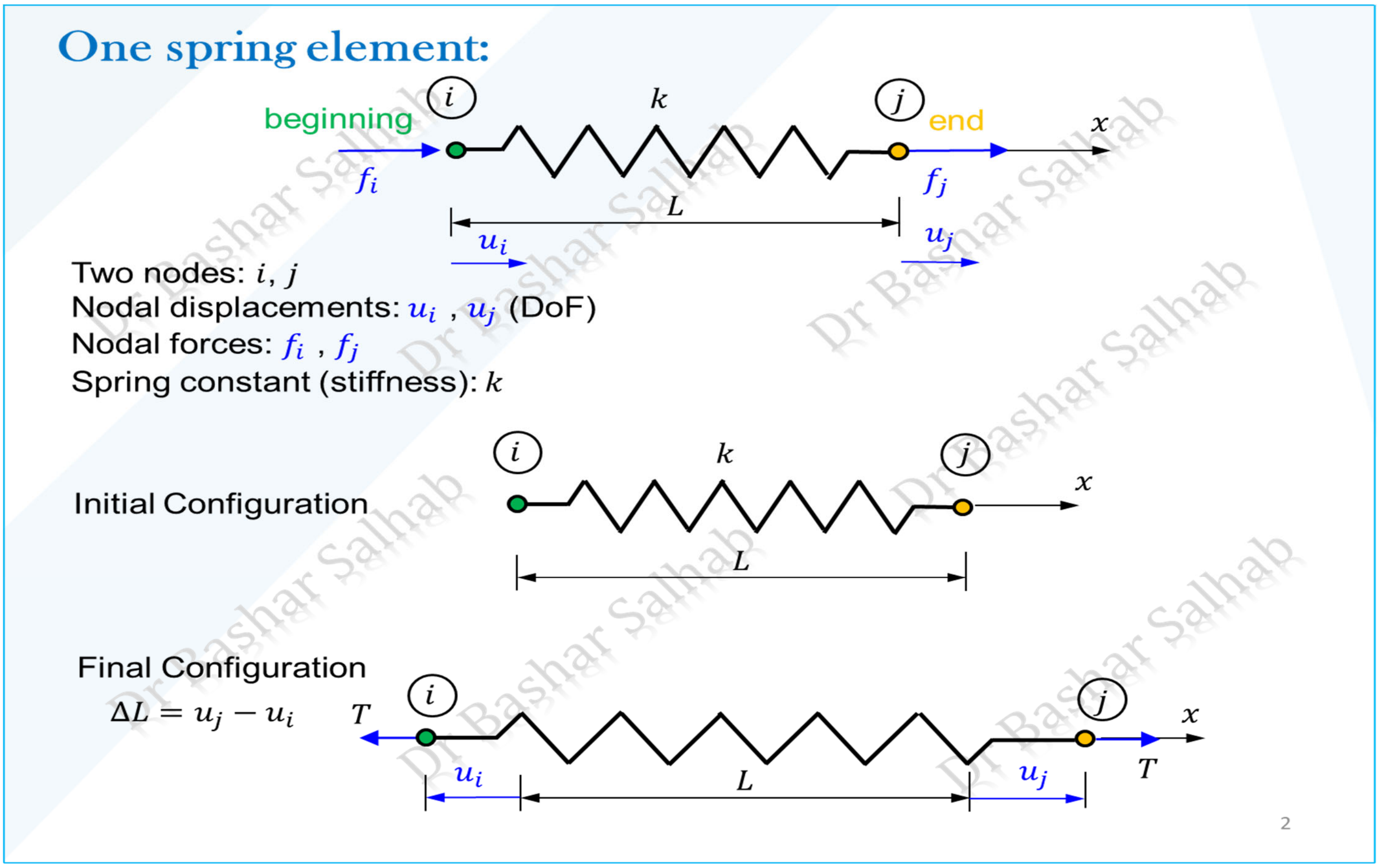


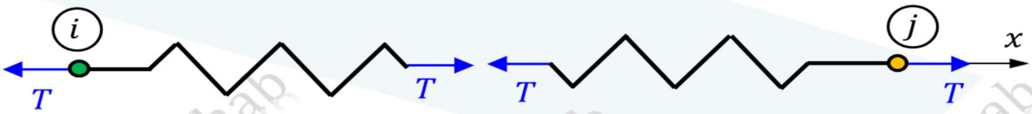
One spring element:

Two nodes: i, j
Nodal displacements: u_i , u_j (DoF)
Nodal forces: f_i , f_j
Spring constant (stiffness): k

Initial Configuration

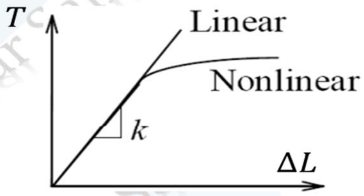
Final Configuration

$\Delta L = u_j - u_i$



Spring force-displacement relationship: $T = k\Delta L$

$T = k(u_j - u_i)$



$k = T/\Delta L\ (> 0)$ is the force needed to produce a unit stretch

Consider the equilibrium of forces for the spring:

At node i : $f_i = -T = -k(u_j - u_i) = ku_i - ku_j$

At node j : $f_j = T = k(u_j - u_i) = -ku_i + ku_j$

In matrix form $\begin{Bmatrix} f_i \\ f_j \end{Bmatrix} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} u_i \\ u_j \end{Bmatrix}$ or $\begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$ or $\{\mathbf{f}\} = [\mathbf{k}]\{\mathbf{u}\}$

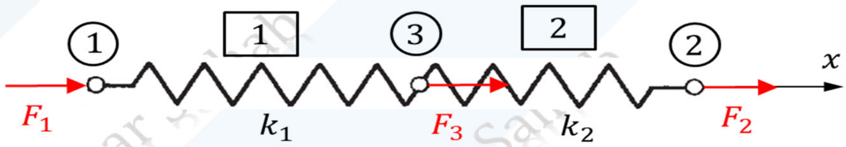
$\mathbf{k} =$ (element) stiffness matrix
$\mathbf{u} =$ (element nodal) displacement vector
$\mathbf{f} =$ (element nodal) force vector

k_{ij} represent the force f_i in the ith degree of freedom due to a unit displacement d_j in the jth degree of freedom while all other displacements are zero.

Notes about the stiffness matrix $[\mathbf{k}]$:

- ☐ It is square, because number of forces is the same as the number of DoF
- ☐ $|\mathbf{k}| = 0 \Longrightarrow$ It is singular & there is no inverse. i.e. can we solve the equation?
- ☐ $\mathbf{k}$ is symmetric.
- ☐ Diagonals are positive (force produces deformation in its direction)

Spring Assemblage:



$\begin{Bmatrix} f_i \\ f_j \end{Bmatrix} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} u_i \\ u_j \end{Bmatrix}$

The local x of the elements is the same as the global x of the assemblage ⇒ No transformation

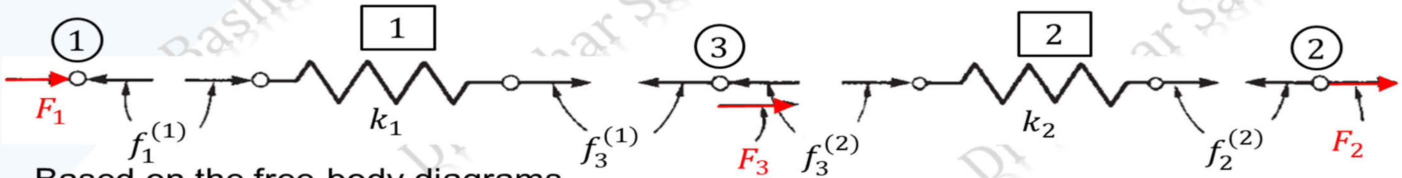
This structure has 3 DoF: u_1, u_2, u_3

For element # (1) $\begin{Bmatrix} f_1^{(1)} \\ f_3^{(1)} \end{Bmatrix} = \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \begin{Bmatrix} u_1^{(1)} \\ u_3^{(1)} \end{Bmatrix}$

For element # (2) $\begin{Bmatrix} f_3^{(2)} \\ f_2^{(2)} \end{Bmatrix} = \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} u_3^{(2)} \\ u_2^{(2)} \end{Bmatrix}$

where $f_i^{(e)}$ is the (internal) force acting on local node i of element e $(e = 1$ or $2)$

Elements 1 and 2 must remain connected at the common node 3 throughout the displacement (continuity or compatibility requirement), i.e. $u_3^{(1)} = u_3^{(2)} = u_3$



Based on the free-body diagrams

$F_1 = f_1^{(1)}$ $\quad F_3 = f_3^{(1)} + f_3^{(2)} = -f_1^{(1)} - f_2^{(2)}$ $\quad F_2 = f_2^{(2)}$

Assemble the stiffness matrix for the whole system:

$$\left.\begin{aligned} F_1 &= k_1u_1 - k_1u_3 \\ F_2 &= k_2u_2 - k_2u_3 \\ F_3 &= -k_1u_1 - k_2u_2 + (k_1+k_2)u_3 \end{aligned}\right\}$$

In matrix form: $\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = \begin{bmatrix} k_1 & 0 & -k_1 \\ 0 & k_2 & -k_2 \\ -k_1 & -k_2 & k_1+k_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$

Or: an alternative way of assembling the whole stiffness matrix:

Element # (1) Connects 1 → 3	Element # (2) Connects 3 → 2
$\begin{Bmatrix} f_1^{(1)} \\ f_3^{(1)} \end{Bmatrix} = \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \begin{Bmatrix} u_1^{(1)} \\ u_3^{(1)} \end{Bmatrix}$	$\begin{Bmatrix} f_3^{(2)} \\ f_2^{(2)} \end{Bmatrix} = \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} u_3^{(2)} \\ u_2^{(2)} \end{Bmatrix}$

No transformation because local & global are the same

Global stiffness matrix (Element)

$$\begin{Bmatrix} F_1 \\ F_3 \end{Bmatrix} = \begin{array}{c} \begin{matrix} u_1 & \quad u_3 \end{matrix} \\ \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \end{array} \begin{Bmatrix} u_1 \\ u_3 \end{Bmatrix} \qquad \begin{Bmatrix} F_3 \\ F_2 \end{Bmatrix} = \begin{array}{c} \begin{matrix} u_3 & \quad u_2 \end{matrix} \\ \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \end{array} \begin{Bmatrix} u_3 \\ u_2 \end{Bmatrix}$$

“Enlarging” the stiffness matrices for elements 1 and 2, we have

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = \begin{bmatrix} k_1 & 0 & -k_1 \\ 0 & 0 & 0 \\ -k_1 & 0 & k_1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \qquad \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

Assembling the Total Stiffness Matrix by adding the two matrix equations i.e. superposition (Direct Stiffness Method)

Global stiffness matrix (Structure)

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = \begin{array}{c} \begin{matrix} u_1 & \quad u_2 & \quad u_3 \end{matrix} \\ \begin{bmatrix} k_1 & 0 & -k_1 \\ 0 & k_2 & -k_2 \\ -k_1 & -k_2 & k_1+k_2 \end{bmatrix} \end{array} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

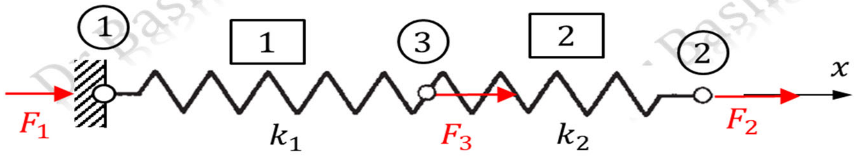
symmetry

This is the same equation we derived by using the force equilibrium concept.

total or global or system stiffness matrix $[\mathbf{K}] = \begin{bmatrix} k_1 & 0 & -k_1 \\ 0 & k_2 & -k_2 \\ -k_1 & -k_2 & k_1+k_2 \end{bmatrix}$

Boundary and load conditions:

$u_1 = 0$



Without BC , the $[\mathbf{K}]$ is singular, it has no invers, we cannot calculate the displacements from forces, nothing works.
BCs make a structure stable. They stop rigid body movement.
Number of BCs to make $[\mathbf{K}]$ non-singular are the number of possible rigid body movements.

$u_1 = 0$

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = \begin{bmatrix} k_1 & 0 & -k_1 \\ 0 & k_2 & -k_2 \\ -k_1 & -k_2 & k_1 + k_2 \end{bmatrix} \begin{Bmatrix} 0 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$\left.\begin{aligned} F_1 &= k_1(0) + (0)u_2 - k_1 u_3 \\ F_2 &= 0(0) + k_2 u_2 - k_2 u_3 \\ F_3 &= -k_1(0) - k_2 u_2 + (k_1 + k_2)\, u_3 \end{aligned}\right\} \Longrightarrow \boxed{\begin{Bmatrix} F_2 \\ F_3 \end{Bmatrix} = \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_1 + k_2 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}} \ \& \ F_1 = -k_1 u_3$$

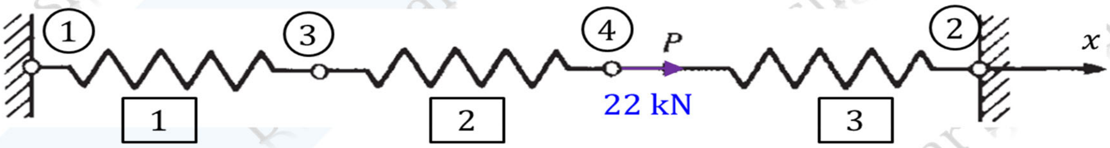
this is solvable, and as if we have removed these row and column

The unknowns are $\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}$ and the reaction force F_1

Note: the destroyed row and column correspond to BC=0

EXAMPLE (Spring assemblage)

For the spring assemblage with arbitrarily numbered nodes shown in the figure, obtain: (a) the global stiffness matrix, (b) the displacements of nodes 3 and 4, (c) the reaction forces at nodes 1 and 2, and (d) the forces in each spring. A force of 22 kN is applied at node 4 in the x direction. The spring constants are given in the figure. Nodes 1 and 2 are fixed.



$k_1 = 200\ \mathrm{N/mm}$ $k_2 = 400\ \mathrm{N/mm}$ $k_3 = 600\ \mathrm{N/mm}$

Solution:

$$\begin{Bmatrix} f_i \\ f_j \end{Bmatrix} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} u_i \\ u_j \end{Bmatrix}$$

The element stiffness matrices are:

element # (1) (node 1 → node 3)

$$[k^{(1)}] = \begin{array}{c} \begin{array}{cc} 1 & 3 \end{array} \\ \begin{bmatrix} 200 & -200 \\ -200 & 200 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 3 \end{array}$$

element # (2) (node 3 → node 4)

$$[k^{(2)}] = \begin{array}{c} \begin{array}{cc} 3 & 4 \end{array} \\ \begin{bmatrix} 400 & -400 \\ -400 & 400 \end{bmatrix} \end{array} \begin{array}{c} 3 \\ 4 \end{array}$$

element # (3) (node 4 → node 2)

$$[k^{(3)}] = \begin{array}{c} \begin{array}{cc} 4 & 2 \end{array} \\ \begin{bmatrix} 600 & -600 \\ -600 & 600 \end{bmatrix} \end{array} \begin{array}{c} 4 \\ 2 \end{array}$$

Local axis is the same as global axis no transformation needed.

Applying the superposition concept, we obtain the global stiffness matrix for the spring system:

$$[K] = \begin{array}{cccc|c} u_1 & u_2 & u_3 & u_4 & \\ 200 & 0 & -200 & 0 & u_1 \\ 0 & 600 & 0 & -600 & u_2 \\ -200 & 0 & 200+400 & -400 & u_3 \\ 0 & -600 & -400 & 400+600 & u_4 \end{array}$$

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix} = \begin{bmatrix} 200 & 0 & -200 & 0 \\ 0 & 600 & 0 & -600 \\ -200 & 0 & 200+400 & -400 \\ 0 & -600 & -400 & 400+600 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

Applying the BC: $u_1 = u_2 = 0$. Also, we know that $F_3 = 0$ and $F_4 = 22000\text{ N}$

$$\begin{Bmatrix} 0 \\ 22000 \end{Bmatrix} = \begin{bmatrix} 600 & -400 \\ -400 & 1000 \end{bmatrix} \begin{Bmatrix} u_3 \\ u_4 \end{Bmatrix} \Rightarrow \quad u_3 = 20\text{ mm} \quad , \qquad u_4 = 30\text{ mm}$$

Let's find the nodal forces:

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix} = \begin{bmatrix} 200 & 0 & -200 & 0 \\ 0 & 600 & 0 & -600 \\ -200 & 0 & 600 & -400 \\ 0 & -600 & -400 & 1000 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 20 \\ 30 \end{Bmatrix} \Rightarrow$$

$F_1 = -4000\text{ N}$

$F_2 = -18000$

$F_3 = 0$

$F_4 = 22000$

Note that the reactions equal the externally applied force

What are the internal forces?

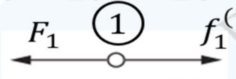
1 get the local displacements (which is the same as global displacement)

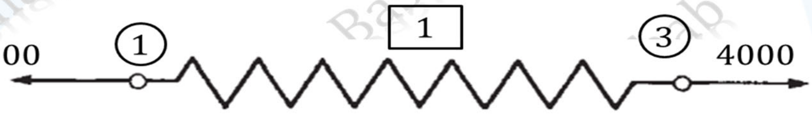
2 use the local stiffness matrix

element # (1)

$$\begin{Bmatrix} f_1^{(1)} \\ f_3^{(1)} \end{Bmatrix} = \begin{bmatrix} 200 & -200 \\ -200 & 200 \end{bmatrix} \begin{Bmatrix} 0 \\ 20 \end{Bmatrix}$$

$f_1^{(1)} = -4000\text{ N}$ $\qquad$ $f_3^{(1)} = 4000\text{ N}$

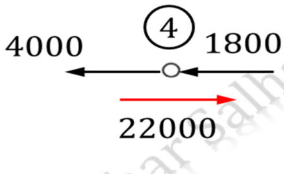




element # (2)

$$\begin{Bmatrix} f_3^{(2)} \\ f_4^{(2)} \end{Bmatrix} = \begin{bmatrix} 400 & -400 \\ -400 & 400 \end{bmatrix} \begin{Bmatrix} 20 \\ 30 \end{Bmatrix}$$

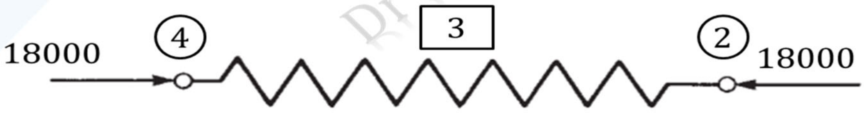
$f_3^{(2)} = -4000\text{ N}$ $\qquad$ $f_4^{(2)} = 4000\text{ N}$

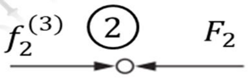


element # (3)

$$\begin{Bmatrix} f_4^{(3)} \\ f_2^{(3)} \end{Bmatrix} = \begin{bmatrix} 600 & -600 \\ -600 & 600 \end{bmatrix} \begin{Bmatrix} 30 \\ 0 \end{Bmatrix}$$

$f_4^{(3)} = 18000\text{ N}$ $\qquad$ $f_2^{(3)} = -18000\text{ N}$

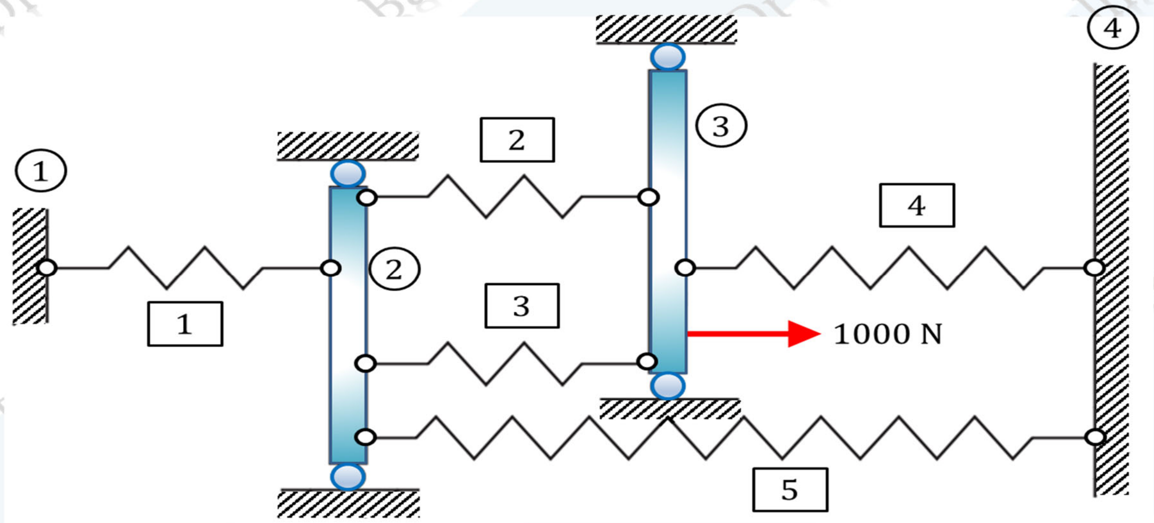




EXAMPLE **(Spring assemblage)**

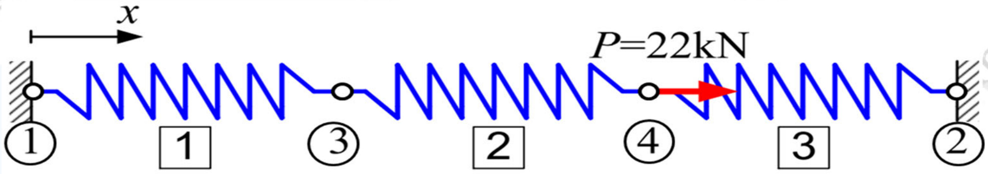
For the five-spring assemblage shown in the figure, determine the displacements, reactions, and the internal forces. Assume the rigid vertical bars at nodes 2 and 3 (connecting the springs) remain vertical at all times, but are free to slide or displace left or right. There is an applied force at node 3 of 1000 N to the right.

$k^{(1)} = 500\ \text{N/mm}$ $\quad$ $k^{(2)} = k^{(3)} = 300\ \text{N/mm}$ $\quad$ $k^{(4)} = k^{(5)} = 400\ \text{N/mm}$



HOMEWORK 2.1

For the spring assemblage with arbitrarily numbered nodes, as shown below, obtain (a) the global stiffness matrix, (b) the displacements of nodes 3 and 4, (c) the reaction forces at nodes 1 and 2, and (d) the forces in each spring. A force of 22 kN is applied at node 4 in the x-direction. The spring constants are given in the figure. Nodes 1 and 2 are fixed



$k^{(1)} = 175\ \text{kN/m}$ $\quad$ $k^{(2)} = 350\ \text{kN/m}$ $\quad$ $k^{(3)} = 525\ \text{kN/m}$

HOMEWORK 2.2

For the spring assemblage shown in the figure below, formulate the global stiffness matrix and equations for solution of the unknown global displacement and forces. The spring constants for the elements are $k^{(1)}$, $k^{(2)}$, and $k^{(3)}$; P is an applied force at node 2 (same values as in Homework 2.1)