Undergraduate Course

FINITE ELEMENT METHODS

Truss Element (Simple Bar)

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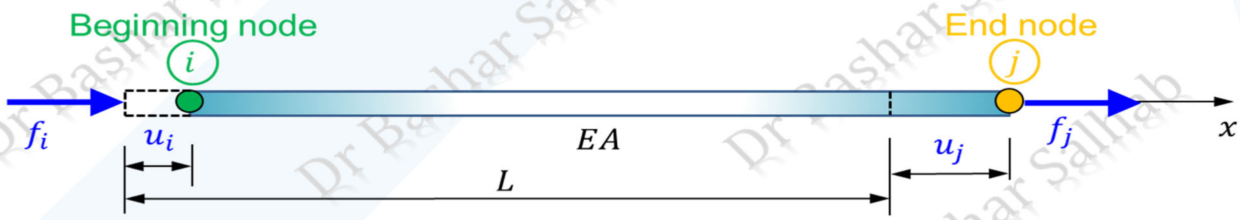
Truss Element in 1D

Derivation of the Stiffness Matrix for a truss element in local coordinates

Consider a uniform prismatic bar:



Two nodes: i, j
Nodal displacements: u_i, u_j (DoF)
Nodal forces: f_i , f_j Equilibrium $f_i = -f_j$
Geometric & Material Properties (Element constants): L, EA

Assumptions:

E, A constants
Bar only has axial forces i.e. $f_{iy} = f_{jy} = M_i = M_j = 0$ (Truss=Axially loaded)
Transverse displacement is ignored (the relative movement of one node with respect to the other in y doesn't create stresses
Loads are applied on nodes

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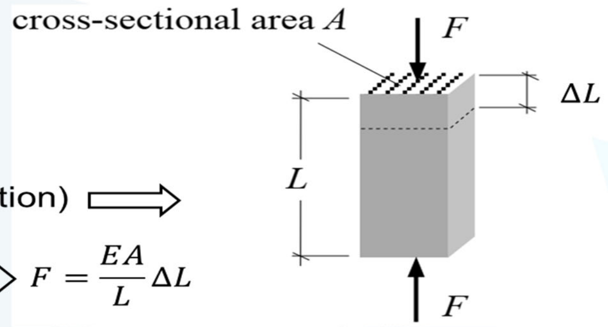
Element Stiffness:



Strain-displacement relation: $\varepsilon = \dfrac{du}{dx}$

Stress-strain relation: $\sigma = E\varepsilon$

We have $\varepsilon = \dfrac{du}{dx} = \dfrac{u_j - u_i}{L} = \dfrac{\Delta L}{L}$ (ΔL elongation) $\Longrightarrow$

$\sigma = E\varepsilon = E\dfrac{\Delta L}{L}$ We also have $\sigma = \dfrac{F}{A}$ $\Longrightarrow$ $F = \dfrac{EA}{L}\Delta L$

or $\Delta L = \varepsilon\, L = \dfrac{\sigma L}{E} = \dfrac{F\,L}{EA}$ $\Longrightarrow$ $F = \dfrac{EA}{L}\Delta L$

hence when $\Delta L = 1$ (i.e. unit displacement) then the axial stiffness $k(=F) = \dfrac{EA}{L}$

$k = \dfrac{EA}{L}$ The 'axial' stiffness of this column

Kinematic response: The local displacements u_i of node b , u_j of node e , & $\Delta L = u_j - u_i$ the elongation of the element (Assuming $u_j > u_i$)

Static response: The local forces f_i at node b, f_j at node e , & the internal axial forces N of the element.

Derivation of the stiffness matrix for a simple bar in local coordinates:

by considering the equilibrium of the forces at the two nodes:

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The force/displacement relationships for this element are:

$$\begin{cases} f_i = \dfrac{EA}{L}(u_i - u_j) \\ f_j = -\dfrac{EA}{L}(u_i - u_j) \end{cases}$$

Element equilibrium equation is $\overset{\rightarrow +}{\sum} F_x = 0 \Longrightarrow f_i = -f_j$

$$\left.\begin{aligned} f_i &= +\frac{EA}{L}u_i - \frac{EA}{L}u_j \\ f_j &= -\frac{EA}{L}u_i + \frac{EA}{L}u_j \end{aligned}\right\}$$

Grouping in a matrix form $\Longrightarrow$

$$\boxed{\begin{Bmatrix} f_i \\ f_j \end{Bmatrix} = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_i \\ u_j \end{Bmatrix}}$$

The bar is acting like a spring in this case and we conclude that element stiffness matrix is:

$$\mathbf{k} = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Physical Meaning of the Coefficients in $\mathbf{k}$: The jth column of k (here $j = 1$ or 2) represents the forces applied to the bar to maintain a deformed shape with unit displacement at node j and zero displacement at the other node

Or symbolically $\{f\}_{2\times 1} = [k]_{2\times 2}\ \{u\}_{2\times 1}$

Local known externally applied loads at the nodes

Local element stiffness matrix

Local element unknown nodal displacements

Member Stiffness Relations

In matrix stiffness method of structural analysis, the primary unknowns, the joint displacements of the structure are determined by solving a system of simultaneous equations as the following one

$$\overline{\boldsymbol{P}} = \mathbf{S} \times \mathbf{DoF}$$

DoF joint unknown displacement vector

$\overline{\boldsymbol{P}}$ joint external applied load vector

S the structure stiffness matrix, obtained by assembling the stiffness matrices **k** of the individual members , **S** is used to express the end forces f_i of the member in term of its joint displacements u_i

System stiffness matrix [S] by direct method

We can construct the global stiffness matrix for the entire structure by adding the stiffness coefficients of the local matrix to the global matrix based on the DoF.

Consider a structure such as pin-jointed truss consists of several rod elements:

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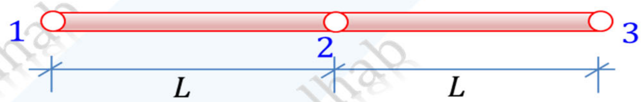
For 2 linear truss elements with the same EA, L



Element (1):

$$\mathbf{k}^{(1)} = \frac{EA}{L}\begin{array}{c} \begin{array}{cc} DoF_1 & DoF_2 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} DoF_1 \\ DoF_2 \end{array}$$

Element (2):

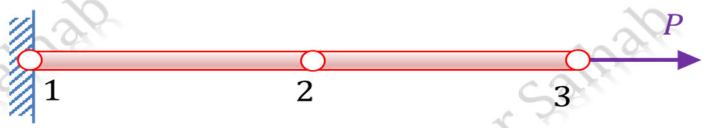
$$\mathbf{k}^{(2)} = \frac{EA}{L}\begin{array}{c} \begin{array}{cc} DoF_2 & DoF_3 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} DoF_2 \\ DoF_3 \end{array}$$

The DoF corresponds to the index of the global stiffness matrix

$$[\mathbf{K}] = \frac{EA}{L}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{EA}{L}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} = \frac{EA}{L}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

If we want to solve the system with BC and force as shown:

$$\{\boldsymbol{F}\} = [\mathbf{K}]\{\boldsymbol{u}\} \qquad \{\boldsymbol{F}\} = \begin{Bmatrix} R \\ 0 \\ P \end{Bmatrix}$$



$$\begin{Bmatrix} R \\ 0 \\ P \end{Bmatrix} = \frac{EA}{L}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \xRightarrow{u_1 = 0} \begin{Bmatrix} 0 \\ P \end{Bmatrix} = \frac{EA}{L}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} \qquad \overline{\boldsymbol{P}} = \mathbf{S}\,\boldsymbol{DoF}$$

We can solve for u_2 , u_3

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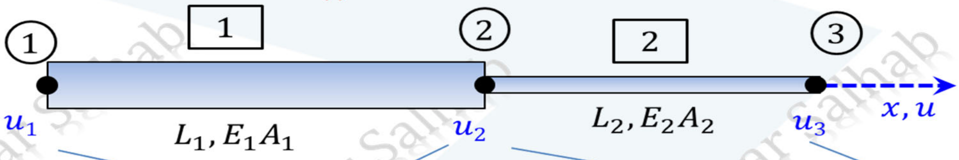
For 2 linear truss elements with different EA, L



Element (1):

$$\mathbf{k}^{(1)} = \frac{E_1A_1}{L_1}\begin{array}{c} \begin{array}{cc} u_1 & u_2 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ u_1 \\ u_2 \end{array}$$

Element (2):

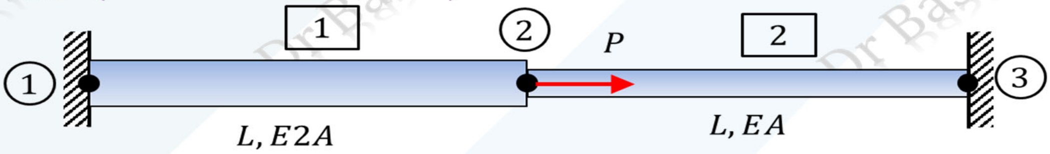
$$\mathbf{k}^{(2)} = \frac{E_2A_2}{L_2}\begin{array}{c} \begin{array}{cc} u_2 & u_3 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ u_2 \\ u_3 \end{array}$$

The system stiffness matrix $[\mathbf{K}]$ is obtained by superimposing the coefficients of stiffness of the elemental stiffness matrices:

$$[\mathbf{K}] = \begin{array}{c} \begin{array}{ccc} u_1 & u_2 & u_3 \end{array} \\ \begin{bmatrix} E_1A_1/L_1 & -E_1A_1/L_1 & 0 \\ -E_1A_1/L_1 & E_1A_1/L_1 + E_2A_2/L_2 & -E_2A_2/L_2 \\ 0 & -E_2A_2/L_2 & E_2A_2/L_2 \end{bmatrix} \end{array} \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \end{array}$$

Solution of $\{\boldsymbol{F}\} = [\mathbf{K}]\{\boldsymbol{u}\}$ cannot be carried out, as $[\mathbf{K}]$ is singular, i.e. the structure is floating in space and has not been constrained.

EXAMPLE (2 linear truss elements)



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Find the stresses in the two bar assembly which is loaded with force P, and firmly fixed at the two ends, as shown in the figure.

Solution:

Element (1):

$$\mathbf{k}^{(1)} = \frac{EA}{L}\begin{array}{c} \begin{array}{cc} 0 & DoF \end{array} \\ \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \end{array} \begin{array}{c} \\ 0 \\ DoF \end{array}$$

Element (2):

$$\mathbf{k}^{(2)} = \frac{EA}{L}\begin{array}{c} \begin{array}{cc} DoF & 0 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ DoF \\ 0 \end{array}$$

We can assemble the global FE equation as follow

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = \frac{EA}{L}\underbrace{\begin{array}{c} \begin{array}{ccc} 0 & DoF & 0 \end{array} \\ \begin{bmatrix} 2 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix} \end{array}}_{[\mathbf{K}]} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

Load and boundary conditions (BC) are, $u_1 = u_3 = 0$, $F_2 = P$

$$\begin{Bmatrix} F_1 \\ P \\ F_3 \end{Bmatrix} = \frac{EA}{L}\begin{bmatrix} 2 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ u_2 \\ 0 \end{Bmatrix}$$

Deleting the 1st row and column, and the 3rd row and column, we obtain:

$$P = \frac{EA}{L}(3)u_2 \Longrightarrow u_2 = \frac{PL}{3EA}$$

$$\left.\begin{array}{l} \{\boldsymbol{f}\} = \dfrac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\{\mathbf{d}\} \\ \sigma = \dfrac{f'}{A} \end{array}\right\} \{\boldsymbol{\sigma}\} = \frac{E}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\{\mathbf{d}\}$$

Stress in element (1):

$$\sigma^{(1)} = \frac{E}{L}(u_2 - u_1) = \frac{E}{L}\left(\frac{PL}{3EA} - 0\right) = \frac{P}{3A}$$

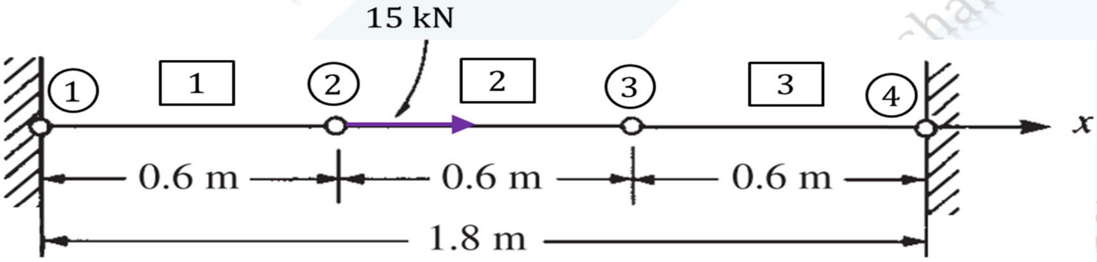
Stress in element (2):

$$\sigma^{(2)} = \frac{E}{L}(u_3 - u_2) = \frac{E}{L}\left(0 - \frac{PL}{3EA}\right) = -\frac{P}{3A}$$

which indicates it is in compression

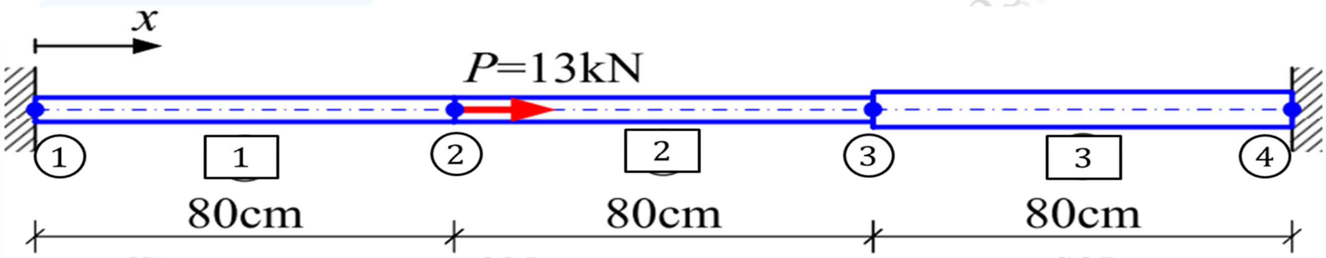
EXAMPLE **Three-bar assemblage**

For the three-bar assemblage shown in the figure, determine:
(a) the global stiffness matrix, (b) the displacements of nodes 2 and 3, and (c) the reactions at nodes 1 and 4. A force of 15 kN is applied in the x direction at node 2. The length of each element is 0.6 m. $E = 2 \times 10^{11}$ Pa, $A = 6 \times 10^{-4} \text{m}^2$ for elements 1 and 2, and $E = 1 \times 10^{11}$ Pa, $A = 12 \times 10^{-4} \text{m}^2$ for element 3. Nodes 1 and 4 are fixed.



HOMEWORK 3.1

For the three-bar assemblage shown in the figure below, determine:
(a) the global stiffness matrix,
(b) the displacements of nodes 2 and 3, and
(c) the reaction forces at nodes 1 and 4.
A force of 13 kN is applied at node 2 in the x direction. The length of each element is 80 cm. $E = 200$ GPa and $A = 6.5 \text{ cm}^2$ for elements 1 and 2, and $E = 100$ GPa and $A = 13 \text{ cm}^2$ for element 3. Nodes 1 and 4 are fixed.



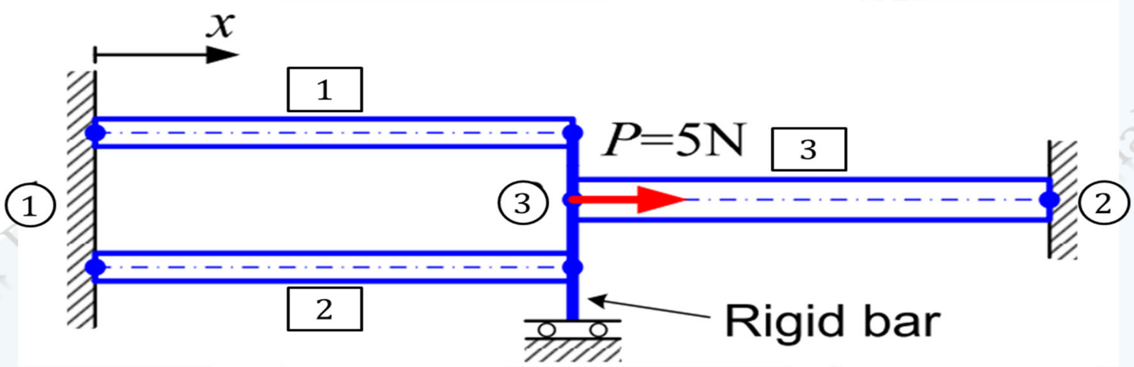
HOMEWORK 3.2

Three bars are joined as shown in the figure below. The left and right ends are both constrained. There is a force of 5 N acting on the middle node. Determine:

(a) the global stiffness matrix,

(b) the displacement of node 3, and

(c) the reaction forces at nodes 1 and 2.



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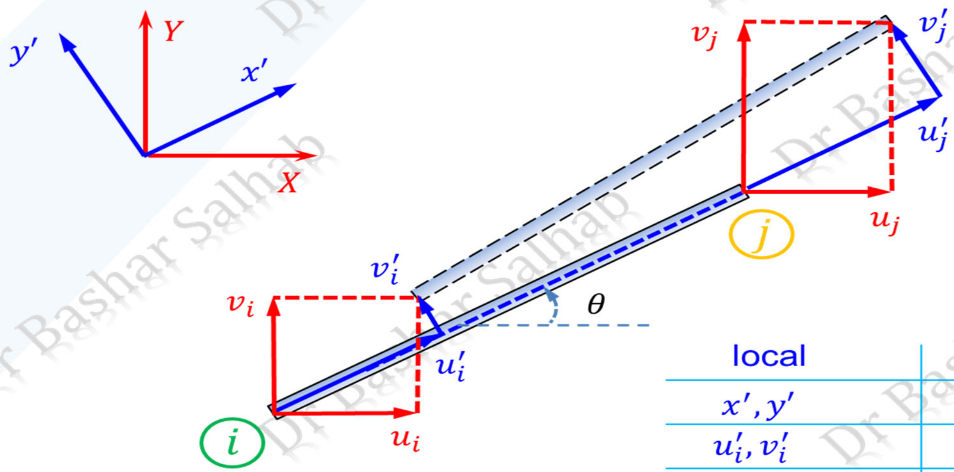
Transformation of vectors of bar element in 2D

A 2D truss element in the global coordinate system XY and local x' y'

We need transformation matrix i.e. cosine & sine of your stuff. It transforms something from one coordinates system to the other.

The angle θ is defined as positive in the counterclockwise sense.



local	global
x', y'	X, Y
u'_i, v'_i	u_i, v_i
1 DoF at node	2 DoF's at node

$u' = u\cos\theta + v\sin\theta = [c \quad s]\begin{Bmatrix} u \\ v \end{Bmatrix}$

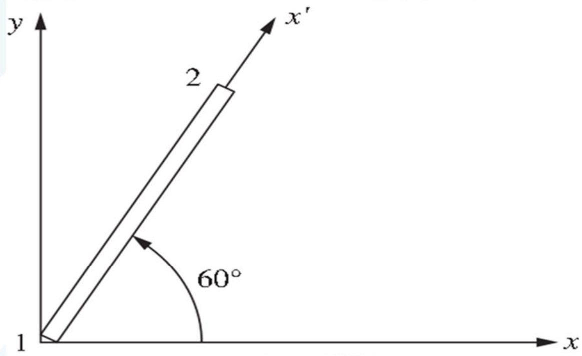
$v' = -u\sin\theta + v\cos\theta = [-s \quad c]\begin{Bmatrix} u \\ v \end{Bmatrix}$

$c = \cos\theta \qquad s = \sin\theta$

$$\Longrightarrow \begin{Bmatrix} u' \\ v' \end{Bmatrix} = \underbrace{\begin{bmatrix} c & s \\ -s & c \end{bmatrix}}_{\text{transformation matrix}} \begin{Bmatrix} u \\ v \end{Bmatrix}$$

EXAMPLE **(Bar element with local axis x' acting along the element)**

The global nodal displacements at node 2 have been determined to be $u_2 = 2.5$ mm and $v_2 = 5$ mm for the bar element shown in the figure. Determine the local x displacement at node 2



Solution:

$u' = u\cos\theta + v\sin\theta = [c \quad s]\begin{Bmatrix} u \\ v \end{Bmatrix} \Longrightarrow u'_2 = (\cos 60°)(2.5) + (\sin 60°)(5) = 5.58 \text{ mm}$

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Element Global Stiffness Matrix $\boldsymbol{k}$ for truss in 2D

Note that at each member end has 2 DoF and two end forces are needed in global coordinates to represent the components of the member axial displacement and axial force, respectively.

Transformation from global to local coordinates

$$\boxed{\begin{aligned} c &= \cos\theta = (X_j - X_i)/L \\ s &= \sin\theta = (Y_j - Y_i)/L \end{aligned}} \qquad L^2 = (x_j - x_i)^2 + (y_j - y_i)^2$$

$u_i'^2 = u_i^2 + v_i^2$

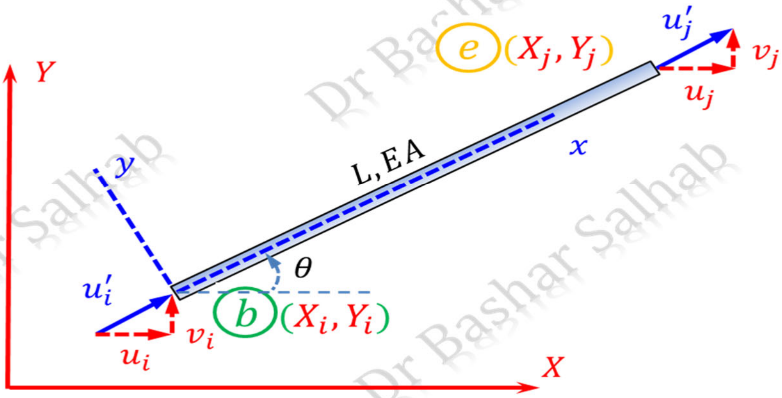
$\Longrightarrow u_i' = \frac{u_i}{u_i'}u_i + \frac{v_i}{u_i'}v_i$

$\Longrightarrow u_i' = cu_i + sv_i$

بالمثل: $u_j' = cu_j + sv_j$

$$\underset{\text{local}}{\begin{Bmatrix} u_i' \\ u_j' \end{Bmatrix}} = \begin{bmatrix} c & s & 0 & 0 \\ 0 & 0 & c & s \end{bmatrix} \underset{\text{global}}{\begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \end{Bmatrix}}$$

$\Longrightarrow \mathbf{d'}_{2\times1} = \mathbf{T}_{2\times4}\mathbf{d}_{4\times1}$



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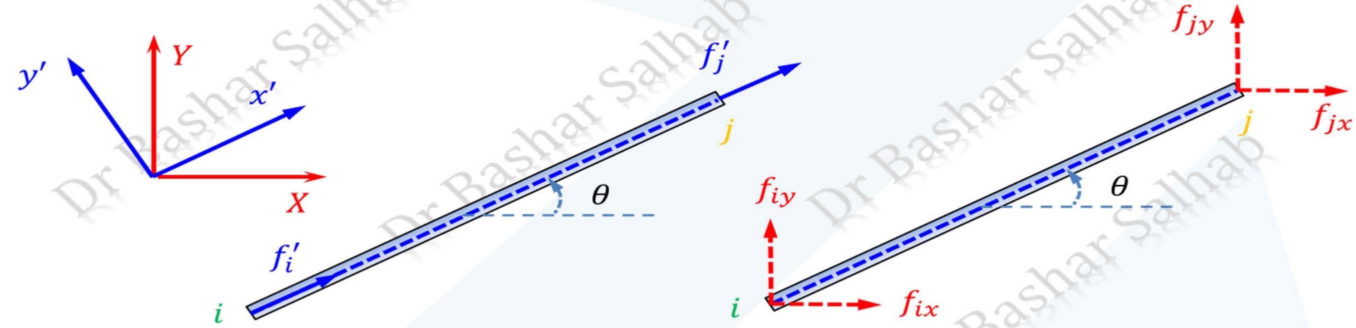
Similarly, the nodal forces are transformed in the same way $\left.\begin{aligned} f'_i &= c f_{ix} + s f_{iy} \\ f'_j &= c f_{jx} + s f_{jy} \end{aligned}\right\} \mathbf{f}' = \mathbf{T}\,\mathbf{f}$

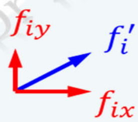


For local system, a truss does not care about forces perpendicular to bar element

Transformation from local to global coordinates

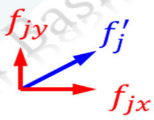
$f_{ix} = c\, f'_i$

$f_{iy} = s\, f'_i$



$f_{jx} = c\, f'_j$

$f_{jy} = s\, f'_j$



$$\Longrightarrow \underbrace{\begin{Bmatrix} f_{ix} \\ f_{iy} \\ f_{jx} \\ f_{jy} \end{Bmatrix}}_{\text{global}} = \begin{bmatrix} c & 0 \\ s & 0 \\ 0 & c \\ 0 & s \end{bmatrix} \underbrace{\begin{Bmatrix} f'_i \\ f'_j \end{Bmatrix}}_{\text{local}}$$

$$\Longrightarrow \mathbf{f}_{4\times 1} = \mathbf{T}^{\mathrm{T}}_{4\times 2}\mathbf{f}'_{2\times 1}$$

Similarly, $\mathbf{d} = \mathbf{T}^{T}\,\mathbf{d}'$

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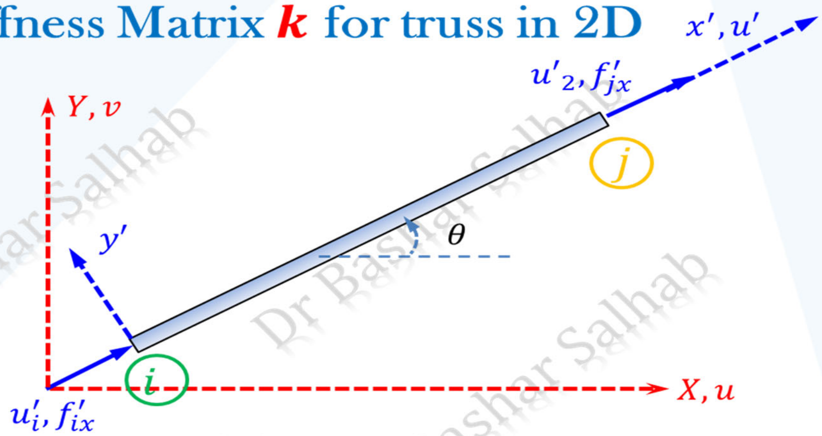
Element Global Stiffness Matrix $\boldsymbol{k}$ for truss in 2D

local $\{f'\} = [k']\{d'\} \Longleftrightarrow$

$$\begin{Bmatrix} f'_i \\ f'_j \end{Bmatrix} = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u'_i \\ u'_j \end{Bmatrix}$$

Our target: global $\{f\} = [k]\{d\}$

$$\Longleftrightarrow \begin{Bmatrix} f_{ix} \\ f_{iy} \\ f_{jx} \\ f_{jy} \end{Bmatrix} = [k]\begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \end{Bmatrix}$$



Remember:

$u'_i = u_i\, c + v_i\, s$

$u'_j = u_j\, c + v_j\, s$

$$\Longrightarrow \begin{Bmatrix} u'_i \\ u'_j \end{Bmatrix} = \begin{bmatrix} c & s & 0 & 0 \\ 0 & 0 & c & s \end{bmatrix}\begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \end{Bmatrix} \Longleftrightarrow \{d'\} = [T]\{d\}$$

$$T = \begin{bmatrix} c & s & 0 & 0 \\ 0 & 0 & c & s \end{bmatrix}$$

Similarly: $\begin{Bmatrix} f'_{ix} \\ f'_{jx} \end{Bmatrix} = \begin{bmatrix} c & s & 0 & 0 \\ 0 & 0 & c & s \end{bmatrix}\begin{Bmatrix} f_{ix} \\ f_{iy} \\ f_{jx} \\ f_{jy} \end{Bmatrix} \Longleftrightarrow \{f'\} = [T]\{f\}$

$\{f'\} = [k']\{d'\}$

$[T]\{f\} = [k'][T]\{d\}$

$\{f\} = [T]^{-1}[k'][T]\{d\} \Longleftrightarrow \{f\} = [k]\{d\} \Longrightarrow [k] = [T]^{-1}[k'][T]$

We write T in the expanded form so we can invert it

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Or simply:

Combining Static & kinematic transformations with the Local Element Eq.

$\mathbf{f}_{4\times1} = \mathbf{T}^{\mathrm{T}}_{4\times2}\mathbf{f}'_{2\times1}$ $\quad\quad$ $\mathbf{f}'_{2\times1} = \mathbf{k}'_{2\times2}\mathbf{d}'_{2\times1}$ $\quad\quad$ $\mathbf{d}'_{2\times1} = \mathbf{T}_{2\times4}\mathbf{d}_{4\times1}$

$$\Longrightarrow \quad \mathbf{f}_{4\times1} = \mathbf{T}^{\mathrm{T}}_{4\times2}\mathbf{k}'_{2\times2}\mathbf{T}_{2\times4}\mathbf{d}_{4\times1} \quad \Longleftrightarrow \quad \mathbf{f}_{4\times1} = \mathbf{k}_{4\times4}\mathbf{d}_{4\times1}$$

$\mathbf{k}_{4\times4} = \mathbf{T}^{\mathrm{T}}_{4\times2}\mathbf{k}'_{2\times2}\mathbf{T}_{2\times4}$ the global 2D truss element matrix

$$\mathbf{k}_{4\times4} = \begin{bmatrix} c & 0 \\ s & 0 \\ 0 & c \\ 0 & s \end{bmatrix}\left(\frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\right)\begin{bmatrix} c & s & 0 & 0 \\ 0 & 0 & c & s \end{bmatrix} = \boxed{\frac{EA}{L}\begin{bmatrix} c^2 & c\,s & -c^2 & -c\,s \\ c\,s & s^2 & -c\,s & -s^2 \\ -c^2 & -c\,s & c^2 & c\,s \\ -c\,s & -s^2 & c\,s & s^2 \end{bmatrix}}$$

k is symmetric and singular

$$\begin{Bmatrix} f_{ix} \\ f_{iy} \\ f_{jx} \\ f_{jy} \end{Bmatrix} = \frac{EA}{L}\begin{bmatrix} c^2 & c\,s & -c^2 & -c\,s \\ c\,s & s^2 & -c\,s & -s^2 \\ -c^2 & -c\,s & c^2 & c\,s \\ -c\,s & -s^2 & c\,s & s^2 \end{bmatrix}\begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \end{Bmatrix}$$

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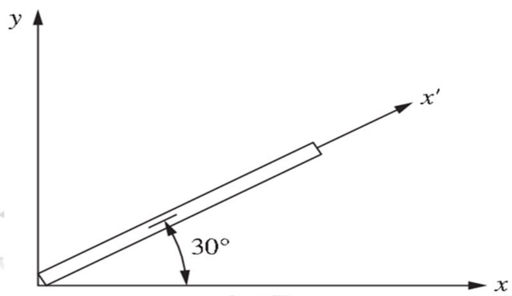
EXAMPLE **(Bar element for stiffness matrix evaluation)**

For the bar element shown in the figure, evaluate the global stiffness matrix with respect to the $x-y$ coordinate system. Let the bar's cross-sectional area equal to $6\times10^{-4}\mathrm{m}^2$, length equal $1.2\mathrm{m}$, and modulus of elasticity equal $2\times10^{11}\mathrm{Pa}$. The angle of the bar makes with the x axis 30º.



Solution:

$$\theta = 30^\circ \qquad C = \cos 30^\circ = \frac{\sqrt{3}}{2} \qquad S = \sin 30^\circ = \frac{1}{2}$$

$$[k] = \frac{(6\times10^{-4})(2\times10^{11})}{1.2}\begin{bmatrix} \frac{3}{4} & \frac{\sqrt{3}}{4} & \frac{-3}{4} & \frac{-\sqrt{3}}{4} \\ & \frac{1}{4} & \frac{-\sqrt{3}}{4} & \frac{-1}{4} \\ & & \frac{3}{4} & \frac{\sqrt{3}}{4} \\ \text{Symmetry} & & & \frac{1}{4} \end{bmatrix}\frac{\mathrm{N}}{\mathrm{m}} = 10^8\begin{bmatrix} 0.75 & 0.433 & -0.75 & -0.433 \\ & 0.25 & -0.433 & -0.25 \\ & & 0.75 & 0.433 \\ \text{Symmetry} & & & 0.25 \end{bmatrix}\frac{\mathrm{N}}{\mathrm{m}}$$

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Computation of Stress for a Bar in 2D

The usual definition of axial tensile stress is axial force divided by cross-sectional area Simply calculate the local force f' then divide it by the area A, but you have to decide if f' tension or compression.

$$\{f'\} = [k']\{d'\} \Longleftrightarrow \{f'\} = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\{d'\}$$

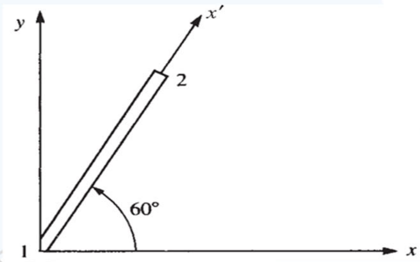
$$\{d'\} = [T]\{d\} \qquad T = \begin{bmatrix} c & s & 0 & 0 \\ 0 & 0 & c & s \end{bmatrix} \qquad \sigma = \frac{f'}{A}$$

$$\sigma^{(e)} = \frac{E}{L}\begin{bmatrix} -c & -s & c & s \end{bmatrix}\begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \end{Bmatrix}$$

EXAMPLE **(Bar element for stress evaluation)**

For the bar shown in the figure, determine the axial stress. $A = 4 \times 10^{-4}\,\mathrm{m}^2$, $E = 210$ GPa, and $L = 2$ m. Assume the global displacements have been previously determined to be $u_1 = 0.25$ mm, $v_1 = 0$, $u_2 = 0.5$ mm, and $v_2 = 0.75$ mm

Solution:



$$\{d\} = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix} = \begin{Bmatrix} 0.25 \times 10^{-3}\ \mathrm{m} \\ 0.0 \\ 0.50 \times 10^{-3}\ \mathrm{m} \\ 0.75 \times 10^{-3}\ \mathrm{m} \end{Bmatrix}$$

$c = \cos 60 = 0.5$

$s = \sin 60 = \sqrt{3}/2$

$$\sigma_x = \frac{210 \times 10^6}{2}\begin{bmatrix} \frac{-1}{2} & \frac{-\sqrt{3}}{2} & \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}\begin{Bmatrix} 0.25 \\ 0.0 \\ 0.50 \\ 0.75 \end{Bmatrix} \times 10^{-3} = 81.32 \times 10^3\ \mathrm{kN/m^2} = 81.32\ \mathrm{MPa}$$

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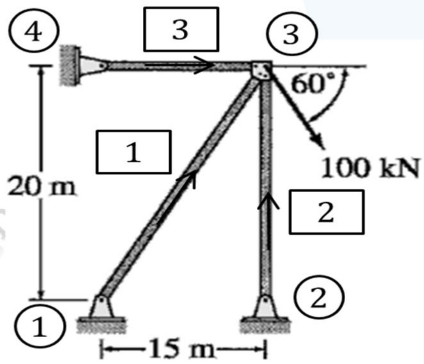
EXAMPLE

For the given truss use the matrix method to:

1. Construct the Analytical model & determine the global degrees of freedom.
2. For each member construct the global stiffness matrix.
3. Then establish the truss stiffness matrix & solve for the global degrees of freedom.
4. Finally Determine the member internal forces and the reactions. $A = 9\ \mathrm{m}^2$, $E = 2417$ kPa

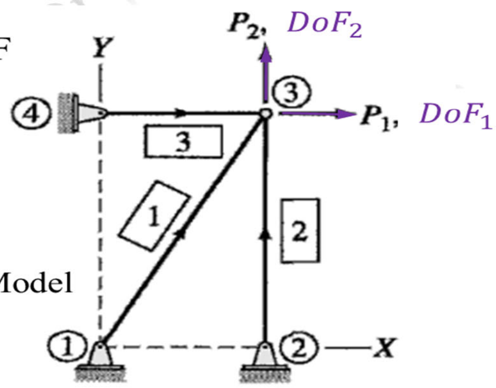


Solution:

❑ Analytical Model & Degrees of Freedom:

From the Analytical model, the truss has only 2 DoF DoF_1 & DoF_2 (the global translations of joint 3).

$$\mathbf{DoF} = \begin{Bmatrix} DoF_1 \\ DoF_2 \end{Bmatrix}$$



Analytical Model

- ❑ Element Stiffness matrices:

For element 1:

$$L=\sqrt{(15-0)^2+(20-0)^2}=25\ \text{m}$$

$$\cos\theta=0.6 \qquad \sin\theta=0.8$$

$$EA/L=870\ \text{kN/m}$$

$$\mathbf{k}^{(1)}=\begin{array}{c}\begin{array}{cccc}0 & 0 & DoF_1 & DoF_2\end{array}\\ \begin{bmatrix}313.2 & 417.6 & -313.2 & -417.6\\ 417.6 & 556.8 & -417.6 & -556.8\\ -313.2 & -417.6 & \boxed{313.2} & \boxed{417.6}\\ -417.6 & -556.8 & \boxed{417.6} & \boxed{556.8}\end{bmatrix}\end{array}\begin{array}{c}\\0\\0\\DoF_1\\DoF_2\end{array}$$

For element 2:

$$L=20\,\text{m}\,,c=0\,,s=1$$

$$EA/L=1088\,\text{kN/m}$$

$$\mathbf{k}^{(2)}=\begin{array}{c}\begin{array}{cccc}0 & 0 & DoF_1 & DoF_2\end{array}\\ \begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 1088 & 0 & -1088\\ 0 & 0 & \boxed{0} & \boxed{0}\\ 0 & -1088 & \boxed{0} & \boxed{1088}\end{bmatrix}\end{array}\begin{array}{c}\\0\\0\\DoF_1\\DoF_2\end{array}$$

For element 3:

$$L=15\,\text{m}\,,c=1\,,s=0$$

$$EA/L=1450\ \text{kN/m}$$

$$\mathbf{k}^{(3)}=\begin{array}{c}\begin{array}{cccc}0 & 0 & DoF_1 & DoF_2\end{array}\\ \begin{bmatrix}1450 & 0 & -1450 & 0\\ 0 & 0 & 0 & 0\\ -1450 & 0 & \boxed{1450} & \boxed{0}\\ 0 & 0 & \boxed{0} & \boxed{0}\end{bmatrix}\end{array}\begin{array}{c}\\0\\0\\DoF_1\\DoF_2\end{array}$$

- ❑ Structural (truss) Stiffness matrix $\mathbf{S}$:

1- Member Matrices assembling

$$\mathbf{S}=\begin{array}{c}\begin{array}{cc}DoF_1 & DoF_2\end{array}\\ \begin{bmatrix}(313.2+0+1450) & (417.6+0+0)\\ (417.6+0+0) & (556.8+1088+0)\end{bmatrix}\end{array}\begin{array}{c}\\DoF_1\\DoF_2\end{array}=\begin{array}{c}\begin{array}{cc}DoF_1 & DoF_2\end{array}\\ \begin{bmatrix}1763.2 & 417.6\\ 417.6 & 1644.8\end{bmatrix}\end{array}\begin{array}{c}\\DoF_1\\DoF_2\end{array}$$

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2- Solving for the global degrees of freedom $\overline{\mathbf{P}}=\mathbf{Sd}$

$$P_1=100\cos 60=50\ \text{kN}$$

$$P_2=-100\sin 60=\ -86.6\ \text{kN}$$

$$\overline{\mathbf{P}}=\begin{Bmatrix}50\\-86.6\end{Bmatrix}$$

$$\begin{Bmatrix}50\\-86.6\end{Bmatrix}=\begin{bmatrix}1763.2 & 417.6\\ 417.6 & 1644.8\end{bmatrix}\begin{Bmatrix}DoF_1\\DoF_2\end{Bmatrix}\Longrightarrow\ \mathbf{DoF}=\begin{Bmatrix}0.0434\ \text{m}\\-0.0637\ \text{m}\end{Bmatrix}$$

- ❑ Member End Displacements & End Forces:

Global Displacement Vector

element 1:

$$\mathbf{d}^{(1)}=\begin{Bmatrix}u_1\\v_1\\u_3\\v_3\end{Bmatrix}=\begin{Bmatrix}0\\0\\DoF_1\\DoF_2\end{Bmatrix}=\begin{Bmatrix}0\\0\\0.0434\text{m}\\-0.0637\text{m}\end{Bmatrix}$$

element 2:

$$\mathbf{d}^{(2)}=\begin{Bmatrix}u_2\\v_2\\u_3\\v_3\end{Bmatrix}=\begin{Bmatrix}0\\0\\DoF_1\\DoF_2\end{Bmatrix}=\begin{Bmatrix}0\\0\\0.0434\text{m}\\-0.0637\text{m}\end{Bmatrix}$$

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element 3:

$$\mathbf{d}^{(3)} = \begin{Bmatrix} u_4 \\ v_4 \\ u_3 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ DoF_1 \\ DoF_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0.0434\text{m} \\ -0.0637\text{m} \end{Bmatrix}$$

Local Displacement Vector

$$\mathbf{d}'^{(1)} = \mathbf{T}^{(1)}\mathbf{d}^{(1)} = \begin{Bmatrix} u'_1 \\ u'_3 \end{Bmatrix} = \begin{bmatrix} 0.6 & 0.8 & 0 & 0 \\ 0 & 0 & 0.6 & 0.8 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0.0434 \\ -0.0637 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -0.0249 \end{Bmatrix} \text{ m}$$

$$\mathbf{d}'^{(2)} = \mathbf{T}^{(2)}\mathbf{d}^{(2)} = \begin{Bmatrix} u'_2 \\ u'_3 \end{Bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0.0434 \\ -0.0637 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -0.0637 \end{Bmatrix} \text{ m}$$

$$\mathbf{d}'^{(3)} = \mathbf{T}^{(3)}\mathbf{d}^{(3)} = \begin{Bmatrix} u'_4 \\ u'_3 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0.0434 \\ -0.0637 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0.0434 \end{Bmatrix} \text{ m}$$

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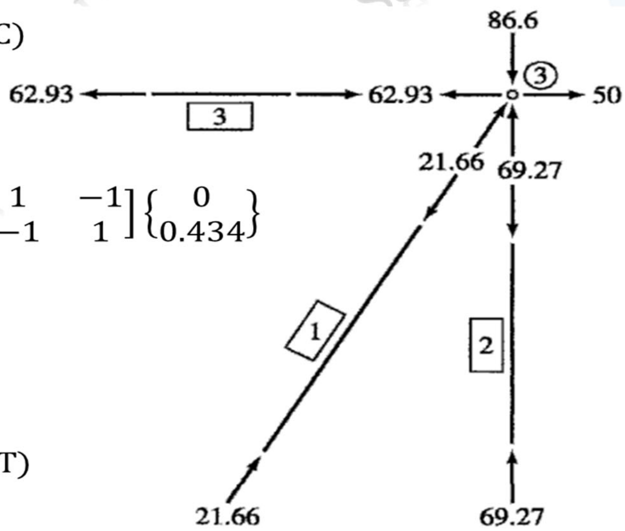
Local End Force Vector

$$\mathbf{f}'^{(1)} = \mathbf{k}'^{(1)}\mathbf{d}'^{(1)} = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\mathbf{d}'^{(1)} = 870\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ -0.0249 \end{Bmatrix} = \begin{Bmatrix} 21.66 \\ -21.66 \end{Bmatrix} \text{ kN}$$

The axial force in member 1: $N_1 = 21.66 \text{ kN (C)}$

$$\mathbf{f}'^{(2)} = \mathbf{k}'^{(2)}\mathbf{d}'^{(2)} = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\mathbf{d}'^{(2)} = 1088\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ -0.0637 \end{Bmatrix} = \begin{Bmatrix} 69.27 \\ -69.27 \end{Bmatrix} \text{ kN}$$

The axial force in member2: $N_2 = 69.27 \text{ kN (C)}$



$$\mathbf{f}'^{(3)} = \mathbf{k}'^{(3)}\mathbf{d}'^{(3)} = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\mathbf{d}'^{(3)} = 1450\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ 0.434 \end{Bmatrix}$$

$$= \begin{Bmatrix} -62.93 \\ 62.93 \end{Bmatrix} \text{ kN}$$

The axial force in member3: $N_3 = 62.93 \text{ kN (T)}$

Global End Force Vector

$$\mathbf{f}^{(1)}=\mathbf{T}^{T(1)}\mathbf{f}'^{(1)}=\begin{Bmatrix} f_{1x}\\ f_{1y}\\ f_{3x}\\ f_{3y}\end{Bmatrix}=\begin{bmatrix}0.6 & 0\\ 0.8 & 0\\ 0 & 0.6\\ 0 & 0.8\end{bmatrix}\begin{Bmatrix}21.66\\ -21.66\end{Bmatrix}=\begin{Bmatrix}\boxed{\begin{matrix}13\\ 17.33\end{matrix}}\\ -13\\ -17.33\end{Bmatrix}\text{kN}$$

Reactions at Support 1

$$\mathbf{R}_1=\begin{Bmatrix}13 \text{ kN}\\ 17.33 \text{ kN}\end{Bmatrix}$$

$$\mathbf{f}^{(2)}=\mathbf{T}^{T(2)}\mathbf{f}'^{(2)}=\begin{Bmatrix} f_{2x}\\ f_{2y}\\ f_{3x}\\ f_{3y}\end{Bmatrix}=\begin{bmatrix}0 & 0\\ 1 & 0\\ 0 & 0\\ 0 & 1\end{bmatrix}\begin{Bmatrix}69.27\\ -69.27\end{Bmatrix}=\begin{Bmatrix}\boxed{\begin{matrix}0\\ 69.27\end{matrix}}\\ 0\\ -69.27\end{Bmatrix}\text{kN}$$

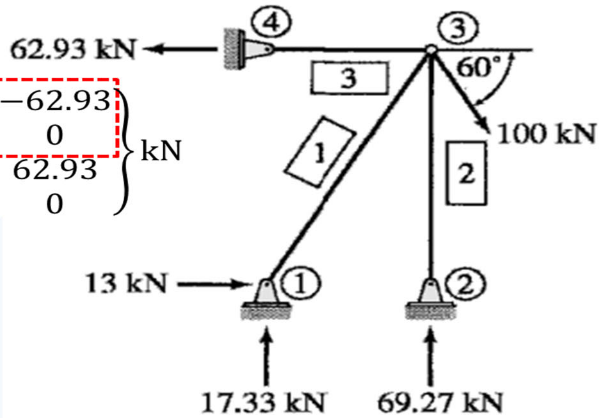
Reactions at Support 2

$$\mathbf{R}_2=\begin{Bmatrix}0\\ 69.27 \text{ kN}\end{Bmatrix}$$

$$\mathbf{f}^{(3)}=\mathbf{T}^{T(3)}\mathbf{f}'^{(3)}=\begin{Bmatrix} f_{4x}\\ f_{4y}\\ f_{3x}\\ f_{3y}\end{Bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 0\\ 0 & 1\\ 0 & 0\end{bmatrix}\begin{Bmatrix}-62.93\\ 62.93\end{Bmatrix}=\begin{Bmatrix}\boxed{\begin{matrix}-62.93\\ 0\end{matrix}}\\ 62.93\\ 0\end{Bmatrix}\text{kN}$$

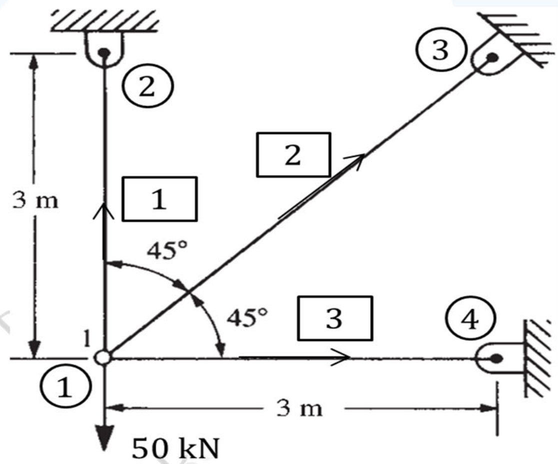
Reactions at Support 4

$$\mathbf{R}_3=\begin{Bmatrix}-62.93 \text{ kN}\\ 0\end{Bmatrix}$$



EXAMPLE (2D Truss)

For the plane truss composed of the three elements shown in the figure & subjected to a downward force of 50 kN applied at node 1, determine the x and y displacements at node 1 and the stresses in each element. $E = 200$ GPa, $A = 6\times10^{-4}\text{m}^2$ for all elements.



Solution:

$c = cos\theta = (X_j - X_i)/L$

$s = sin\theta = (Y_j - Y_i)/L$

Element	$b\,(X_i, Y_i) \longrightarrow e\,(X_j, Y_j)$	L (m)	θ	c	s	$\frac{EA}{L}$
1	1 (0,0) ⟶ 2 (0,3)	3	90°	0	1	4×10^7
2	1 (0,0) ⟶ 3 $(0,3\sqrt{2})$	$3\sqrt{2}$	45°	$\sqrt{2}/2$	$\sqrt{2}/2$	$4\times10^7\times0.707$
3	1 (0,0) ⟶ 4 (0,3)	3	0°	1	0	4×10^7

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For element 1:

$$[k^{(1)}] = \underbrace{\frac{(2\times10^{11})(6\times10^{-4})}{3}}_{4\times10^{7}} \begin{array}{cccc} u_1 & v_1 & u_2 & v_2 \end{array} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

For element 2:

$$[k^{(2)}] = \underbrace{\frac{(2\times10^{11})(6\times10^{-4})}{3\times\sqrt{2}}}_{4\times10^{7}\times0.707} \begin{array}{cccc} u_1 & v_1 & u_3 & v_3 \end{array} \begin{bmatrix} 0.5 & 0.5 & -0.5 & -0.5 \\ 0.5 & 0.5 & -0.5 & -0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \end{bmatrix}$$

For element 3:

$$[k^{(3)}] = \underbrace{\frac{(2\times10^{11})(6\times10^{-4})}{3}}_{4\times10^{7}} \begin{array}{cccc} u_1 & v_1 & u_4 & v_4 \end{array} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{Bmatrix} 0 \\ -50{,}000 \end{Bmatrix} = (4\times10^{7}) \begin{bmatrix} 1.354 & 0.354 \\ 0.354 & 1.354 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \end{Bmatrix}$$

$$u_1 = 2.59075\times10^{4}\ \text{m} \qquad v_1 = -9.90925\times10^{-4}\ \text{m}$$

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$$\sigma^{(1)} = \frac{2\times10^{11}}{3}[0\quad -1\quad 0\quad 1]\begin{Bmatrix} u_1 = 2.59075\times10^{-4} \\ v_1 = -9.90925\times10^{-4} \\ u_2 = 0 \\ v_2 = 0 \end{Bmatrix} = 66.06\ \text{MPa}$$

$$\sigma^{(2)} = \frac{2\times10^{11}}{3\sqrt{2}}\left[\frac{-\sqrt{2}}{2}\quad \frac{-\sqrt{2}}{2}\quad \frac{\sqrt{2}}{2}\quad \frac{\sqrt{2}}{2}\right]\begin{Bmatrix} u_1 = 2.59075\times10^{-4} \\ v_1 = -9.90925\times10^{-4} \\ u_3 = 0 \\ v_3 = 0 \end{Bmatrix} = 24.4\ \text{MPa}$$

$$\sigma^{(3)} = \frac{2\times10^{11}}{3}[-1\quad 0\quad 1\quad 0]\begin{Bmatrix} u_1 = 2.59075\times10^{-4} \\ v_1 = -9.90925\times10^{-4} \\ u_4 = 0 \\ v_4 = 0 \end{Bmatrix} = -17.27\ \text{MPa}$$

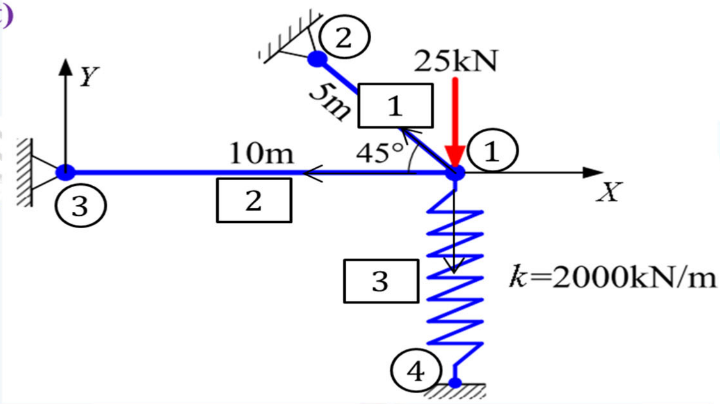
We now verify our results by examining force equilibrium at node 1; that is, summing forces in the global x and y directions, we obtain:

$$\sum F_x = 0 \qquad (24.4\ \text{MPa})(6\times10^{-4}\ \text{m}^2)\frac{\sqrt{2}}{2} - (17.27\ \text{MPa})(6\times10^{-4}\ \text{m}^2) = 0$$

$$\sum F_y = 0 \qquad (66.06\ \text{MPa})(6\times10^{-4}\ \text{m}^2) + (24.4\ \text{MPa})(6\times10^{-4}\ \text{m}^2)\frac{\sqrt{2}}{2} - 50{,}000 = 0$$

EXAMPLE (2D Truss with spring support)

the two-bar truss supported by a spring as shown in the figure. Both bars have $E = 210$ GPa and $A = 5\times10^{-4}\text{m}^2$. Bar one has a length of 5 m and bar two a length of 10 m. The spring stiffness is $k = 2000$ kN/m



Solution:

Element 1: $\theta^{(1)} = 135°, \quad \cos\theta^{(1)} = -\sqrt{2}/2, \quad \sin\theta^{(1)} = \sqrt{2}/2$

$$[k^{(1)}] = \frac{(5.0\times10^{-4}\text{ m}^2)(210\times10^6\text{ kN/m}^2)}{5\text{ m}}\begin{bmatrix} 0.5 & -0.5 & -0.5 & 0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \\ 0.5 & -0.5 & -0.5 & 0.5 \end{bmatrix}$$

$$= 105\times10^2 \begin{array}{c} \begin{array}{cccc} u_1 & v_1 & u_2 & v_2 \end{array} \\ \begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \end{array}$$

Element 2: $\theta^{(2)} = 180°, \quad \cos\theta^{(2)} = -1.0, \quad \sin\theta^{(2)} = 0$

$$[k^{(2)}] = \frac{(5\times10^{-4}\text{ m}^2)(210\times10^6\text{ kN/m}^2)}{10\text{ m}}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = 105\times10^2 \begin{array}{c} \begin{array}{cccc} u_1 & v_1 & u_3 & v_3 \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Element 3: $\theta^{(3)} = 270°, \quad \cos\theta^{(3)} = 0, \quad \sin\theta^{(3)} = -1.0$

$$[k^{(3)}] = 20\times10^2 \begin{array}{c} \begin{array}{cccc} u_1 & v_1 & u_4 & v_4 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \end{array}$$

Applying the boundary conditions $u_2 = v_2 = u_3 = v_3 = u_4 = v_4 = 0$

$$\begin{Bmatrix} F_{1x} = 0 \\ F_{1y} = -25\text{ kN} \end{Bmatrix} = 10^2 \begin{bmatrix} 210 & -105 \\ -105 & 125 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \end{Bmatrix}$$

$$u_1 = -1.724\times10^{-3}\text{ m} \qquad v_1 = -3.448\times10^{-3}\text{ m}$$

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We can obtain the stresses in the bar elements

$$\sigma^{(1)} = \frac{210\times10^3\ \mathrm{MN/m^2}}{5\ \mathrm{m}}\begin{bmatrix}0.707 & -0.707 & -0.707 & 0.707\end{bmatrix}\begin{Bmatrix}-1.724\times10^{-3}\\ -3.448\times10^{-3}\\ 0\\ 0\end{Bmatrix}$$

$$= 51.2\ \mathrm{MPa}\ (T)$$

$$\sigma^{(2)} = \frac{210\times10^3\ \mathrm{MN/m^2}}{10\ \mathrm{m}}\begin{bmatrix}1.0 & 0 & -1.0 & 0\end{bmatrix}\begin{Bmatrix}-1.724\times10^{-3}\\ -3.448\times10^{-3}\\ 0\\ 0\end{Bmatrix} = -36.2\ \mathrm{MPa}\ (\mathrm{C})$$

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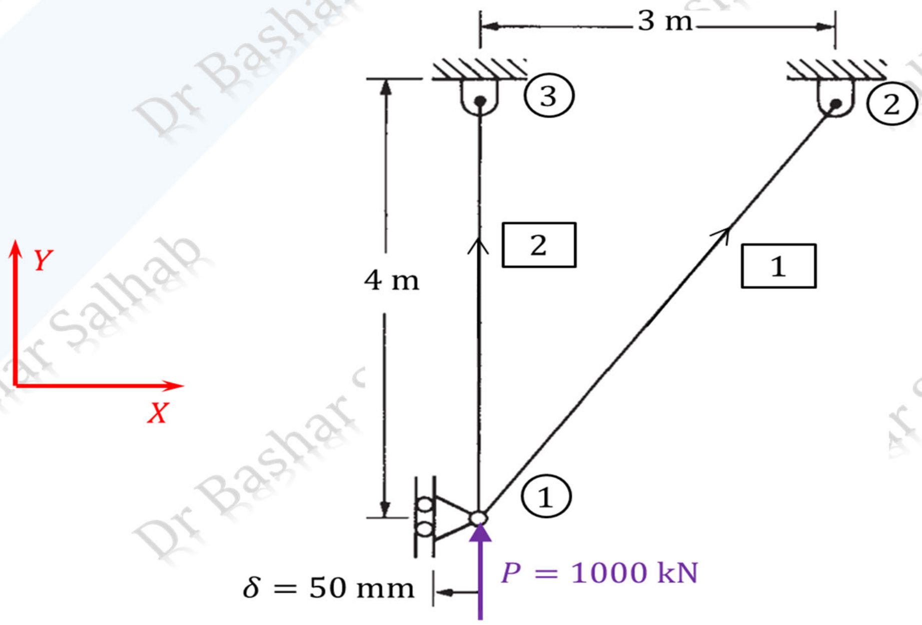
EXAMPLE (2D Truss)

For the two-bar truss shown in the figure, determine the vertical displacement of node 1 and the axial force in each element. A force of $P = 1000$ kN is applied at node 1 while it moves an amount of $\delta = 50$ mm in the negative x direction. $E = 210$ GPa , $A = 6\times10^{-4}\,\mathrm{m}^2$ for each element.



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