Undergraduate Course

FINITE ELEMENT METHODS

Development of Beam Equations & Shape Function

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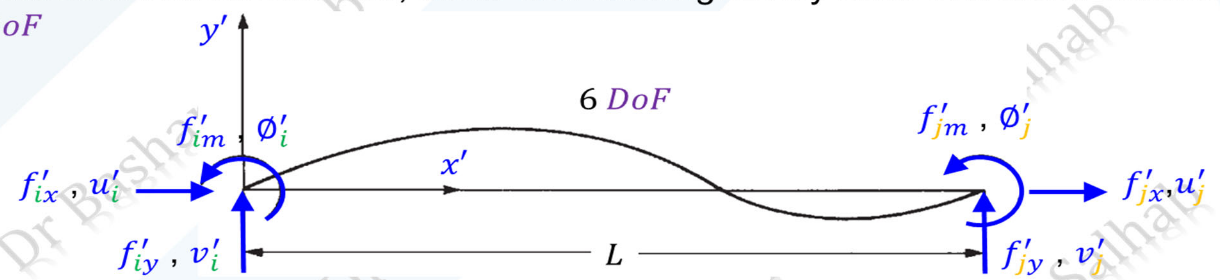
Frame Element:

an element that carries axial, shear and bending. Every node in a frame element has 3 DoF

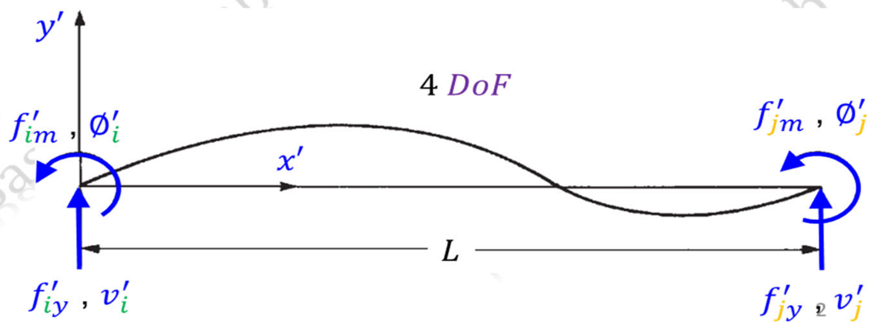


positive nodal displacements, rotations, forces, and moments

At all nodes, the following sign conventions are used: Moments & Rotations are positive in the counterclockwise direction. Vertical Forces & Displacements are positive in the positive y' direction. Horizontal Forces & Displacements are positive in the positive x' direction.

Beam Element:

an element where transversal loads are significantly larger than axial or torsional loading, causing it to be experiencing bending moments and shear forces. Every node in a beam element has 2 DoF



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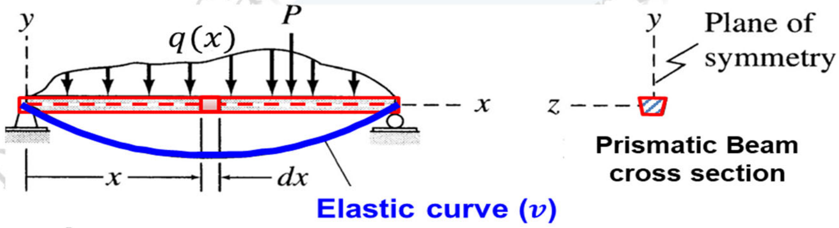
Differential Equation for Euler-Bernoulli beam deflection: (The Exact Solution)

Euler-Bernoulli beam theory i.e. considering bending deformations only



Slope of the elastic curve: $\frac{dv}{dx} = \tan(\emptyset) \approx \emptyset$

Moment deflection relationship: $M = EI\frac{\partial^2 v}{\partial x^2}$

$$\boxed{\frac{d^2 v}{dx^2} = \frac{M(x)}{EI} = \frac{d\emptyset}{dx}}$$

Differential Equation for Euler-Bernoulli Beam Deflection (Euler-Bernoulli beam equation)

$$q(x) = \frac{\partial V}{\partial x} = \frac{\partial^2 M}{\partial x^2} = \frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 v}{\partial x^2}\right) \Longrightarrow q(x) = EI\left(\frac{\partial^4 v}{\partial x^4}\right)$$

$$\boxed{\frac{q(x)}{EI} = \frac{\partial^4 v}{\partial x^4}}$$

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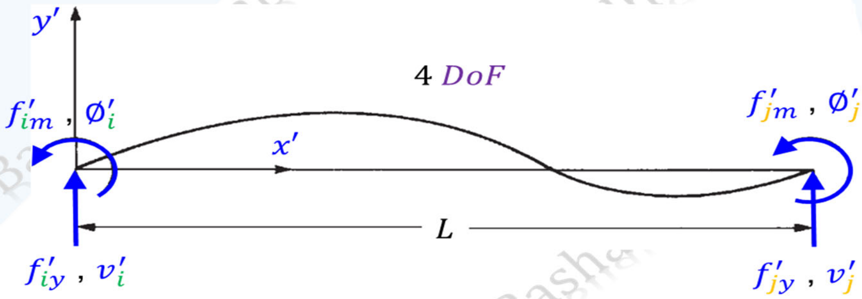
FEM using an approximate solution for continuous beam

Step 1 Select the Element Type



Step 2 Select a Displacement Function

Step 3 Define the Strain/Displacement & Stress/Strain Relationships

Step 4 Derive the Element Stiffness Matrix and Equations

Determine the shear & bending moment function from the deflection function

$$M = EI\frac{\partial^2 v}{\partial x^2} \qquad V = EI\frac{\partial^3 v}{\partial x^3}$$

Determine the stress components from the moment and shear functions

$$\sigma_x = -\frac{My}{I} \qquad \tau_{xy} = \frac{VS}{Ib}$$

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Selecting a deflection Function

By integrating the equation $\frac{q(x)}{EI}=\frac{\partial^4 v}{\partial x^4}$ we get the elastic curve equation $(v(x))$ resulting in 4 constants that can be obtained from BC

For constant EI over the length L , and only nodal forces and moments (i.e. no distributed load between nodes as in FEM which assumes that loads are applied at nodes), we have: $\frac{\partial^4 v}{\partial x^4}=0$ By integration we get: $v=a_0+a_1x+a_2x^2+a_3x^3$

So we assume the displacement function in the form: $v(x)=a_0+a_1x+a_2x^2+a_3x^3$

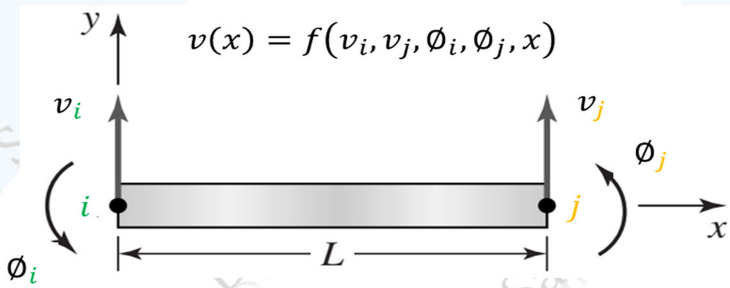
The choice of a cubic displacement function is appropriate because:

- There are 4 DoF & 4 BC, no more than 4 constants in the assumed displacement function can be determined.
- The loads are applied only at the element nodes, the bending moment varies linearly. Hence $\frac{\partial 2v(x)}{\partial x^2}$ is linear.

a truss has 2 DoF which are the nodal displacements (u_1, u_2), thus we choose the displacement function $u=a_0+a_1x$

Shape functions of continuous beam element

Subjected to BC:
$$\begin{cases} v(x=0)=v_i \\ v(x=L)=v_j \\ \left.\frac{dv}{dx}\right|_{x=0}=\emptyset_i \\ \left.\frac{dv}{dx}\right|_{x=L}=\emptyset_j \end{cases}$$



Beam element nodal displacements shown in positive directions

Application of the BC : $v(x)=a_0+a_1x+a_2x^2+a_3x^3$

$$v(0)=v_i=a_0 \qquad \left.\frac{dv}{dx}\right|_{x=0}=\emptyset_i=a_1$$

$$v(L)=v_j=a_0+a_1L+a_2L^2+a_3L^3 \qquad \left.\frac{dv}{dx}\right|_{x=L}=\emptyset_j=a_1+2a_2L+3a_3L^2$$

Solving them simultaneously to obtain the coefficients in terms of the nodal degrees of freedom as: $\Rightarrow$

$$\begin{cases} a_1=\emptyset_i \\ a_0=v_i \\ a_2=\frac{3}{L^2}(v_2-v_1)-\frac{1}{L}(2\emptyset_i+\emptyset) \\ a_3=\frac{2}{L^3}(v_1-v_2)+\frac{1}{L^2}(\emptyset_i+\emptyset_j) \end{cases}$$

substituting into $v(x)$

$$v(x) = \left(1 - \frac{3x^2}{L^2} + \frac{2x^3}{L^3}\right)v_i + \left(x - \frac{2x^2}{L} + \frac{x^3}{L^2}\right)\emptyset_i + \left(\frac{3x^2}{L^2} - \frac{2x^3}{L^3}\right)v_j + \left(\frac{x^3}{L^2} - \frac{x^2}{L}\right)\emptyset_j$$

Which is of the form: $v(x) = N_1(x)v_i + N_2(x)\emptyset_i + N_3(x)v_j + N_4(x)\emptyset_j$

In matrix form: $v(x) = \{N_1 \quad N_2 \quad N_3 \quad N_4\}\begin{Bmatrix} v_i \\ \emptyset_i \\ v_j \\ \emptyset_j \end{Bmatrix} = \mathbf{N\,d}$

$$\left.\begin{aligned} N_1 &= \frac{1}{L^3}(2x^3 - 3x^2L + L^3) \\ N_2 &= \frac{1}{L^3}(x^3L - 2x^2L^2 + xL^3) \\ N_3 &= \frac{1}{L^3}(-2x^3 + 3x^2L) \\ N_4 &= \frac{1}{L^3}(x^3L - x^2L^2) \end{aligned}\right\}$$

N_1, N_2, N_3, and N_4 are called the shape functions for a beam element

For the beam element, $N_1 = 1$ when evaluated at node 1 and $N_1 = 0$ when evaluated at node 2. Because N_2 is associated with $\emptyset_i$, we have, from the second of Eqs. ($\partial N_2/\partial x = 1$ when evaluated at node 1. Shape functions N_3 and N_4 have analogous results for node 2.

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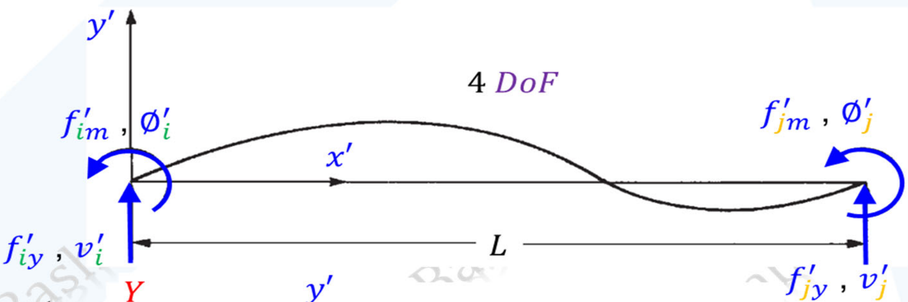
Beam Stiffness Matrix

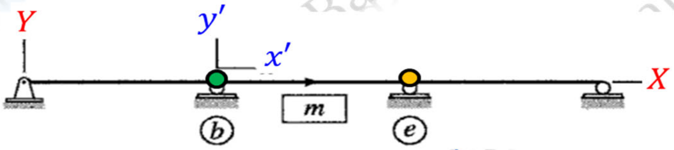
$$M = EI\frac{\partial^2 v}{\partial x^2}$$

$$V = EI\frac{\partial^3 v}{\partial x^3}$$



For Continuous Beam elements:

$\mathbf{F} = \mathbf{f'}$ $\qquad$ $\mathbf{d} = \mathbf{d'}$



$$\left.\begin{aligned} f_{iy} &= V = EI\frac{\partial^3 v(0)}{\partial x^3} = \frac{EI}{L^3}(12v_i + 6L\emptyset_i - 12v_j + 6L\emptyset_j) \\ f_{im} &= -M = -EI\frac{\partial^2 v(0)}{\partial x^2} = \frac{EI}{L^3}(6Lv_i + 4L^2\emptyset_i - 6Lv_j + 2L^2\emptyset_j) \\ f_{jy} &= -V = -EI\frac{\partial^3 v(L)}{\partial x^3} = \frac{EI}{L^3}(-12v_i - 6L\emptyset_i + 12v_j - 6L\emptyset_j) \\ f_{jm} &= M = EI\frac{\partial^2 v(L)}{\partial x^2} = \frac{EI}{L^3}(6Lv_i + 2L^2\emptyset_i - 6Lv_j + 4L^2\emptyset_j) \end{aligned}\right\} \Longrightarrow \mathbf{f'} = \mathbf{k'}\,\mathbf{d'}$$

$$\Longleftrightarrow \quad \begin{Bmatrix} f_{iy} \\ f_{im} \\ f_{jy} \\ f_{jm} \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} v_i \\ \emptyset_i \\ v_j \\ \emptyset_j \end{Bmatrix}$$

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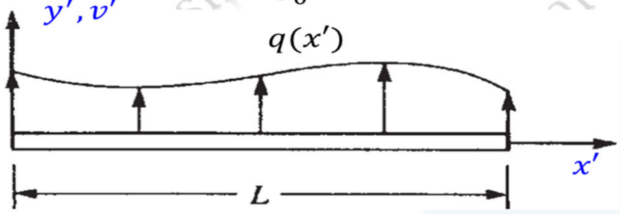
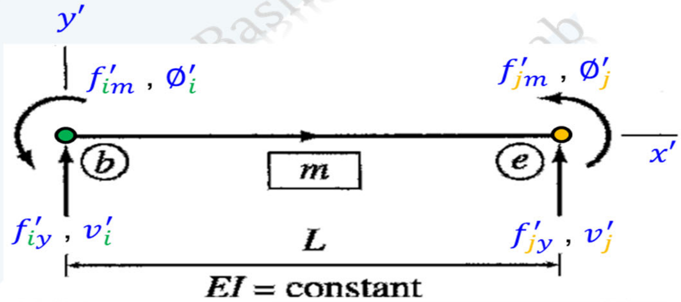
Fixed End Moments $\mathbf{f}'_0$

If we have loads between nodes, then: $\mathbf{f}' = \mathbf{k}'\,\mathbf{d}' + \mathbf{f}'_0$

using work-equivalence method to obtain $\mathbf{f}'_0$:

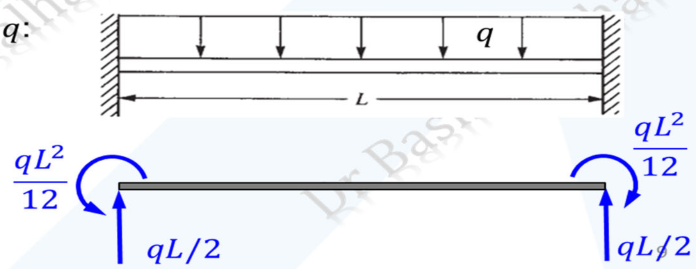
$$W_{distributed} = \int_0^L q(x')v'(x')dx' \qquad W_{discrete} = f'_{im}\,\phi'_i + f'_{jm}\phi'_j + f'_{iy}\,v'_i + f'_{jy}\,v'_j$$

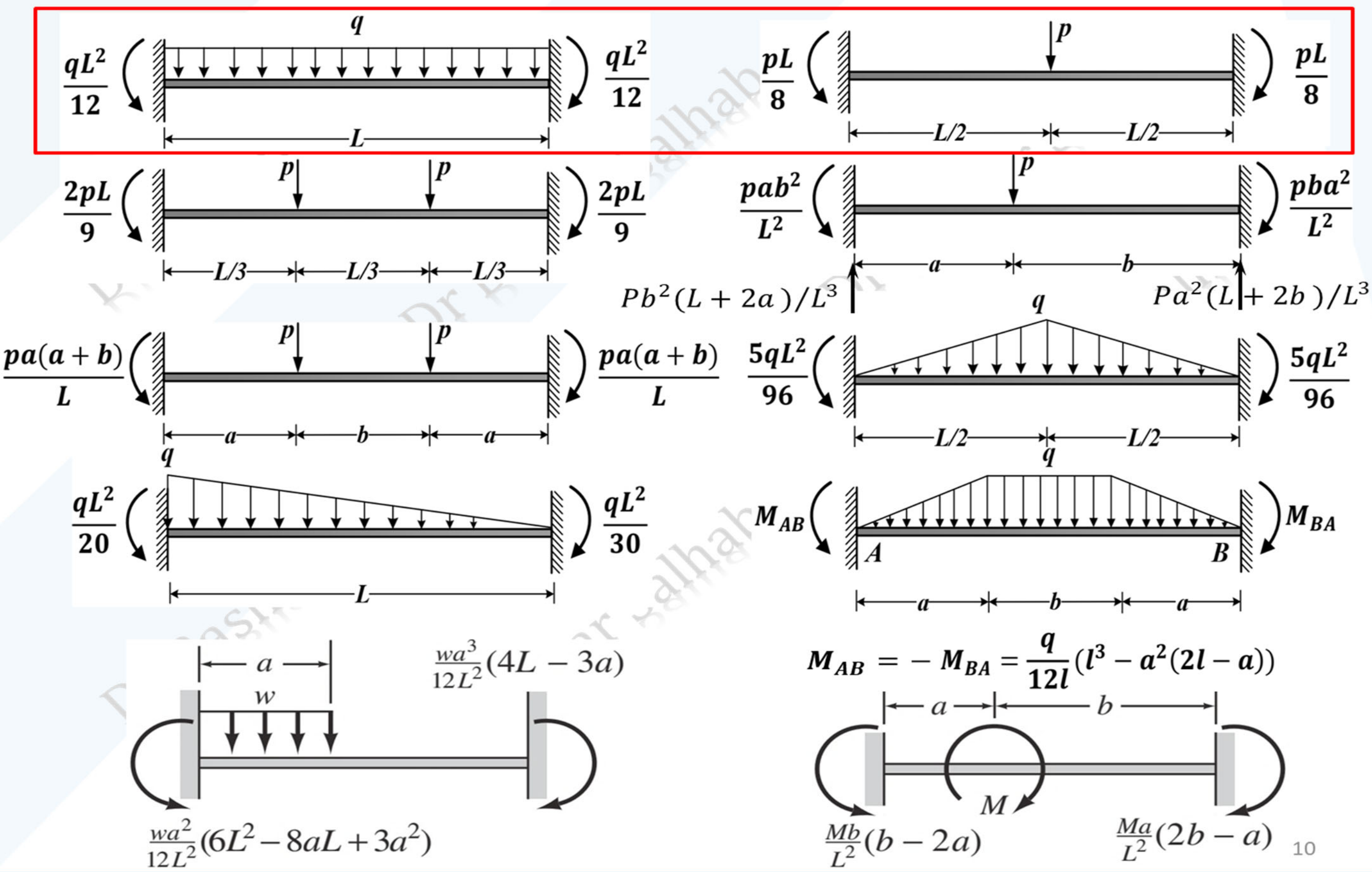




If we take uniformly Distributed Loading q:

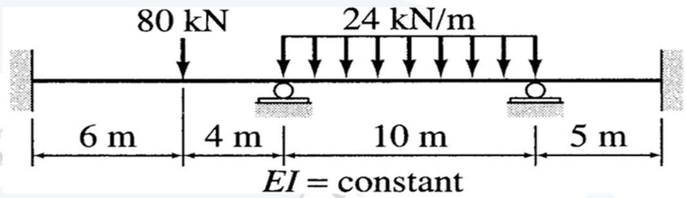
equivalent nodal forces: $\mathbf{f}'_0 = \begin{Bmatrix} \frac{ql}{2} \\ \frac{qL^2}{12} \\ \frac{ql}{2} \\ -\frac{qL^2}{12} \end{Bmatrix}$





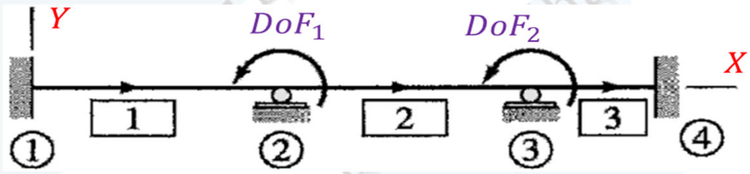
EXAMPLE (Continuous beam)

Determine the reactions and the member end forces for the three-span continuous beam using the matrix stiffness method



Solution:

From the analytical model the two unknown degrees of freedom, DoF_1 & DoF_2, the two rotations at nodes 2 & 3



The element stiffness matrix: $$\mathbf{k}' = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

$$\begin{Bmatrix} f'_{iy} \\ f'_{im} \\ f'_{jy} \\ f'_{jm} \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} v'_i \\ \emptyset'_i \\ v'_j \\ \emptyset'_j \end{Bmatrix} + \begin{Bmatrix} f'_{iy0} \\ f'_{im0} \\ f'_{jy0} \\ f'_{jm0} \end{Bmatrix}$$

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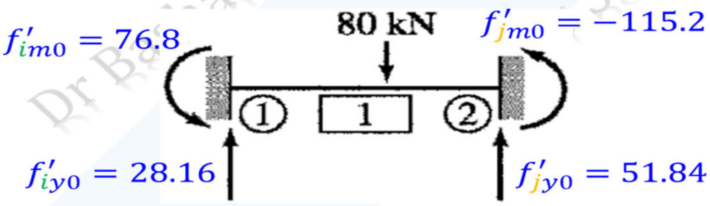
Element (1): using $L = 10$ m & the fixed-end forces from the table

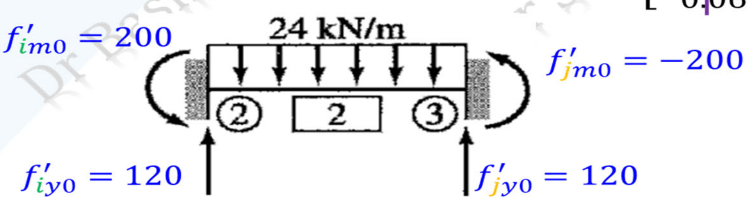
$$\mathbf{K}^{(1)} = \mathbf{k}'^{(1)} = EI \begin{array}{c} \begin{array}{cccc} 0 & 0 & 0 & DoF_1 \end{array} \\ \begin{bmatrix} 0.012 & 0.06 & -0.012 & 0.06 \\ 0.06 & 0.4 & -0.06 & 0.2 \\ -0.012 & -0.06 & 0.012 & -0.06 \\ 0.06 & 0.2 & -0.06 & 0.4 \end{bmatrix} \end{array} \begin{array}{c} 0 \\ 0 \\ 0 \\ DoF_1 \end{array}$$



$$\begin{Bmatrix} f'_{iy0} \\ f'_{im0} \\ f'_{jy0} \\ f'_{jm0} \end{Bmatrix} = \begin{Bmatrix} 28.16 \\ 76.80 \\ 51.84 \\ -115.2 \end{Bmatrix} \begin{array}{c} 0 \\ 0 \\ 0 \\ DoF_1 \end{array}$$

Element (2): using $L = 10$ m & the fixed-end forces from the table

$$\mathbf{K}^{(2)} = \mathbf{k}'^{(2)} = EI \begin{array}{c} \begin{array}{cccc} 0 & DoF_1 & 0 & DoF_2 \end{array} \\ \begin{bmatrix} 0.012 & 0.06 & -0.012 & 0.06 \\ 0.06 & 0.4 & -0.06 & 0.2 \\ -0.012 & -0.06 & 0.012 & -0.06 \\ 0.06 & 0.2 & -0.06 & 0.4 \end{bmatrix} \end{array} \begin{array}{c} 0 \\ DoF_1 \\ 0 \\ DoF_2 \end{array}$$



$$\begin{Bmatrix} f'_{iy0} \\ f'_{im0} \\ f'_{jy0} \\ f'_{jm0} \end{Bmatrix} = \begin{Bmatrix} 120 \\ 200 \\ 120 \\ -200 \end{Bmatrix} \begin{array}{c} 0 \\ DoF_1 \\ 0 \\ DoF_2 \end{array}$$

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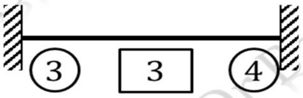
Element (3): using $L = 5$ m & the fixed-end forces from the table

$$\mathbf{K}^{(3)} = \mathbf{k}'^{(3)} = EI \begin{array}{c} \begin{array}{cccc} 0 & DoF_2 & 0 & 0 \end{array} \\ \begin{bmatrix} 0.096 & 0.24 & -0.096 & 0.24 \\ 0.24 & 0.8 & -0.24 & 0.4 \\ -0.096 & -0.24 & 0.096 & -0.24 \\ 0.24 & 0.4 & 0.24 & 0.8 \end{bmatrix} \end{array} \begin{array}{c} 0 \\ DoF_2 \\ 0 \\ 0 \end{array}$$



$$\begin{Bmatrix} f'_{iy0} \\ f'_{im0} \\ f'_{jy0} \\ f'_{jm0} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \begin{array}{c} 0 \\ DoF_2 \\ 0 \\ 0 \end{array}$$

Joint Load Vector $\overline{\boldsymbol{P}}$: Since no external moments are applied to the beam at joints 2 and 3 : $\overline{\boldsymbol{P}} = 0$

$$\mathbf{S} = EI \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} (0.4+0.4) & 0.2 \\ 0.2 & (0.4+0.8) \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \end{array} = EI \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} 0.8 & 0.2 \\ 0.2 & 1.2 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \end{array} \qquad \mathbf{P}_f = \begin{bmatrix} (-115.2+200) \\ -200 \end{bmatrix} \begin{array}{c} 1 \\ 2 \end{array} = \begin{bmatrix} 84.8 \\ -200 \end{bmatrix} \begin{array}{c} 1 \\ 2 \end{array}$$

$$\overline{\boldsymbol{P}} - \boldsymbol{P}_f = \boldsymbol{S}\,\boldsymbol{DoF} \Longrightarrow \begin{Bmatrix} -84.8 \\ 200 \end{Bmatrix} = EI \begin{bmatrix} 0.8 & 0.2 \\ 0.2 & 1.2 \end{bmatrix} \begin{Bmatrix} DoF_1 \\ DoF_2 \end{Bmatrix} \Longrightarrow \begin{Bmatrix} DoF_1 \\ DoF_2 \end{Bmatrix} = \frac{1}{EI} \begin{Bmatrix} -154.09 \\ 192.35 \end{Bmatrix}$$

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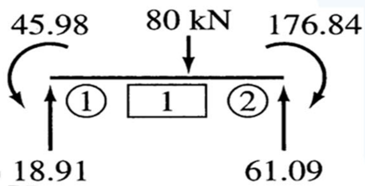
Member End Displacements and End Forces

$$\mathbf{f}' = \mathbf{k}'\,\mathbf{d}' + \mathbf{f}'_0$$

Element (1):

$$\mathbf{d}'^{(1)} = \mathbf{d}^{(1)} = \begin{Bmatrix} v_1 \\ \phi_1 \\ v_2 \\ \phi_2 \end{Bmatrix} \begin{array}{c} 0 \\ 0 \\ 0 \\ 1 \end{array} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ DoF_1 \end{Bmatrix} = \frac{1}{EI} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -154.09 \end{Bmatrix}$$

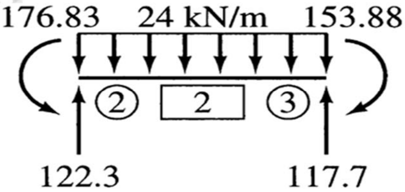


$$\mathbf{f}^{(1)} = \mathbf{f}'^{(1)} =$$

$$= EI \begin{bmatrix} 0.012 & 0.06 & -0.012 & 0.06 \\ 0.06 & 0.4 & -0.06 & 0.2 \\ -0.012 & -0.06 & 0.012 & -0.06 \\ 0.06 & 0.2 & -0.06 & 0.4 \end{bmatrix} \frac{1}{EI} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -154.09 \end{Bmatrix} + \begin{Bmatrix} 28.16 \\ 76.80 \\ 51.84 \\ -115.2 \end{Bmatrix} = \begin{Bmatrix} 18.91 \\ 45.98 \\ 61.09 \\ -176.84 \end{Bmatrix} \begin{array}{l} \text{kN} \\ \text{kN.m} \\ \text{kN} \\ \text{kN.m} \end{array}$$

Element (2):

$$\mathbf{d}'^{(2)} = \mathbf{d}^{(2)} = \begin{Bmatrix} v_2 \\ \phi_2 \\ v_3 \\ \phi_3 \end{Bmatrix} \begin{array}{c} 0 \\ 1 \\ 0 \\ 2 \end{array} = \begin{Bmatrix} 0 \\ DoF_1 \\ 0 \\ DoF_2 \end{Bmatrix} = \frac{1}{EI} \begin{Bmatrix} 0 \\ -154.09 \\ 0 \\ 192.35 \end{Bmatrix}$$



$$\mathbf{f}^{(2)} = \mathbf{f}'^{(2)} =$$

$$= EI \begin{bmatrix} 0.012 & 0.06 & -0.012 & 0.06 \\ 0.06 & 0.4 & -0.06 & 0.2 \\ -0.012 & -0.06 & 0.012 & -0.06 \\ 0.06 & 0.2 & -0.06 & 0.4 \end{bmatrix} \frac{1}{EI} \begin{Bmatrix} 0 \\ -154.09 \\ 0 \\ 192.35 \end{Bmatrix} + \begin{Bmatrix} 120 \\ 200 \\ 120 \\ -200 \end{Bmatrix} = \begin{Bmatrix} 122.3 \\ 176.83 \\ 117.7 \\ -153.88 \end{Bmatrix} \begin{array}{l} \text{kN} \\ \text{kN.m} \\ \text{kN} \\ \text{kN.m} \end{array}$$

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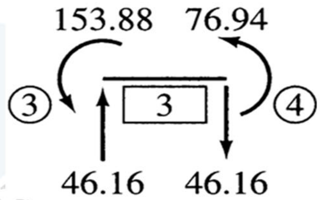
Element (3):

$$\mathbf{d}'^{(3)} = \mathbf{d}^{(3)} = \begin{Bmatrix} v_3 \\ \emptyset_3 \\ v_4 \\ \emptyset_4 \end{Bmatrix} \begin{matrix} 0 \\ 2 \\ 0 \\ 0 \end{matrix} = \begin{Bmatrix} 0 \\ DoF_2 \\ 0 \\ 0 \end{Bmatrix} = \frac{1}{EI} \begin{Bmatrix} 0 \\ 192.35 \\ 0 \\ 0 \end{Bmatrix}$$



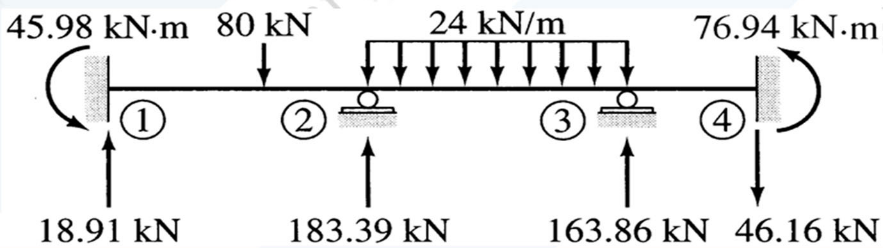
$$\mathbf{f}^{(3)} = \mathbf{f}'^{(3)} =$$

$$= EI \begin{bmatrix} 0.096 & 0.24 & -0.096 & 0.24 \\ 0.24 & 0.8 & -0.24 & 0.4 \\ -0.096 & -0.24 & 0.096 & -0.24 \\ 0.24 & 0.4 & -0.24 & 0.8 \end{bmatrix} \frac{1}{EI} \begin{Bmatrix} 0 \\ 192.35 \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 46.16 \\ 153.88 \\ -46.16 \\ 76.94 \end{Bmatrix} \begin{matrix} \text{kN} \\ \text{kN.m} \\ \text{kN} \\ \text{kN.m} \end{matrix}$$

Support Reactions:

$$\mathbf{R}_{①} = \begin{Bmatrix} 18.91 \text{ kN} \\ 45.98 \text{ kN.m} \end{Bmatrix} \qquad \mathbf{R}_{②} = \begin{Bmatrix} 61.09 \\ -176.84 \end{Bmatrix} + \begin{Bmatrix} 122.3 \\ 176.83 \end{Bmatrix} = \begin{Bmatrix} 183.39 \text{ kN} \\ 0 \end{Bmatrix}$$

$$\mathbf{R}_{③} = \begin{Bmatrix} 117.7 \\ -153.88 \end{Bmatrix} + \begin{Bmatrix} 46.16 \\ 153.88 \end{Bmatrix} = \begin{Bmatrix} 163.86 \text{ kN} \\ 0 \end{Bmatrix} \qquad \mathbf{R}_{④} = \begin{Bmatrix} -46.16 \text{ kN} \\ 76.94 \text{ kN.m} \end{Bmatrix}$$



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Frame Element

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