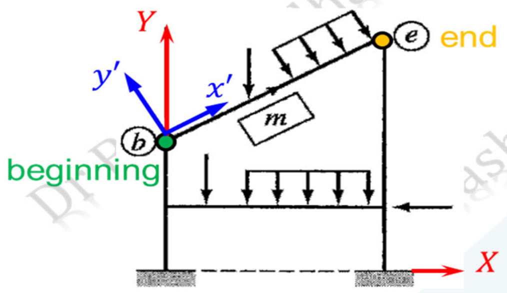
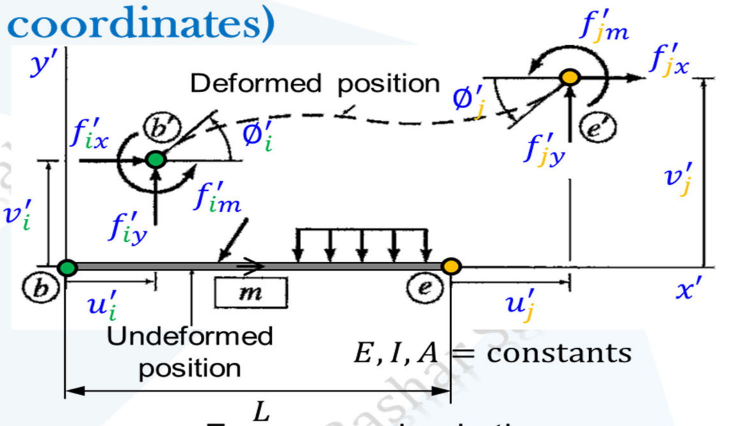
Planar frame elements (local coordinates)



Frame



Frame member in the Local Coordinates $x'y'$

x' & y' are the local coordinates system of the undeformed frame member be. Under external actions the member move to the deformed position $b'e'$

vector $\mathbf{d}'$ is the local nodal displacements $\mathbf{d}' \equiv \{d'\} = \begin{Bmatrix} u'_i \\ v'_i \\ \emptyset'_i \\ u'_j \\ v'_j \\ \emptyset'_j \end{Bmatrix}$

vector $\mathbf{f}'$ is the local nodal forces $\mathbf{f}' \equiv \{f'\} = \begin{Bmatrix} f'_{ix} \\ f'_{iy} \\ f'_{im} \\ f'_{jx} \\ f'_{jy} \\ f'_{jm} \end{Bmatrix}$

Stiffness coefficients

$\sigma = E\varepsilon$ Constitutive Equation

$f_i = \sigma A$ Internal Equilibrium Equation

$\Delta L = L\varepsilon = u_i$ Compatibility Equation

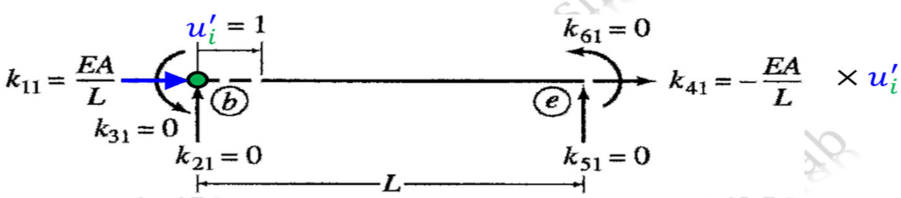
$f_i = (EA/L)u_i = k_{11}u_i$

$+\rightarrow \sum F_x = 0$ External Equilibrium Equation $f_i = -F$

For Beginning node i:

$k_{11} = -k_{41} = EA/L$

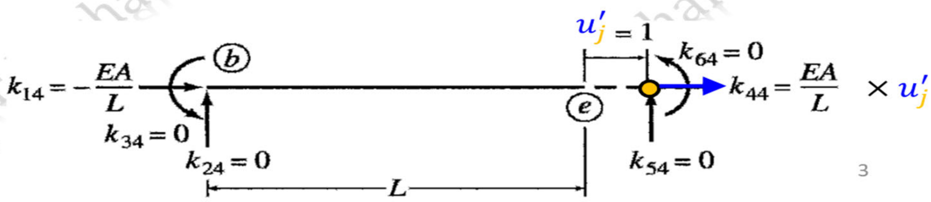
$k_{21} = k_{31} = k_{51} = k_{61} = 0$



Similarly for End node j:

$k_{14} = -k_{44} = -EA/L$

$k_{24} = k_{34} = k_{54} = k_{64} = 0$



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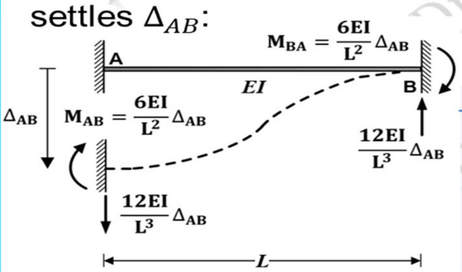
For Beginning node i:

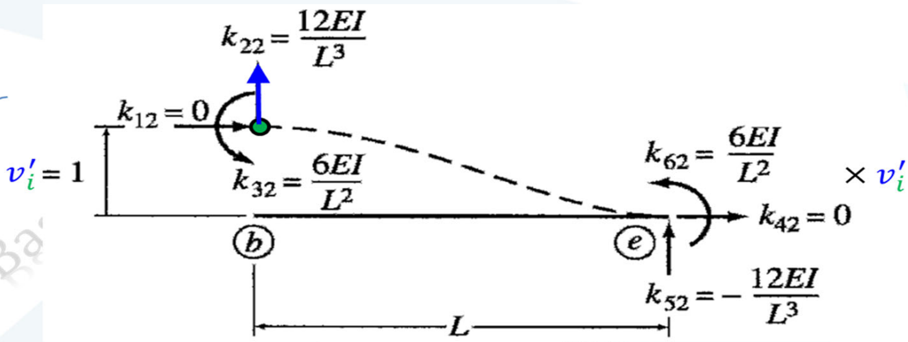
$k_{22} = -k_{52} = 12EI/L^3$

$k_{12} = k_{42} = 0$

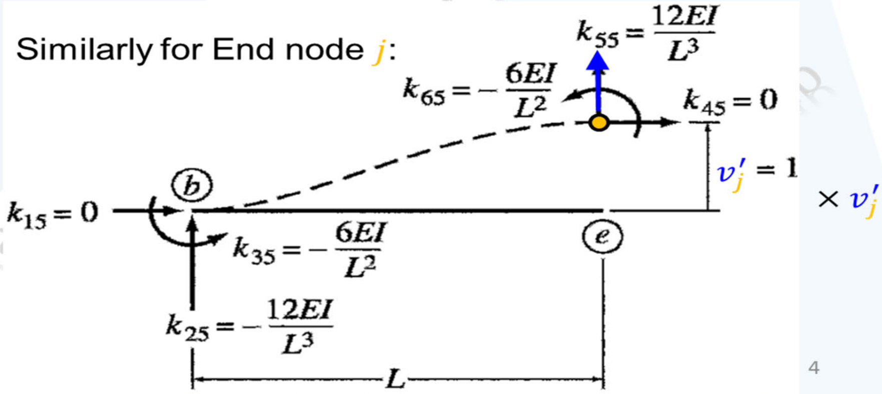
$k_{32} = k_{62} = 6EI/L^2$

stiffness coefficients if A settles Δ_{AB}:





Similarly for End node j:



$k_{25} = -k_{55} = -12EI/L^3$

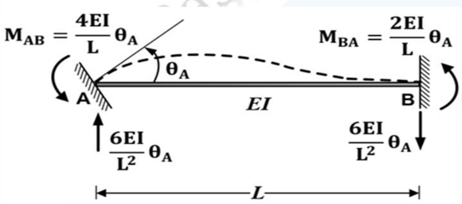
$k_{15} = k_{45} = 0$

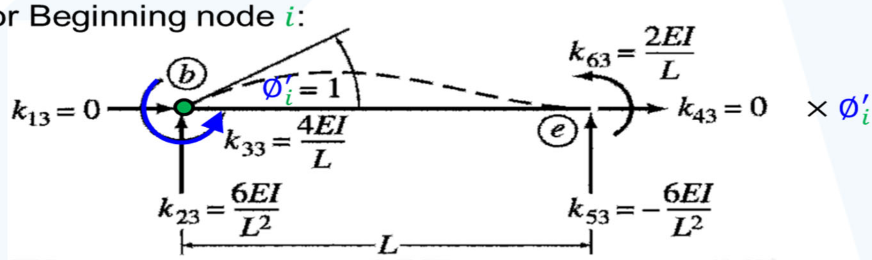
$k_{35} = k_{65} = -6EI/L^2$

4

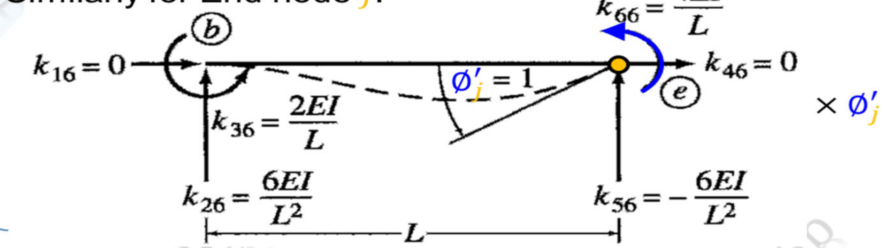
For Beginning node i:

stiffness coefficients if A rotates θ_A:

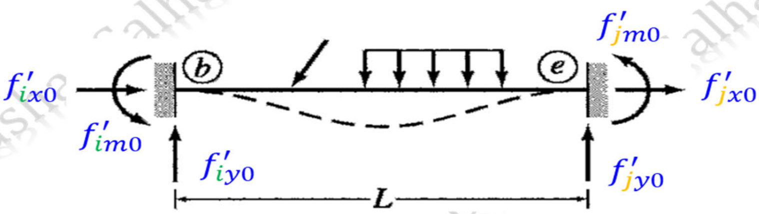




Similarly for End node j:



+ Fixed End Moments:



For beam & frame elements: $\mathbf{f'} = \mathbf{k'}\,\mathbf{d'} + \mathbf{f'_0}$

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Frame stiffness matrix:

end force vector (All in local coordinates), end displacement vector, fixed-end force vector

$$\begin{Bmatrix} f'_{ix} \\ f'_{iy} \\ f'_{im} \\ f'_{jx} \\ f'_{jy} \\ f'_{jm} \end{Bmatrix} = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} & k_{15} & k_{16} \\ k_{21} & k_{22} & k_{23} & k_{24} & k_{25} & k_{26} \\ k_{31} & k_{32} & k_{33} & k_{34} & k_{35} & k_{36} \\ k_{41} & k_{42} & k_{43} & k_{44} & k_{45} & k_{46} \\ k_{51} & k_{52} & k_{53} & k_{54} & k_{55} & k_{56} \\ k_{61} & k_{62} & k_{63} & k_{64} & k_{65} & k_{66} \end{bmatrix} \begin{Bmatrix} u'_i \\ v'_i \\ \emptyset'_i \\ u'_j \\ v'_j \\ \emptyset'_j \end{Bmatrix} + \begin{Bmatrix} f'_{ix0} \\ f'_{iy0} \\ f'_{im0} \\ f'_{jx0} \\ f'_{jy0} \\ f'_{jm0} \end{Bmatrix}$$

The stiffness matrix $\mathbf{k}$ (6×6) of the 2D frame element in local coordinates written as:

$$\mathbf{k}' = \frac{EI}{L^3}\begin{bmatrix} \frac{AL^2}{I} & 0 & 0 & -\frac{AL^2}{I} & 0 & 0 \\ 0 & 12 & 6L & 0 & -12 & 6L \\ 0 & 6L & 4L^2 & 0 & -6L & 2L^2 \\ -\frac{AL^2}{I} & 0 & 0 & \frac{AL^2}{I} & 0 & 0 \\ 0 & -12 & -6L & 0 & 12 & -6L \\ 0 & 6L & 2L^2 & 0 & -6L & 4L^2 \end{bmatrix}$$

can be separated into the 2 following matrices:

The stiffness matrix $\mathbf{k}'$ (4×4) of the 2D **beam member** in local coordinates.

Beam equation (shear & moment)

$$\mathbf{k}' = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

& the stiffness matrix $\mathbf{k}'$ (2×2) of 2D truss member in local coordinates.

Truss equation (axial)

$$\mathbf{k}' = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

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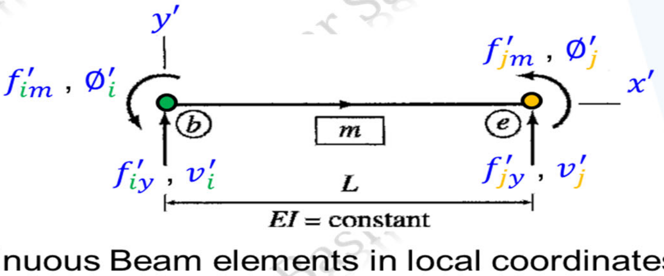
Continuous Beam stiffness matrix:

we do not need to consider the degrees of freedom in the direction of the member's centroidal axis in the analysis. Thus, only four degrees of freedom need to be considered.

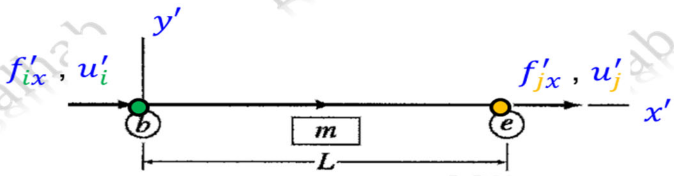
$$\mathbf{k}' = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$



Continuous Beam elements in local coordinates

Truss stiffness matrix:

$$\mathbf{k}' = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$



The beam member matrix can be used in global coordinates but the frame & truss matrices must be transformed from local to global coordinates using $\mathbf{T}$ & $\mathbf{T}^T$.

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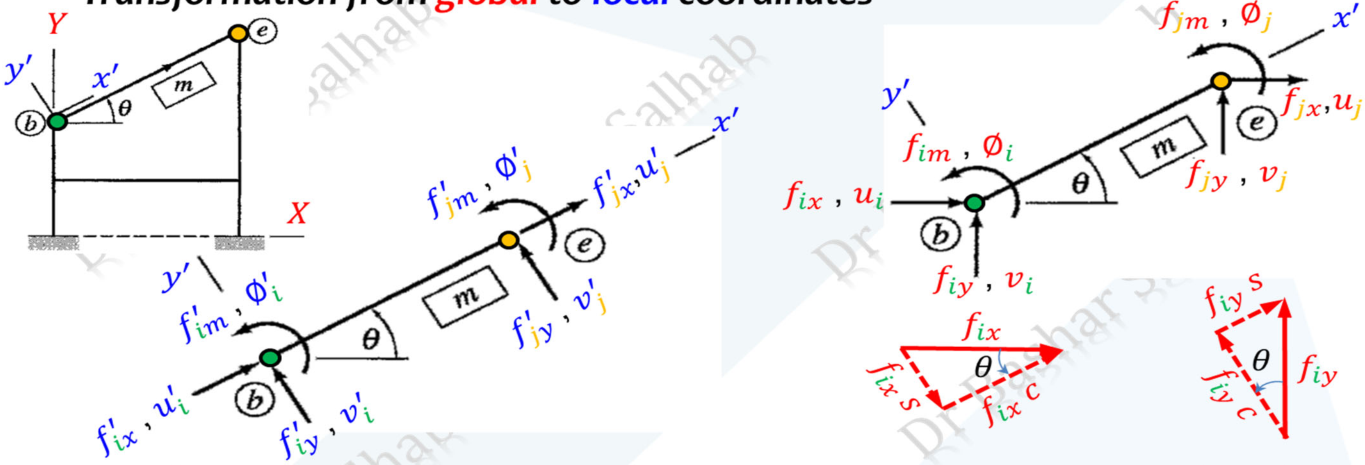
Coordinate Transformations for Frame Members

Transformation from global to local coordinates



Member end forces $\mathbf{f}'$ & end displacement $\mathbf{d}$ in local coordinates

Member end forces $\mathbf{f}$ & end displacement $\mathbf{d}$ in Global coordinates

$$\left.\begin{aligned} f'_{ix} &= f_{ix}\,c + f_{iy}\,s \\ f'_{iy} &= -f_{ix}\,s + f_{iy}\,c \\ f'_{im} &= f_{im} \\ f'_{jx} &= f_{jx}\,c + f_{jy}\,s \\ f'_{jy} &= -f_{jx}\,s + f_{jy}\,c \\ f'_{jm} &= f_{jm} \end{aligned}\right\} \quad \begin{Bmatrix} f'_{ix} \\ f'_{iy} \\ f'_{im} \\ f'_{jx} \\ f'_{jy} \\ f'_{jm} \end{Bmatrix} = \begin{bmatrix} c & s & 0 & 0 & 0 & 0 \\ -s & c & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & 0 & 0 \\ 0 & 0 & 0 & -s & s & 0 \\ 0 & 0 & 0 & 0 & c & 1 \end{bmatrix} \begin{Bmatrix} f_{ix} \\ f_{iy} \\ f_{im} \\ f_{jx} \\ f_{jy} \\ f_{jm} \end{Bmatrix} \Longleftrightarrow \mathbf{f}' = \mathbf{T}\,\mathbf{F}$$

Similarly: $\mathbf{d}' = \mathbf{T}\,\mathbf{d}$

$$\begin{Bmatrix} u'_i \\ v'_i \\ \phi'_i \\ u'_j \\ v'_j \\ \phi'_j \end{Bmatrix} = \begin{bmatrix} c & s & 0 & 0 & 0 & 0 \\ -s & c & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & s & 0 \\ 0 & 0 & 0 & -s & c & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_i \\ v_i \\ \phi_i \\ u_j \\ v_j \\ \phi_j \end{Bmatrix} \qquad \begin{Bmatrix} \phi'_i \\ \phi'_j \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \phi_i \\ \phi_j \end{Bmatrix}$$

transformation matrix from global to local for frame element

Note that, the rotation needs no transformation in 2D, because the rotation vector is perpendicular to the xy plane in 2D problems

Transformation from local to global coordinates

$$\left.\begin{aligned} f_{ix} &= f'_{ix}\,c - f'_{iy}\,s \\ f_{iy} &= f'_{ix}\,s + f'_{iy}\,c \\ f_{im} &= f'_{im} \\ f_{jx} &= f'_{jx}c - f'_{jy}s \\ f_{jy} &= f'_{jx}s + f'_{jy}c \\ f_{jm} &= f'_{jm} \end{aligned}\right\} \quad \begin{Bmatrix} f_{ix} \\ f_{iy} \\ f_{im} \\ f_{jx} \\ f_{jy} \\ f_{jm} \end{Bmatrix} = \begin{bmatrix} c & -s & 0 & 0 & 0 & 0 \\ s & c & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & -s & 0 \\ 0 & 0 & 0 & s & c & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} f'_{ix} \\ f'_{iy} \\ f'_{im} \\ f'_{jx} \\ f'_{jy} \\ f'_{jm} \end{Bmatrix} \Longleftrightarrow \mathbf{F} = \mathbf{T}^{\mathrm{T}}\mathbf{f}'$$

Similarly: $\mathbf{d} = \mathbf{T}^{\mathrm{T}}\mathbf{d}'$

Member Stiffness Relations in Global Coordinates

Frame Members:

$$\left.\begin{aligned}\mathbf{f}' &= \mathbf{k}'\,\mathbf{d}' + \mathbf{f}'_0\\ \mathbf{F} &= \mathbf{T}^{\mathrm{T}}\mathbf{f}'\end{aligned}\right\}\quad \mathbf{F} = \mathbf{T}^{\mathrm{T}}(\mathbf{k}'\mathbf{d}' + \mathbf{f}'_0) = \mathbf{T}^{\mathrm{T}}\mathbf{k}'\mathbf{d}' + \mathbf{T}^{\mathrm{T}}\mathbf{f}'_0$$

$$\mathbf{d}' = \mathbf{T}\,\mathbf{d} \Longrightarrow \mathbf{F} = \underbrace{\mathbf{T}^{\mathrm{T}}\mathbf{k}'\mathbf{T}}\,\mathbf{d} + \mathbf{T}^{\mathrm{T}}\mathbf{f}'_0$$

the member stiffness matrix in global coordinates: $\mathbf{K} = \mathbf{T}^{\mathrm{T}}\mathbf{k}'\mathbf{T}$

the member fixed-end force vector in global coordinates: $\mathbf{F}_0 = \mathbf{T}^{\mathrm{T}}\mathbf{f}'_0$

$$\mathbf{F} = \mathbf{K}\mathbf{d} + \mathbf{F}_0$$

Continuous Beam Members:

The member stiffness relations in the local and global coordinates are the same.

Local coordinates are identical to the global ones: $\mathbf{T} = \mathbf{T}^{\mathrm{T}} = \mathbf{I}$

Truss Members: $\mathbf{F}_0 = 0$ $\quad\quad \mathbf{F} = \mathbf{K}\,\mathbf{d}$

$$\mathbf{K} = \begin{bmatrix} c & 0\\ s & 0\\ 0 & c\\ 0 & s\end{bmatrix}\frac{EA}{L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{bmatrix} c & s & 0 & 0\\ 0 & 0 & c & s\end{bmatrix} = \frac{EA}{L}\begin{bmatrix} c^2 & c\,s & -c^2 & -c\,s\\ c\,s & s^2 & -c\,s & -s^2\\ -c^2 & -c\,s & c^2 & c\,s\\ -c\,s & -s^2 & c\,s & s^2\end{bmatrix}$$

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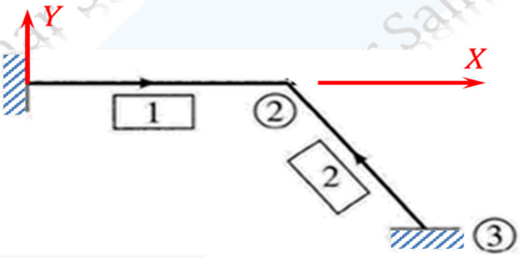
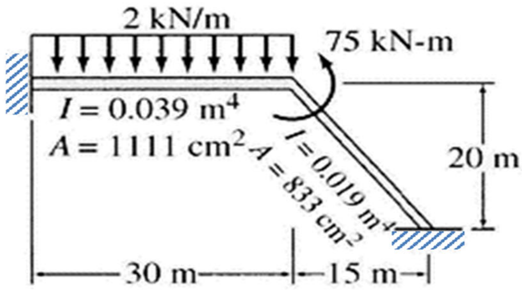
EXAMPLE

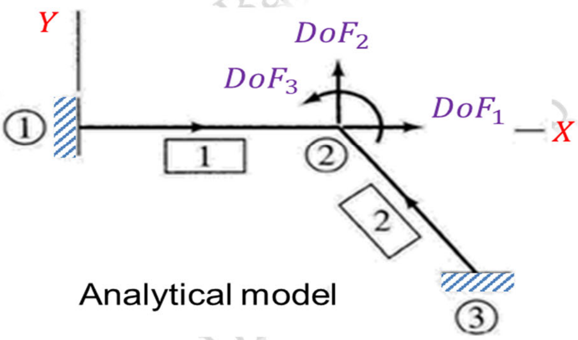
Determine the reactions and the member end forces for the frame shown in the figure by using the matrix stiffness method.





Solution:

Joint Load Vector $\mathbf{P} = \begin{bmatrix}0\\0\\75\end{bmatrix}$



Analytical model

Degrees of Freedom: the frame has 3 degrees of freedom: the translations DoF_1 and DoF_2 in the X and Y directions, respectively, and the rotation DoF_3 of joint 2

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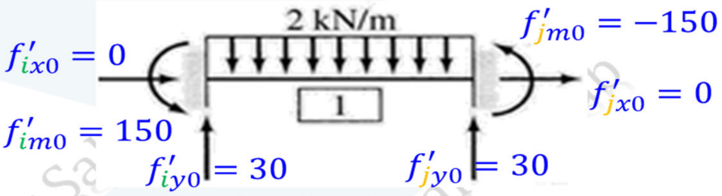
Element (1):

$$\mathbf{F}_0^{(1)} = \mathbf{f}_0^{\prime(1)} = \begin{array}{c} \left[\begin{array}{r} 0 \\ 30 \\ 150 \\ \hline 0 \\ 30 \\ -150 \end{array}\right] \end{array} \begin{array}{c} 0 \\ 0 \\ 0 \\ 1 \\ 2 \\ 3 \end{array}$$



Member fixed-end forces

$$\mathbf{K}^{(1)} = \mathbf{k}^{\prime(1)} = \begin{array}{c} \begin{array}{cccccc} 0 & 0 & 0 & 1 & 2 & 3 \end{array} \\ \left[\begin{array}{rrrrrr} 15{,}466.67 & 0 & 0 & -15{,}466.67 & 0 & 0 \\ 0 & 71.6 & 1{,}074.07 & 0 & -71.6 & 1{,}074.07 \\ 0 & 1{,}074.07 & 21{,}481.48 & 0 & -1{,}074.07 & 10{,}740.74 \\ -15{,}466.67 & 0 & 0 & 15{,}466.67 & 0 & 0 \\ 0 & -71.6 & -1{,}074.07 & 0 & 71.6 & -1{,}074.07 \\ 0 & 1{,}074.07 & 10{,}740.74 & 0 & -1{,}074.07 & 21{,}481.48 \end{array}\right] \end{array} \begin{array}{c} \\ 0 \\ 0 \\ 0 \\ 1 \\ 2 \\ 3 \end{array}$$

Element (2):

$$\mathbf{k}^{\prime(2)} = \left[\begin{array}{rrrrrr} 13{,}920 & 0 & 0 & -13{,}920 & 0 & 0 \\ 0 & 61.87 & 773.33 & 0 & -61.87 & 773.33 \\ 0 & 773.33 & 12{,}888.89 & 0 & -773.33 & 6{,}444.44 \\ -13{,}920 & 0 & 0 & 13{,}920 & 0 & 0 \\ 0 & -61.87 & -773.33 & 0 & 61.87 & -773.33 \\ 0 & 773.33 & 6{,}444.44 & 0 & -773.33 & 12{,}888.89 \end{array}\right]$$

$$\mathbf{f}_0^{\prime(2)} = \mathbf{0}$$

By using the global coordinates of the beginning joint 3 and the end joint 2, we determine the direction cosines of member 2 as

$$\cos\theta = \frac{X_2 - X_3}{L} = \frac{30 - 45}{25} = -0.6$$

$$\sin\theta = \frac{Y_2 - Y_3}{L} = \frac{0 - (-20)}{25} = 0.8$$

$$\mathbf{T}^{(2)} = \left[\begin{array}{rrrrrr} -0.6 & 0.8 & 0 & 0 & 0 & 0 \\ -0.8 & -0.6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.6 & 0.8 & 0 \\ 0 & 0 & 0 & -0.8 & -0.6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\mathbf{K} = \mathbf{T}^T\mathbf{k}\mathbf{T}$$

$$\mathbf{K}^{(2)} = \begin{array}{c} \begin{array}{cccccc} 0 & 0 & 0 & 1 & 2 & 3 \end{array} \\ \left[\begin{array}{rrrrrr} 5{,}050.8 & -6{,}651.9 & -618.67 & -5{,}050.8 & 6{,}651.9 & -618.67 \\ -6{,}651.9 & 8{,}931.07 & -464 & 6{,}651.9 & -8{,}931.07 & -464 \\ -618.67 & -464 & 12{,}888.89 & 618.67 & 464 & 6{,}444.44 \\ -5{,}050.8 & 6{,}651.9 & 618.67 & 5{,}050.8 & -6{,}651.9 & 618.67 \\ 6{,}651.9 & -8{,}931.07 & 464 & -6{,}651.9 & 8{,}931.07 & 464 \\ -618.67 & -464 & 6{,}444.44 & 618.67 & 464 & 12{,}888.89 \end{array}\right] \end{array} \begin{array}{c} \\ 0 \\ 0 \\ 0 \\ 1 \\ 2 \\ 3 \end{array}$$

The assembled stiffness matrix is:

$$\mathbf{S} = \begin{array}{c} \begin{array}{ccc} DoF_1 & DoF_2 & DoF_3 \end{array} \\ \left[\begin{array}{ccc} 15466.67 + 5050.8 & -6651.9 & 618.67 \\ -6651.9 & 71.6 + 8931.07 & -1074.07 + 464 \\ 618.67 & -1074.07 + 464 & 21481.48 + 12888.89 \end{array}\right] \end{array} \begin{array}{c} \\ DoF_1 \\ DoF_2 \\ DoF_3 \end{array}$$

$$= \begin{array}{c} \begin{array}{ccc} DoF_1 & DoF_2 & DoF_3 \end{array} \\ \left[\begin{array}{ccc} 20517.47 & -6651.9 & 618.67 \\ -6651.9 & 9002.67 & -610.07 \\ 618.67 & -610.07 & 34370.37 \end{array}\right] \end{array} \begin{array}{c} \\ DoF_1 \\ DoF_2 \\ DoF_3 \end{array} \qquad P_f = \left\{\begin{array}{c} 0 \\ 30 \\ -150 \end{array}\right\} \begin{array}{c} DoF_1 \\ DoF_2 \\ DoF_3 \end{array}$$

Joint Displacements The stiffness relations for the entire frame, $\{\bar{P}\} = [S]\{DoF\} + \{P_f\}$

$$\begin{Bmatrix} 0 \\ 0 \\ 75 \end{Bmatrix} - \begin{Bmatrix} 0 \\ 30 \\ -150 \end{Bmatrix} = \begin{bmatrix} 20517.47 & -6651.9 & 618.67 \\ -6651.9 & 9002.67 & -610.07 \\ 618.67 & -610.07 & 34370.37 \end{bmatrix} \begin{Bmatrix} DoF_1 \\ DoF_2 \\ DoF_3 \end{Bmatrix}$$

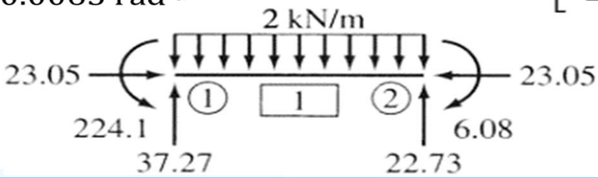
$$\begin{Bmatrix} 0 \\ -30 \\ 225 \end{Bmatrix} = \begin{bmatrix} 20517.47 & -6651.9 & 618.67 \\ -6651.9 & 9002.67 & -610.07 \\ 618.67 & -610.07 & 34370.37 \end{bmatrix} \begin{Bmatrix} DoF_1 \\ DoF_2 \\ DoF_3 \end{Bmatrix} \Longrightarrow \mathbf{DoF} = \begin{Bmatrix} -0.00149 \text{ m} \\ -0.00399 \text{ m} \\ 0.0065 \text{ rad} \end{Bmatrix}$$

Member End Displacements and End Forces

Element (1):

$$\mathbf{d}'^{(1)} = \mathbf{d}^{(1)} = \begin{Bmatrix} u_1 \\ v_1 \\ \emptyset_1 \\ u_1 \\ v_2 \\ \emptyset_2 \end{Bmatrix} \begin{matrix} 0 \\ 0 \\ 0 \\ 1 \\ 2 \\ 3 \end{matrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ DoF_1 \\ DoF_2 \\ DoF_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -0.00149 \text{ m} \\ -0.00399 \text{ m} \\ 0.0065 \text{ rad} \end{Bmatrix}$$

$$\mathbf{f}' = \mathbf{k}'\,\mathbf{d}' + \mathbf{f}'_0 \Longrightarrow \mathbf{F}^{(1)} = \mathbf{f}'^{(1)} = \begin{bmatrix} 23.05 \text{ kN} \\ 37.27 \text{ kN} \\ 224.1 \text{ kN-m} \\ -23.05 \text{ kN} \\ 22.73 \text{ kN} \\ -6.08 \text{ kN-m} \end{bmatrix}$$



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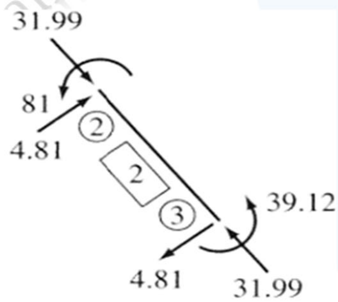
Element (2):

$$\mathbf{d}^{(2)} = \begin{Bmatrix} u_3 \\ v_3 \\ \emptyset_3 \\ u_2 \\ v_2 \\ \emptyset_2 \end{Bmatrix} \begin{matrix} 0 \\ 0 \\ 0 \\ 1 \\ 2 \\ 3 \end{matrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ DoF_1 \\ DoF_2 \\ DoF_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -0.00149 \text{ m} \\ -0.00399 \text{ m} \\ 0.0065 \text{ rad} \end{Bmatrix}$$

$$\mathbf{f}'^{(2)}_0 = \mathbf{0}$$

$$\mathbf{F} = \mathbf{Kd} + \mathbf{F}_0 \Longrightarrow \mathbf{F}^{(2)} = \begin{bmatrix} -23.04 \text{ kN} \\ 22.71 \text{ kN} \\ 39.12 \text{ kN-m} \\ 23.04 \text{ kN} \\ -22.71 \text{ kN} \\ 81 \text{ kN-m} \end{bmatrix} \qquad \mathbf{f}' = \mathbf{T}\,\mathbf{F} \Longrightarrow \mathbf{f}'^{(2)} = \begin{bmatrix} 31.99 \text{ kN} \\ 4.81 \text{ kN} \\ 39.12 \text{ kN-m} \\ -31.99 \text{ kN} \\ -4.81 \text{ kN} \\ 81 \text{ kN-m} \end{bmatrix}$$



Support Reactions

Since support joints 1 & 3 are the beginning joints for elements 1 & 2, respectively, the reaction vectors R_1 & R_3 must be equal to the upper halves of $\mathbf{F}^{(1)}$ & $\mathbf{F}^{(2)}$,respectively.

$$\mathbf{R}_{①} = \begin{bmatrix} 23.05 \text{ kN} \\ 37.27 \text{ kN} \\ 224.1 \text{ kN-m} \end{bmatrix}$$

$$\mathbf{R}_{③} = \begin{bmatrix} -23.04 \text{ kN} \\ 22.71 \text{ kN} \\ 19.12 \text{ kN-m} \end{bmatrix}$$