كلية الصيدلة السنة الأولى

٢٠٢٥-٢٠٢٦

الرياضيات

المحاضرة الثالثة

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بعض التوابع الشهيرة

التّـــابع اللوغاريتمي (Logarithmic Function)

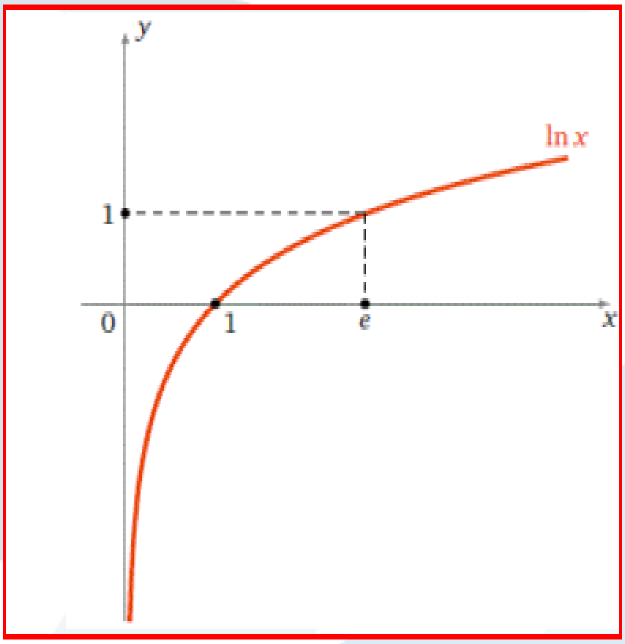
مبرهنة 1 :

يوجد تابع وحيد يرمز له $\ln:]0,+\infty[\to \mathbb{R}$، بحيث :

$$\ln'(x) = \frac{1}{x} \;;\; (\forall x > 0) \quad and \quad \ln(1) = 0$$

كذلك يحقق هذا التابع من أجل كلّ من $a,b > 0$:

1. $\ln(a \times b) = \ln a + \ln b$.
2. $\ln\left(\frac{1}{a}\right) = -\ln(a)$.
3. $\ln(a^n) = n\ln(a) \quad (\forall n \in \mathbb{N})$.
4. $\ln$ تابع مستمر.
5. $\lim_{x\to 0} \frac{\ln(1+x)}{x} = 1$.



ملاحظة :

ln يدعى التّابع اللوغاريتمي الطّبيعيّ أو النيبري، حيث $\ln e = 1$، نعرّف اللوغاريتم بالأساسي a، ورمزه $\log_a$ كالآتي :

$$\log_a(x) = \frac{\ln x}{\ln a}$$

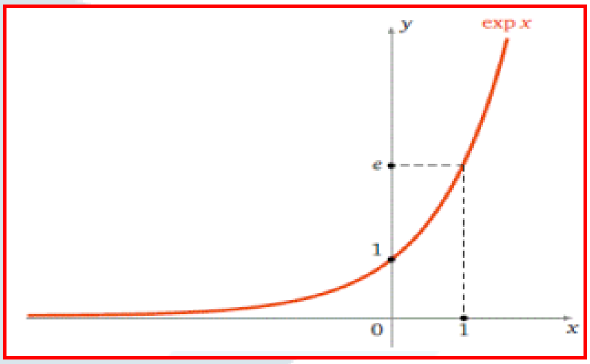
و $\log_a(a) = 1$

وعندما يكون $a = 10$، نحصل على اللوغاريتم العشري $\log_{10}$ الّذي يحقّق $\log_{10}(10) = 1$، ويكون $\log_{10}(10^n) = n$

نستخدم التكافؤ الآتي في التّطبيقات كثيراً :

$$x = 10^y \Leftrightarrow y = \log_{10} x$$

التّـــابع الأسّـــي (Exponential Function)



التّقابل العكسي للتابع $\ln :]0, +\infty[\to \mathbb{R}$، يدعى التّابع الأسيّ، ونرمز له :

$$\exp: \mathbb{R} \to]0, +\infty[$$

من أجل كلّ $x \in \mathbb{R}$، نرمز كذلك للتابع $\exp x$ بالرمز e^x.

من الجدير بالذّكر أنّ التابع الأسيّ، يأخذ قيماً موجبة تماما دوما.

مبرهنة 3 :

يحقّق التّابع الأسيّ العلاقات الآتية :

- $\exp(\ln x) = x \;;\; \forall x > 0$ و $\ln(\exp x) = x \;;\; \forall x \in \mathbb{R}$
- $(\forall a, b \in \mathbb{R})\ \exp(a + b) = \exp a \times \exp b$.
- $\exp(nx) = (\exp x)^n$.
- $\exp: \mathbb{R} \to]0, +\infty[$ تابع مستمرّ، ومتزايد تماماً، ويحقّق : $\lim_{x\to+\infty} \exp\ x = +\infty$ و $\lim_{x\to-\infty} \exp\ x = 0$.
- التّابع الأسيّ قابل للاشتقاق، و $\exp'\ x = \exp\ x \;;\; \forall x \in \mathbb{R}$

وكذلك $\exp\ x \geq 1 + x$.

ملاحظة : التّابع الأسيّ هو التّابع الوحيد الّذي يحقّق أن $\forall x \in \mathbb{R} \quad \exp'\ x = \exp\ x$، ويحقّق $\exp(1) = e$، حيث $e \approx 2.718$ ويحقّق $\ln e = 1$.

تعريف :

بالتّعريف، من أجل $a > 0$ و $b \in \mathbb{R}$ يكون :

$$a^b = \exp(b \ln a)$$

ملاحظة :

- $\sqrt{a} = a^{\frac{1}{2}} = \exp\left(\frac{1}{2}\ln a\right)$
- $\sqrt[n]{a} = a^{\frac{1}{n}} = \exp\left(\frac{1}{n}\ln a\right)$
- التّابع $x \longmapsto a^x$، يمكن كتابته بالشّكل $a^x = \exp(x \ln a)$

مبرهنة 4 :

من أجل أيّ عددين $a, b \in \mathbb{R}$ و $x, y > 0$، يكون :

- $x^{a+b} = x^a . x^b$
- $x^{-a} = \frac{1}{x^a}$
- $(x.y)^a = x^a . y^a$
- $(x^a)^b = x^{ab}$
- $\ln(x^a) = a . \ln x$

EXAMPLE 8 | Solve the equation $e^{5-3x} = 10$.

SOLUTION We take natural logarithms of both sides of the equation

$$\ln(e^{5-3x}) = \ln 10$$
$$5 - 3x = \ln 10$$
$$3x = 5 - \ln 10$$
$$x = \tfrac{1}{3}(5 - \ln 10)$$

What are the values of $e^{\ln 300}$ and $\ln(e^{300})$?

$e^{\ln 300} = 300$ and $\ln(e^{300}) = 300$.

Find the domain of $f(x) = \ln(e^x - 3)$.

(a) We must have $e^x - 3 > 0 \Rightarrow e^x > 3 \Rightarrow x > \ln 3$. Thus, the domain of $f(x) = \ln(e^x - 3)$ is $(\ln 3, \infty)$

Find the exact value of each expression.

35. (a) $\log_5 125$ (b) $\log_3\left(\frac{1}{27}\right)$

36. (a) $\ln(1/e)$ (b) $\log_{10}\sqrt{10}$

38. (a) $e^{-2\ln 5}$ (b) $\ln\left(\ln e^{e^{10}}\right)$

39–41 Express the given quantity as a single logarithm.

39. $\ln 5 + 5 \ln 3$

40. $\ln(a + b) + \ln(a - b) - 2 \ln c$

41. $\frac{1}{3}\ln(x + 2)^3 + \frac{1}{2}[\ln x - \ln(x^2 + 3x + 2)^2]$

47–50 Solve each equation for x.

47. (a) $e^{7-4x} = 6$ (b) $\ln(3x - 10) = 2$

48. (a) $\ln(x^2 - 1) = 3$ (b) $e^{2x} - 3e^x + 2 = 0$

49. (a) $2^{x-5} = 3$ (b) $\ln x + \ln(x - 1) = 1$

50. (a) $\ln(\ln x) = 1$

51–52 Solve each inequality for x.

51. (a) $\ln x < 0$ (b) $e^x > 5$

52. (a) $1 < e^{3x-1} < 2$ (b) $1 - 2 \ln x < 3$

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الحل

35. (a) $\log_5 125 = 3$ since $5^3 = 125$. (b) $\log_3 \frac{1}{27} = -3$ since $3^{-3} = \frac{1}{3^3} = \frac{1}{27}$.

36. (a) $\ln(1/e) = \ln 1 - \ln e = 0 - 1 = -1$ (b) $\log_{10} \sqrt{10} = \log_{10} 10^{1/2} = \frac{1}{2}$ by (2).

38. (a) $e^{-2\ln 5} = \left(e^{\ln 5}\right)^{-2} \overset{(6)}{=} 5^{-2} = \frac{1}{5^2} = \frac{1}{25}$ (b) $\ln\left(\ln e^{e^{10}}\right) \overset{(6)}{=} \ln(e^{10}) \overset{(6)}{=} 10$

39. $$\begin{aligned}\ln 5 + 5\ln 3 &= \ln 5 + \ln 3^5 \\ &= \ln(5 \cdot 3^5) \\ &= \ln 1215\end{aligned}$$

40. $$\begin{aligned}\ln(a+b) + \ln(a-b) - 2\ln c &= \ln[(a+b)(a-b)] - \ln c^2 \\ &= \ln \frac{(a+b)(a-b)}{c^2} \\ &\text{or } \ln \frac{a^2-b^2}{c^2}\end{aligned}$$

41. $$\begin{aligned}\tfrac{1}{3}\ln(x+2)^3 + \tfrac{1}{2}\left[\ln x - \ln(x^2+3x+2)^2\right] &= \ln[(x+2)^3]^{1/3} + \tfrac{1}{2}\ln \frac{x}{(x^2+3x+2)^2} \\ &= \ln(x+2) + \ln \frac{\sqrt{x}}{x^2+3x+2} \\ &= \ln \frac{(x+2)\sqrt{x}}{(x+1)(x+2)} \\ &= \ln \frac{\sqrt{x}}{x+1}\end{aligned}$$

Note that since $\ln x$ is defined for $x > 0$, we have $x+1$, $x+2$, and x^2+3x+2 all positive, and hence their logarithms are defined.

47. (a) $e^{7-4x} = 6 \quad \Leftrightarrow \quad 7 - 4x = \ln 6 \quad \Leftrightarrow \quad 7 - \ln 6 = 4x \quad \Leftrightarrow \quad x = \frac{1}{4}(7 - \ln 6)$

(b) $\ln(3x - 10) = 2 \quad \Leftrightarrow \quad 3x - 10 = e^2 \quad \Leftrightarrow \quad 3x = e^2 + 10 \quad \Leftrightarrow \quad x = \frac{1}{3}(e^2 + 10)$

48. (a) $\ln(x^2 - 1) = 3 \quad \Leftrightarrow \quad x^2 - 1 = e^3 \quad \Leftrightarrow \quad x^2 = 1 + e^3 \quad \Leftrightarrow \quad x = \pm\sqrt{1 + e^3}$.

(b) $e^{2x} - 3e^x + 2 = 0 \Leftrightarrow (e^x - 1)(e^x - 2) = 0 \Leftrightarrow e^x = 1$ or $e^x = 2 \Leftrightarrow x = \ln 1$ or $x = \ln 2$, so $x = 0$ or $\ln 2$.

49. (a) $2^{x-5} = 3 \quad \Leftrightarrow \quad \log_2 3 = x - 5 \quad \Leftrightarrow \quad x = 5 + \log_2 3$.

Or: $2^{x-5} = 3 \quad \Leftrightarrow \quad \ln\left(2^{x-5}\right) = \ln 3 \quad \Leftrightarrow \quad (x - 5)\ln 2 = \ln 3 \quad \Leftrightarrow \quad x - 5 = \dfrac{\ln 3}{\ln 2} \quad \Leftrightarrow \quad x = 5 + \dfrac{\ln 3}{\ln 2}$

(b) $\ln x + \ln(x - 1) = \ln(x(x - 1)) = 1 \quad \Leftrightarrow \quad x(x - 1) = e^1 \quad \Leftrightarrow \quad x^2 - x - e = 0$. The quadratic formula (with $a = 1$, $b = -1$, and $c = -e$) gives $x = \frac{1}{2}\left(1 \pm \sqrt{1 + 4e}\right)$, but we reject the negative root since the natural logarithm is not defined for $x < 0$. So $x = \frac{1}{2}\left(1 + \sqrt{1 + 4e}\right)$.

50. (a) $\ln(\ln x) = 1 \quad \Leftrightarrow \quad e^{\ln(\ln x)} = e^1 \quad \Leftrightarrow \quad \ln x = e^1 = e \quad \Leftrightarrow \quad e^{\ln x} = e^e \quad \Leftrightarrow \quad x = e^e$

51. (a) $\ln x < 0 \quad \Rightarrow \quad x < e^0 \quad \Rightarrow \quad x < 1$. Since the domain of $f(x) = \ln x$ is $x > 0$, the solution of the original inequality is $0 < x < 1$.

(b) $e^x > 5 \quad \Rightarrow \quad \ln e^x > \ln 5 \quad \Rightarrow \quad x > \ln 5$

52. (a) $1 < e^{3x-1} < 2 \quad \Rightarrow \quad \ln 1 < 3x - 1 < \ln 2 \quad \Rightarrow \quad 0 < 3x - 1 < \ln 2 \quad \Rightarrow \quad 1 < 3x < 1 + \ln 2 \quad \Rightarrow$
$\frac{1}{3} < x < \frac{1}{3}(1 + \ln 2)$

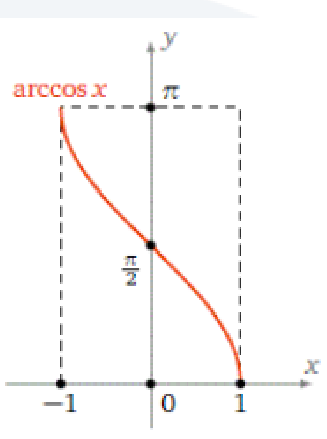
(b) $1 - 2\ln x < 3 \quad \Rightarrow \quad -2\ln x < 2 \quad \Rightarrow \quad \ln x > -1 \quad \Rightarrow \quad x > e^{-1}$

التوابع المثلثية العكسية (Inverse Trigonometric Functions)

التابع arc cos

ليكن لدينا التابع $\cos: \mathbb{R} \to [-1,+1]$: $x \mapsto \cos x$. حتّى يكون هذا التّابع تقابلاً, ينبغي أن نقصر التّابع على المجال $[0,\pi]$, حيث يكون التّابع مستمراً ومتزايداً تماماً، وبهذا التّابع : $\cos : [0,\pi] \to [-1,+1]$ تقابل، ويملك تقابلاً عكسياً arccos المعرّف بالشّكل :

$$\arccos x \ : [-1,+1] \to [0,\pi]$$



وبحسب تعريف التّقابل العكسيّ :

$$\cos(\arccos x) = x \quad , \ \forall x \in [-1,+1]$$

$$\arccos(\cos x) = x \quad , \ \forall x \in [0,\pi]$$

بكلام آخر :

$$\arccos(y) = x \iff \cos x = y \quad , \ \forall x \in [0,\pi]$$

وأخيراً : لنرى مشتقّ arccos

$$\arccos' x = \frac{-1}{\sqrt{1-x^2}} \quad ; \quad \forall x \in \left]-1,+1\right[$$

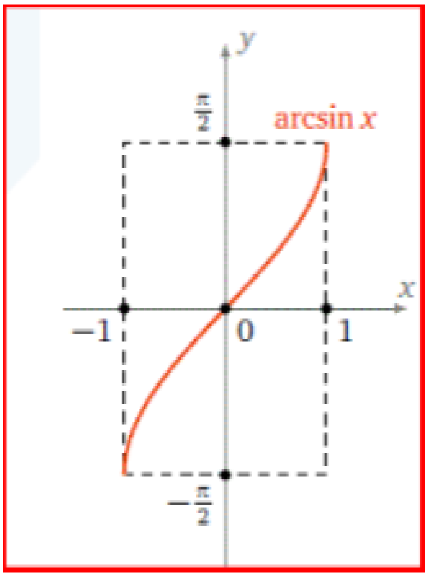
التابع arc sin

مقصور التّابع sin على المجال $\left[-\frac{\pi}{2},+\frac{\pi}{2}\right]$ المعرّف بالشّكل :

$$\sin: \left[-\frac{\pi}{2},+\frac{\pi}{2}\right] \to [-1,+1]$$

يمثّل تقابلاً، وتقابله العكسيّ هو التّابع arcsin

$$\arcsin \ : [-1,+1] \to \left[-\frac{\pi}{2},+\frac{\pi}{2}\right]$$



$$\sin(\arcsin x) = x \quad , \quad \forall x \in [-1,+1]$$

$$\arcsin(\sin x) = x \quad , \quad \forall x \in \left[-\frac{\pi}{2},+\frac{\pi}{2}\right]$$

وبكلام آخر :

$$\arcsin(y) = x \iff \sin x = y \quad , \quad \forall x \in \left[-\frac{\pi}{2},+\frac{\pi}{2}\right]$$

وأخيراً : لنرى مشتق arccos

$$\arcsin' x = \frac{+1}{\sqrt{1-x^2}} \quad ; \quad \forall x \in]-1,+1[$$

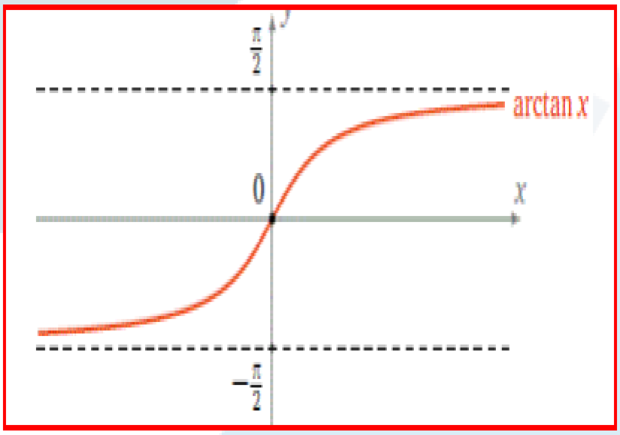
التابع arc tan

مقصور التابع tan على المجال $\left]-\frac{\pi}{2},+\frac{\pi}{2}\right[$ المعرَف بالشَكل :

$$\tan : \left]-\frac{\pi}{2},+\frac{\pi}{2}\right[\to \mathbb{R}$$

يمثّل تقابلاً، وتقابله العكسيّ هو التّابع arctan

$$\arctan : \mathbb{R} \to \left]-\frac{\pi}{2},+\frac{\pi}{2}\right[$$



لدينا حسب تعريف التقابل العكسيّ :

$$\tan(\arctan x) = x \quad , \quad \forall x \in \mathbb{R}$$

$$\arctan(\tan x) = x \quad , \quad \forall x \in \left]-\frac{\pi}{2},+\frac{\pi}{2}\right[$$

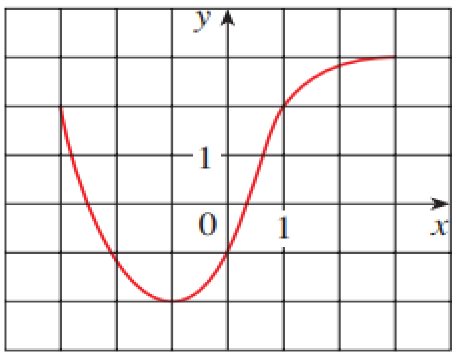
وبكلام آخر :

$$\arctan(y) = x \iff \tan x = y \quad , \quad \forall x \in \left]-\frac{\pi}{2},+\frac{\pi}{2}\right[$$

وأخيراً : لنرى مشتق arccos

$$\arctan' x = \frac{1}{1+x^2} \qquad \forall x \in \mathbb{R}$$

تمرين ١



1. The graph of a function f is given at the left.
   (a) State the value of $f(-1)$.
   (b) Estimate the value of $f(2)$.
   (c) For what values of x is $f(x) = 2$?
   (d) Estimate the values of x such that $f(x) = 0$.
   (e) State the domain and range of f.

الحل

1. (a) Locate -1 on the x-axis and then go down to the point on the graph with an x-coordinate of -1. The corresponding y-coordinate is the value of the function at $x = -1$, which is -2. So, $f(-1) = -2$.

   (b) Using the same technique as in part (a), we get $f(2) \approx 2.8$.

   (c) Locate 2 on the y-axis and then go left and right to find all points on the graph with a y-coordinate of 2. The corresponding x-coordinates are the x-values we are searching for. So $x = -3$ and $x = 1$.

   (d) Using the same technique as in part (c), we get $x \approx -2.5$ and $x \approx 0.3$.

   (e) The domain is all the x-values for which the graph exists, and the range is all the y-values for which the graph exists. Thus, the domain is $[-3, 3]$, and the range is $[-2, 3]$.

تمرين ٢

3. Find the domain of the function.

(a) $f(x) = \dfrac{2x+1}{x^2+x-2}$ (b) $g(x) = \dfrac{\sqrt[3]{x}}{x^2+1}$ (c) $h(x) = \sqrt{4-x} + \sqrt{x^2-1}$

الحل

3. (a) Set the denominator equal to 0 and solve to find restrictions on the domain: $x^2 + x - 2 = 0 \;\Rightarrow$ $(x-1)(x+2) = 0 \;\Rightarrow\; x = 1$ or $x = -2$. Thus, the domain is all real numbers except 1 or -2 or, in interval notation, $(-\infty, -2) \cup (-2, 1) \cup (1, \infty)$.

(b) Note that the denominator is always greater than or equal to 1, and the numerator is defined for all real numbers. Thus, the domain is $(-\infty, \infty)$.

(c) Note that the function h is the sum of two root functions. So h is defined on the intersection of the domains of these two root functions. The domain of a square root function is found by setting its radicand greater than or equal to 0. Now,

$4 - x \geq 0 \;\Rightarrow\; x \leq 4$ and $x^2 - 1 \geq 0 \;\Rightarrow\; (x-1)(x+1) \geq 0 \;\Rightarrow\; x \leq -1$ or $x \geq 1$. Thus, the domain of h is $(-\infty, -1] \cup [1, 4]$.

تمرين ٣

6. Let $f(x) = \begin{cases} 1 - x^2 & \text{if } x \leqslant 0 \\ 2x + 1 & \text{if } x > 0 \end{cases}$

(a) Evaluate $f(-2)$ and $f(1)$. (b) Sketch the graph of f.

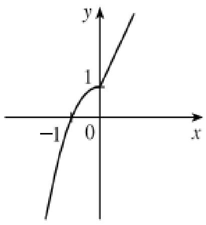
7. If $f(x) = x^2 + 2x - 1$ and $g(x) = 2x - 3$, find each of the following functions.

(a) $f \circ g$ (b) $g \circ f$ (c) $g \circ g \circ g$

الحل

6. (a) $f(-2) = 1 - (-2)^2 = -3$ and $f(1) = 2(1) + 1 = 3$

(b) For $x \leq 0$ plot $f(x) = 1 - x^2$ and, on the same plane, for $x > 0$ plot the graph of $f(x) = 2x + 1$.



7. (a) $(f \circ g)(x) = f(g(x)) = f(2x - 3) = (2x - 3)^2 + 2(2x - 3) - 1 = 4x^2 - 12x + 9 + 4x - 6 - 1 = 4x^2 - 8x + 2$

(b) $(g \circ f)(x) = g(f(x)) = g(x^2 + 2x - 1) = 2(x^2 + 2x - 1) - 3 = 2x^2 + 4x - 2 - 3 = 2x^2 + 4x - 5$

(c) $(g \circ g \circ g)(x) = g(g(g(x))) = g(g(2x - 3)) = g(2(2x - 3) - 3) = g(4x - 9) = 2(4x - 9) - 3$
$= 8x - 18 - 3 = 8x - 21$

تمرين ٤

Determine whether each of the following functions is even, odd, or neither even nor odd.

(a) $f(x) = x^5 + x$ (b) $g(x) = 1 - x^4$ (c) $h(x) = 2x - x^2$

الحل

(a)
$$f(-x) = (-x)^5 + (-x) = (-1)^5x^5 + (-x) = -x^5 - x = -(x^5 + x) = -f(x)$$

Therefore f is an odd function.

(b)
$$g(-x) = 1 - (-x)^4 = 1 - x^4 = g(x)$$

So g is even.

(c)
$$h(-x) = 2(-x) - (-x)^2 = -2x - x^2$$

Since $h(-x) \neq h(x)$ and $h(-x) \neq -h(x)$, we conclude that h is neither even nor odd. ■

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تمارين ٥

19–20 Find the domain of each function.

19. (a) $f(x) = \dfrac{1 - e^{x^2}}{1 - e^{1-x^2}}$ (b) $f(x) = \dfrac{1 + x}{e^{\cos x}}$

20. (a) $g(t) = \sin(e^{-t})$ (b) $g(t) = \sqrt{1 - 2^t}$

23. If $f(x) = 5^x$, show that

$$\frac{f(x+h) - f(x)}{h} = 5^x\left(\frac{5^h - 1}{h}\right)$$

37. If you graph the function

$$f(x) = \frac{1 - e^{1/x}}{1 + e^{1/x}}$$

you'll see that f appears to be an odd function. Prove it.

Find the domain and sketch the graph of the function.

47. $G(x) = \dfrac{3x + |x|}{x}$ **48.** $g(x) = |x| - x$

الحل

19. (a) The denominator is zero when $1 - e^{1-x^2} = 0 \quad \Leftrightarrow \quad e^{1-x^2} = 1 \quad \Leftrightarrow \quad 1 - x^2 = 0 \quad \Leftrightarrow \quad x = \pm 1$. Thus, the function $f(x) = \dfrac{1 - e^{x^2}}{1 - e^{1-x^2}}$ has domain $\{x \mid x \neq \pm 1\} = (-\infty, -1) \cup (-1, 1) \cup (1, \infty)$.

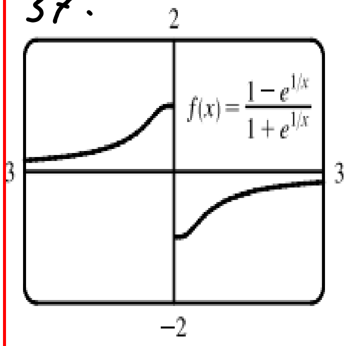
(b) The denominator is never equal to zero, so the function $f(x) = \dfrac{1+x}{e^{\cos x}}$ has domain $\mathbb{R}$, or $(-\infty, \infty)$.

20. (a) The sine and exponential functions have domain $\mathbb{R}$, so $g(t) = \sin(e^{-t})$ also has domain $\mathbb{R}$.

(b) The function $g(t) = \sqrt{1 - 2^t}$ has domain $\{t \mid 1 - 2^t \geq 0\} = \{t \mid 2^t \leq 1\} = \{t \mid t \leq 0\} = (-\infty, 0]$.

23. If $f(x) = 5^x$, then $\dfrac{f(x+h) - f(x)}{h} = \dfrac{5^{x+h} - 5^x}{h} = \dfrac{5^x 5^h - 5^x}{h} = \dfrac{5^x\left(5^h - 1\right)}{h} = 5^x\left(\dfrac{5^h - 1}{h}\right)$.

37:



From the graph, it appears that f is an odd function (f is undefined for $x = 0$).

To prove this, we must show that $f(-x) = -f(x)$.

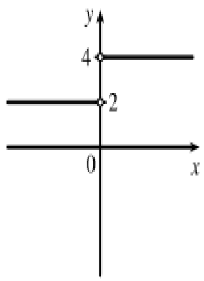
$$f(-x) = \frac{1-e^{1/(-x)}}{1+e^{1/(-x)}} = \frac{1-e^{(-1/x)}}{1+e^{(-1/x)}} = \frac{1-\dfrac{1}{e^{1/x}}}{1+\dfrac{1}{e^{1/x}}} \cdot \frac{e^{1/x}}{e^{1/x}} = \frac{e^{1/x}-1}{e^{1/x}+1}$$

$$= -\frac{1-e^{1/x}}{1+e^{1/x}} = -f(x)$$

so f is an odd function.

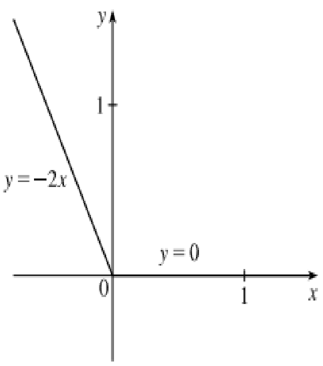
47. $G(x) = \dfrac{3x+|x|}{x}$. Since $|x| = \begin{cases} x & \text{if } x \ge 0 \\ -x & \text{if } x < 0 \end{cases}$, we have

$$G(x) = \begin{cases} \dfrac{3x+x}{x} & \text{if } x > 0 \\ \dfrac{3x-x}{x} & \text{if } x < 0 \end{cases} = \begin{cases} \dfrac{4x}{x} & \text{if } x > 0 \\ \dfrac{2x}{x} & \text{if } x < 0 \end{cases} = \begin{cases} 4 & \text{if } x > 0 \\ 2 & \text{if } x < 0 \end{cases}$$



48. $g(x) = |x| - x = \begin{cases} x - x & \text{if } x \ge 0 \\ -x - x & \text{if } x < 0 \end{cases} = \begin{cases} 0 & \text{if } x \ge 0 \\ -2x & \text{if } x < 0 \end{cases}$.

The domain is $\mathbb{R}$, or $(-\infty, \infty)$.



في 21/12/2025