Polynomial Interpolation
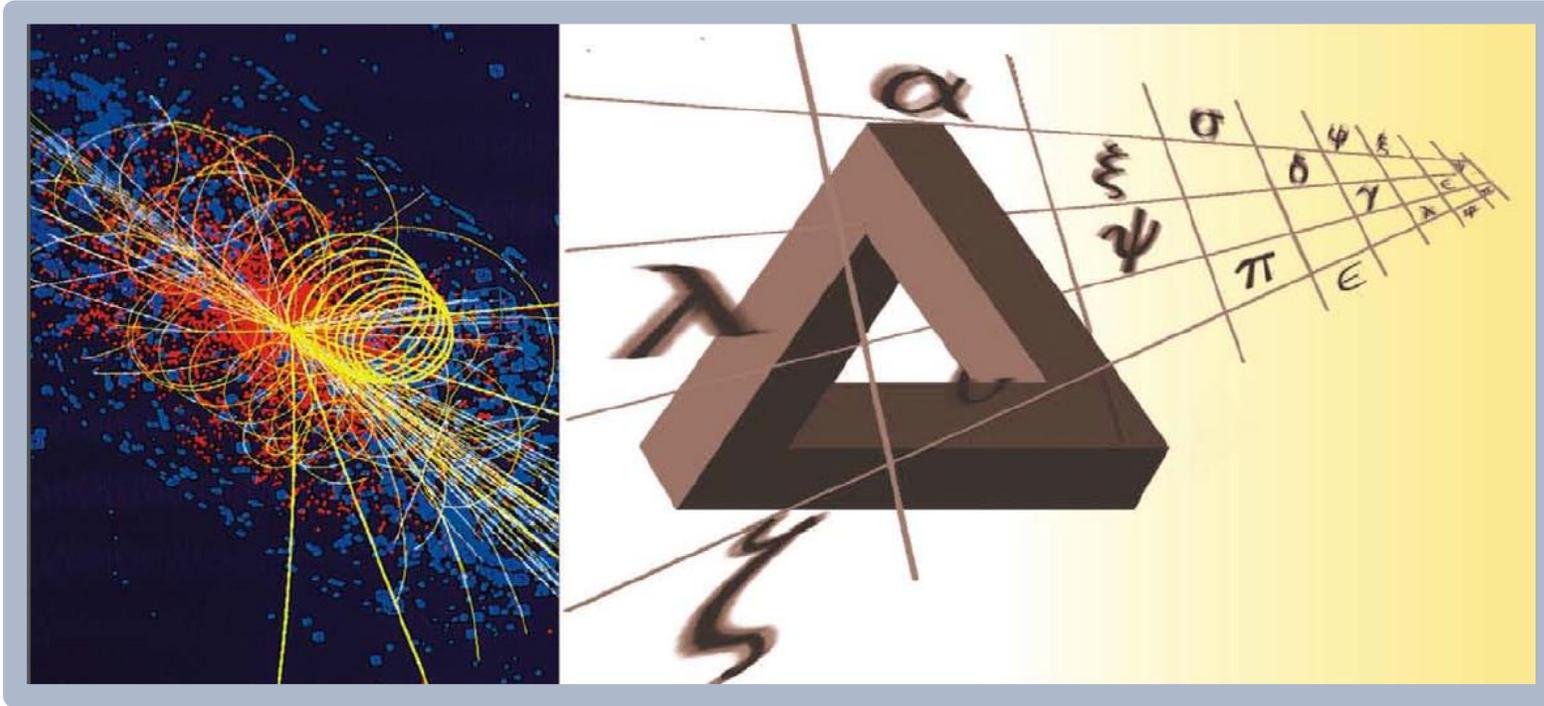
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Contents

Introduction

Lagrange Interpolation Polynomial

Introduction

There are two topics to be dealt with in this chapter, namely, interpolation and curve fitting. Interpolation is to connect discrete data points in a plausible way so that one can get reasonable estimates of data points between the given points. The interpolation curve goes through all data points. Curve fitting, on the other hand, is to find a curve that could best indicate the trend of a given set of data. The curve does not have to go through the data points. In some cases, the data may have diffrent accuracy/reliability/uncertainty and we need the weighted least-squares curve fitting to process such data.

For a given set of $N + 1$ data points $\{(x_0, y_0), (x_1, y_1), \ldots, (x_N, y_N)\}$, we want to find the coefficients of an Nth-degree polynomial function to match them:

$$p_N(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_N x^N$$

The coefficients can be obtained by solving the following system of equations

$$a_0 + x_0 a_1 + x_0^2 a_2 + \cdots + x_0^N a_N = y_0$$

$$a_0 + x_1 a_1 + x_1^2 a_2 + \cdots + x_1^N a_N = y_1$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad .$$

$$a_0 + x_N a_1 + x_N^2 a_2 + \cdots + x_N^N a_N = y_N$$

Lagrange Inetrpolation Polynomial

Suppose we formulate a linear interpolating polynomial as the weighted average of the two values that we are connecting by a straight line:

$$f(x) = L_1 f(x_1) + L_2 f(x_2)$$

where the L's are the weighting coefficients. It is logical that the first weighting coefficient is the straight line that is equal to 1 at x_1 and 0 at x_2:

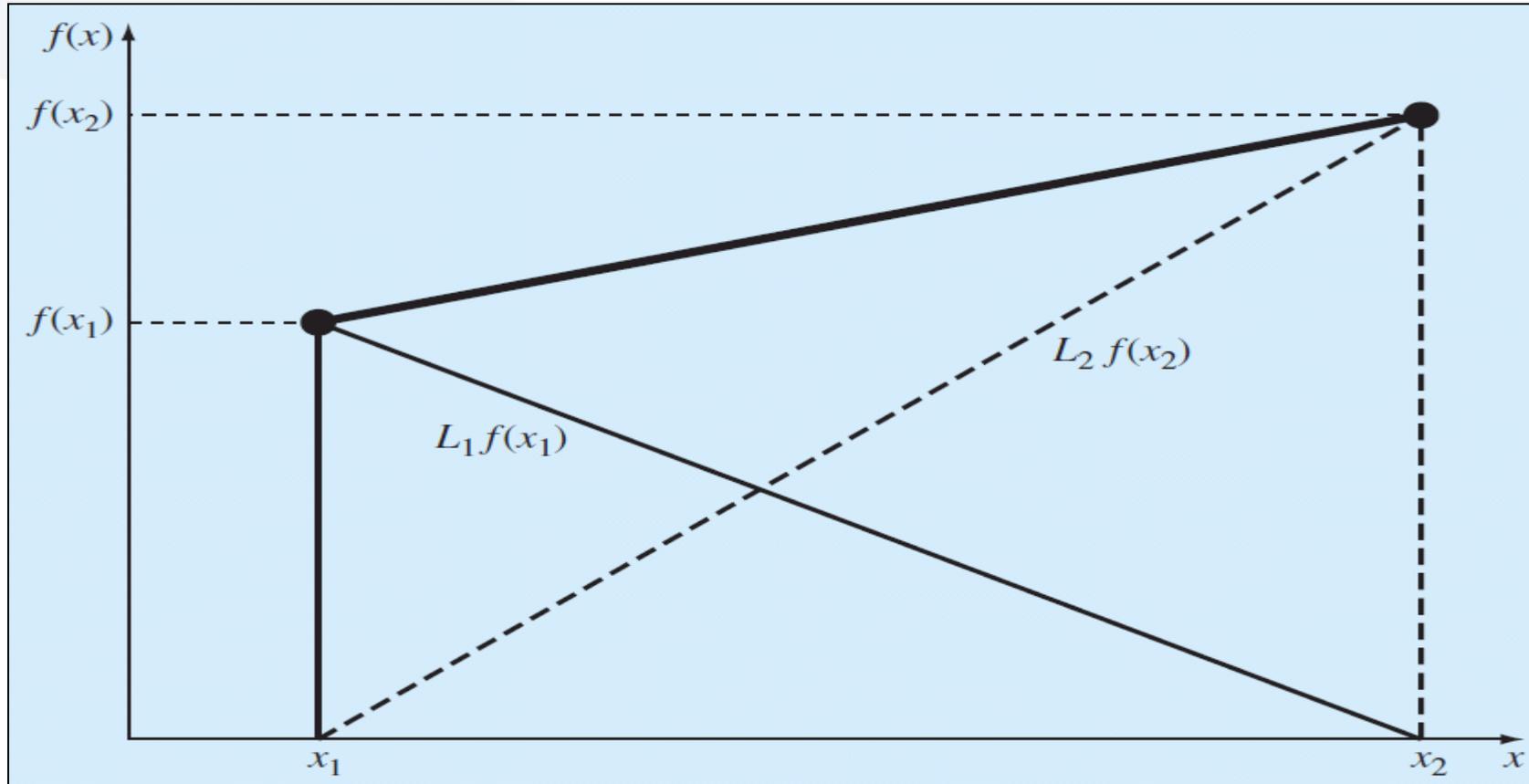$$L_1 = \frac{x - x_2}{x_1 - x_2}$$

Similarly, the second coefficient is the straight line that is equal to 1 at x_2 and 0 at x_1:

$$L_2 = \frac{x - x_1}{x_2 - x_1}$$

Substituting these coefficients yields the straight line that connects the points:

$$f_1(x) = \frac{x - x_2}{x_1 - x_2} f(x_1) + \frac{x - x_1}{x_2 - x_1} f(x_2)$$

where the nomenclature $f_1(x)$ designates that this is a first-order polynomial.

The same strategy can be employed to fit a parabola through three points. For this case three parabolas would be used with each one passing through one of the points and equaling zero at the other two. Their sum would then represent the unique parabola that connects the three points. Such a second-order Lagrange interpolating polynomial can be written as

$$f_2(x) = \frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)}f(x_1) + \frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)}f(x_2) + \frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)}f(x_3)$$

Notice how the first term is equal to $f(x_1)$ at x_1 and is equal to zero at x_2 and x_3. The other terms work in a similar fashion.

Both the first- and second-order versions as well as higher-order Lagrange polynomials can be represented concisely as

$$l_N(x) = y_0 \frac{(x-x_1)(x-x_2)\cdots(x-x_N)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_N)} + y_1 \frac{(x-x_0)(x-x_2)\cdots(x-x_N)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_N)}$$

$$+ \cdots + y_N \frac{(x-x_0)(x-x_1)\cdots(x-x_{N-1})}{(x_N-x_0)(x_N-x_1)\cdots(x_N-x_{N-1})}$$

$$l_N(x) = \sum_{m=0}^{N} y_m L_{N,m}(x) \text{ with } L_{N,m}(x) = \frac{\prod_{k\neq m}^{N}(x-x_k)}{\prod_{k\neq m}^{N}(x_m-x_k)} = \prod_{k\neq m}^{N} \frac{x-x_k}{x_m-x_k}$$

It can easily be shown that the graph of this function matches every data point

$$l_N(x_m) = y_m \qquad \forall \, m = 0, 1, \ldots, N$$

since the Lagrange coefficient polynomial $L_{N,m}(x)$ is 1 only for $x = x_m$ and zero for all other data points $x = x_k$ $(k \neq m)$.
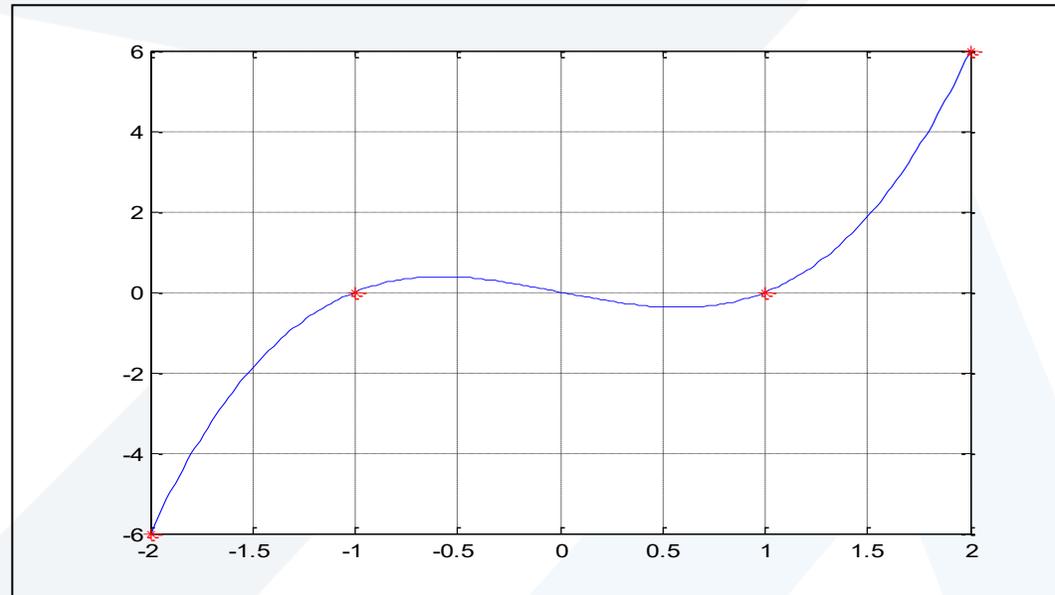
Example

We make the MATLAB program to find the third-degree polynomial $l_3(x)$ which matches the $\{(-2, -6), (-1, 0), (1, 0), (2, 6)\}$

```
clc
clear
x = [-2 -1 1 2];
y = [-6 0 0 6];
N = length(x)-1;
l = 0;
for m = 1:N + 1
    P = 1;
    for k = 1:N + 1
        if k ~= m
            P = conv(P,[1 -x(k)])/(x(m)-x(k));
        end
    end
    l = l + y(m)*P;
end
l
xx = [-2: 0.02 : 2];
yy = polyval(l,xx);
plot(xx,yy,'b', x,y,'r*')
grid
```


l =

    1   0   -1   0

$$l_3(x) = 1 \cdot x^3 + 0 \cdot x^2 - 1 \cdot x + 0$$

Use a Lagrange interpolating polynomial of the first and second order to evaluate the density of unused motor oil at $T = 15\,°C$ based on the following data:

$$x_1 = 0 \qquad f(x_1) = 3.85$$

$$x_2 = 20 \qquad f(x_2) = 0.800$$

$$x_3 = 40 \qquad f(x_3) = 0.212$$

$$f_1(x) = \frac{15 - 20}{0 - 20}3.85 + \frac{15 - 0}{20 - 0}0.800 = 1.5625$$

$$f_2(x) = \frac{(15 - 20)(15 - 40)}{(0 - 20)(0 - 40)}3.85 + \frac{(15 - 0)(15 - 40)}{(20 - 0)(20 - 40)}0.800 + \frac{(15 - 0)(15 - 20)}{(40 - 0)(40 - 20)}0.212$$
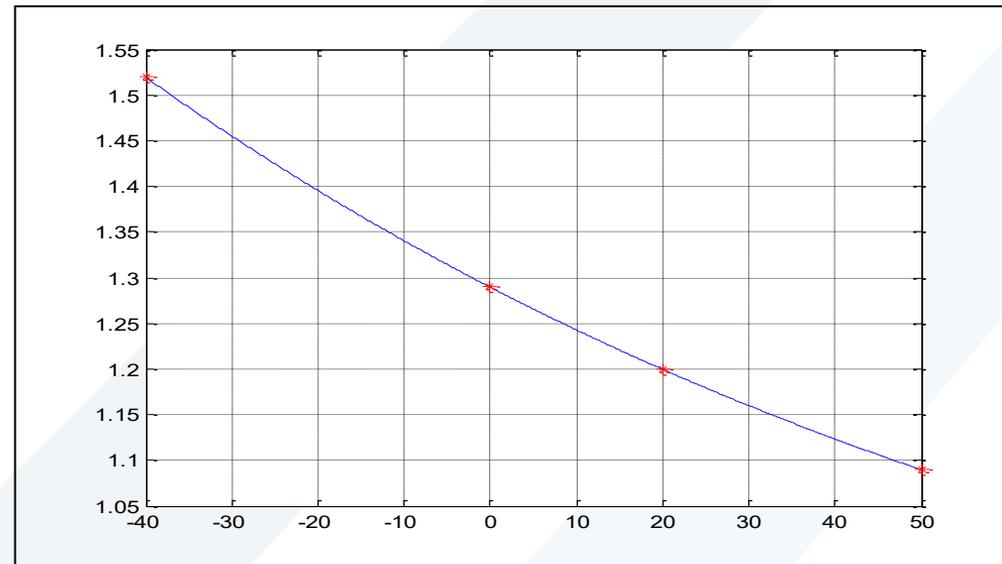
$$= 1.3316875$$

Example

predict the density of air at 1 atm pressure at a temperature of 15 °C based on the four values from Table

T, °C	ρ, kg/m³
−40	1.52
0	1.29
20	1.20
50	1.09

```
clc
clear
x = [-40 0 20 50];
y = [1.52 1.29 1.20 1.09];
N = length(x)-1;
I = 0;
for m = 1:N + 1
    P = 1;
    for k = 1:N + 1
        if k ~= m
            P = conv(P,[1 -x(k)])/(x(m)-x(k));
        end
    end
    I = I + y(m)*P;
end
xx = [-40: 0.02 : 50];
yy = polyval(I,xx);
plot(xx,yy,'b', x,y,'r*')
grid
Density_15=polyval(I,15)
```


Density_15 = 1.2211

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