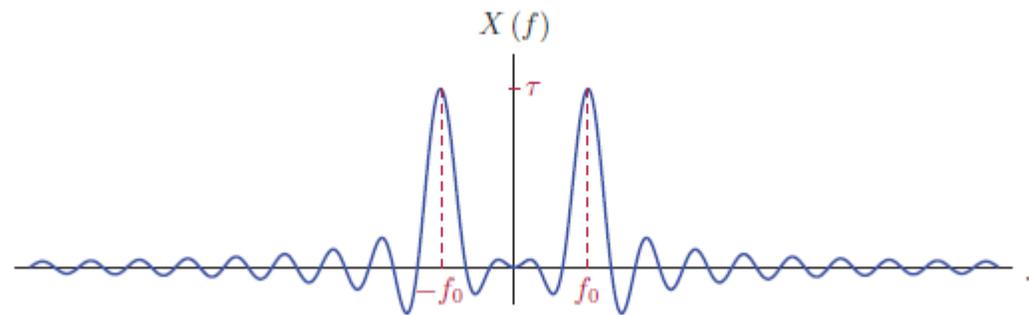




CECC507: Signals and Systems

Lecture Notes 10: Z-Transform for Discrete-Time Signals and Systems



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Chapter 8

Z-Transform for Discrete-Time Signals and Systems

1. Z-Transform
2. Using the Z-Transform with DTLTI Systems
3. Unilateral Z-Transform



Introduction

- The z -transform (ZT) can be viewed as a **generalization of the discrete time Fourier transform**.
- The ZT representation **exists for some sequences that do not have a discrete Fourier transform representation**. So, we can handle some sequences with the ZT that cannot be handled with the DTFT ($x[n] = nu[n]$).

1. Z-Transform

- The z -transform of a discrete-time signal $x[n]$ is defined as:

$$Z\{x[n]\} = X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

where z , the independent variable of the transform is a complex number.



- The z -transform defined is sometimes referred to as the **bilateral** (two sided) z -transform. A simplified variant of the transform termed the **unilateral** (one-sided) z -transform is introduced as an alternative analysis tool.

Relationship Between ZT and Discrete-Time FT

$$X(r, \Omega) = X(z) \Big|_{z=re^{j\Omega}} = \sum_{n=-\infty}^{\infty} x[n](re^{j\Omega})^{-n} = \sum_{n=-\infty}^{\infty} x[n]r^{-n}e^{-j\Omega n} = \mathcal{F}\{r^{-n}x[n]\}$$

$$X(z) \Big|_{z=e^{j\Omega}} = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \mathcal{F}\{x[n]\}$$

- **Example 1:** A simple z -transform example

$$x[n] = \{3.7, 1.3, -1.5, 3.4, 5.2\}$$

\uparrow
 $n=0$



$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = 3.7 + 1.3z^{-1} - 1.5z^{-2} + 3.4z^{-3} + 5.2z^{-4}$$

The transform converges at all points in the complex z -plane except of $z = 0$.

- **Example 2:** z -transform of a non-causal signal

$$x[n] = \{3.7, 1.3, -1.5, 3.4, 5.2\}$$

↑
 $n=0$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = 3.7z^2 + 1.3z^1 - 1.5 + 3.4z^{-1} + 5.2z^{-2}$$

It converges at every point in the z -plane except, the origin and infinity.

- **Example 3:** z -Transform of the unit-impulse

$$X(z) = \mathcal{Z}\{\delta[n]\} = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = x[0]z^0 = 1$$

It converges at every point in the z -plane



- **Example 4:** z -Transform of a time shifted the unit-impulse

$$X(z) = Z\{\delta[n - k]\} = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = z^{-k}$$

1. If $k > 0$ then the transform does not converge at the origin $z = 0$.
2. If $k < 0$ then the transform does not converge at infinity.

Regions of Convergence

- For the z -transform $X(z)$ of $x[n]$ to exist we need that:

$$|X(z)| = \left| \sum_{n=-\infty}^{\infty} x[n]z^{-n} \right| \leq \sum_{n=-\infty}^{\infty} |x[n]| \left| r^{-n} e^{-j\Omega n} \right| = \sum_{n=-\infty}^{\infty} |x[n]| r^{-n} < \infty$$

Thus, the ROC depends only on r and not on Ω .

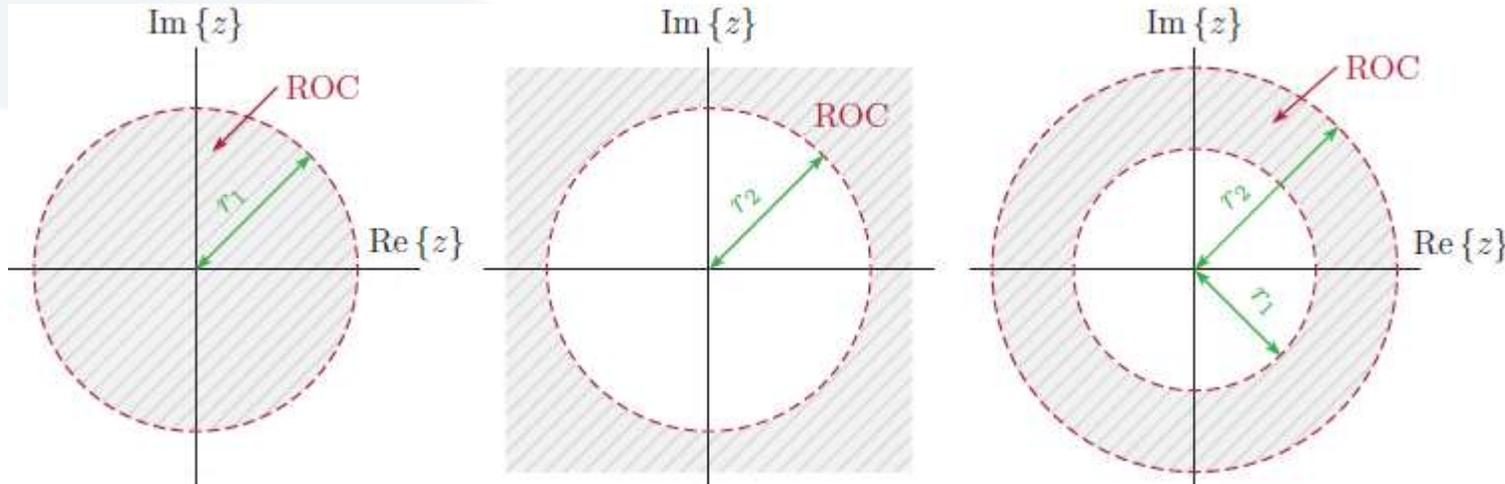


ROC of Finite-Support Signals: The region of convergence (ROC) of the z -transform of a signal $x[n]$ of finite support $[N_0, N_1]$, where $-\infty < N_0 \leq n \leq N_1 < \infty$, is the whole z -plane, excluding the origin $z = 0$ and/or $z = \pm\infty$ depending on N_0 and N_1 .

$$X(z) = \sum_{n=N_0}^{N_1} x[n]z^{-n}$$

ROC of **Infinite-Support** Signals:

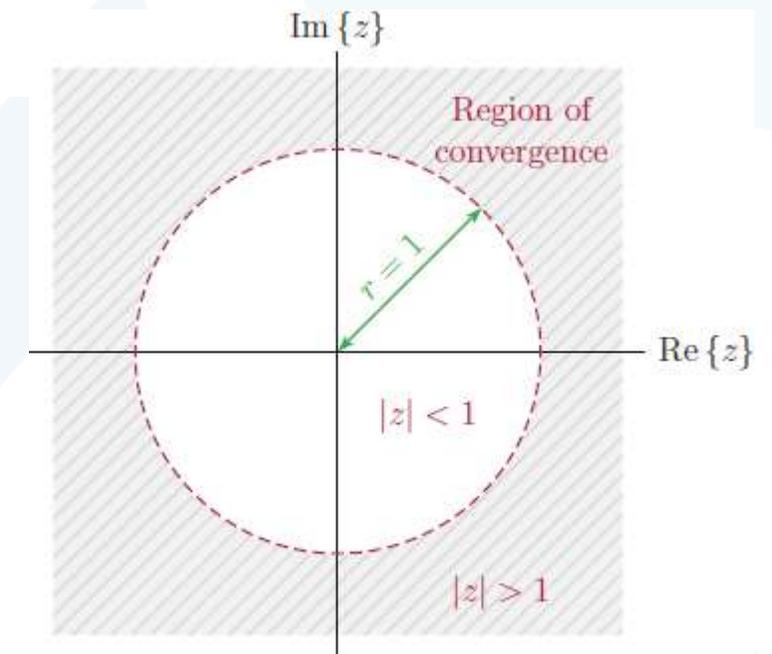
1. **causal** signal $x[n]$ has a region of convergence $|z| > r_1$ where r_1 is the largest radius of the poles of $X(z)$, i.e., the ROC is the outside of a circle of radius r_1 ,
2. **anticausal** signal $x[n]$ has as region of convergence the inside of the circle defined by the smallest radius r_2 of the poles of $X(z)$, or $|z| < r_2$,
3. **noncausal** signal $x[n]$ has as ROC $r_1 < |z| < r_2$, where r_1 and r_2 corresponds to the max and min radii of the poles of $X_c(z)$ and $X_{ac}(z)$.



- **Example 5:** z -Transform of the unit-step signal

$$X(z) = \mathcal{Z}\{u[n]\} = \sum_{n=0}^{\infty} z^{-n} = \frac{1}{1 - z^{-1}} = \frac{z}{z - 1}$$

converge if: $|z^{-1}| < 1 \Rightarrow |z| > 1$



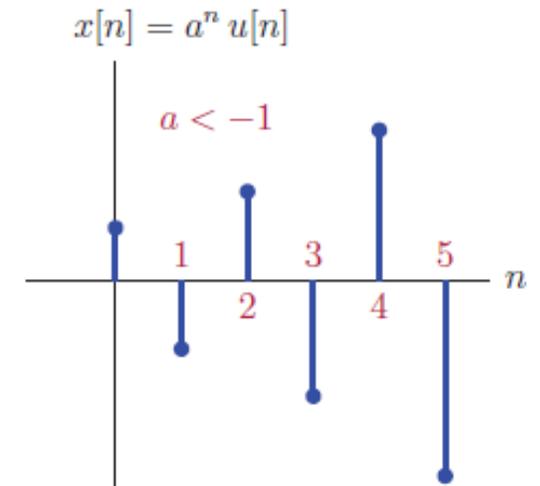
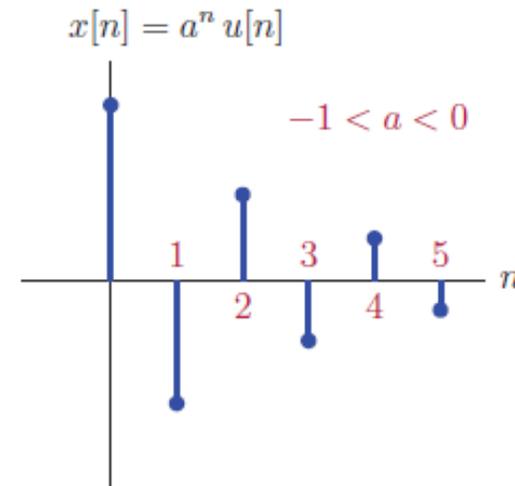
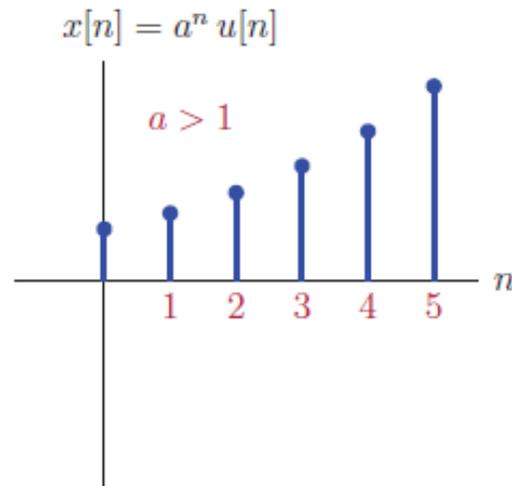
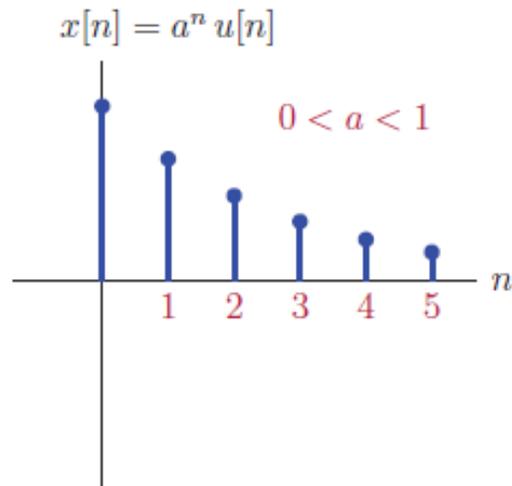


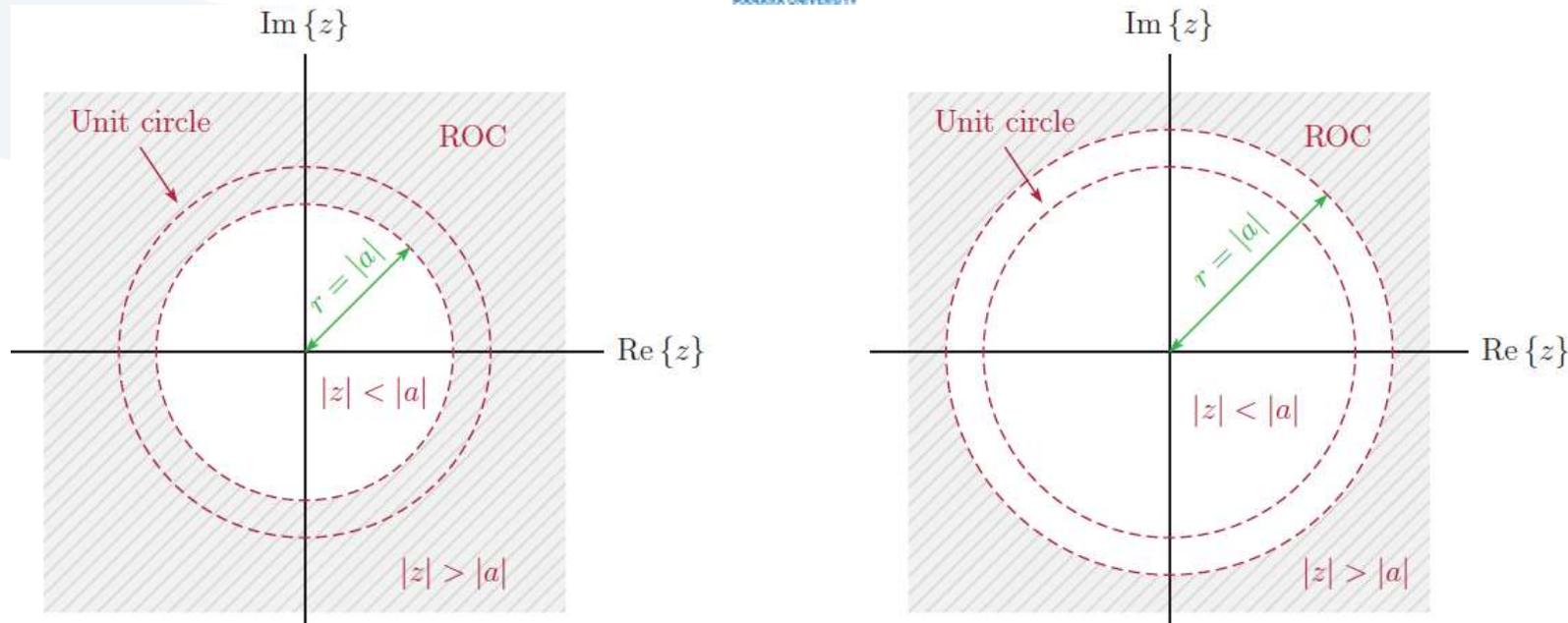
- **Example 6:** z -Transform of a causal exponential signal

$$x[n] = a^n u[n]$$

$$X(z) = \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}$$

converge if: $|az^{-1}| < 1 \Rightarrow |z| > |a|$

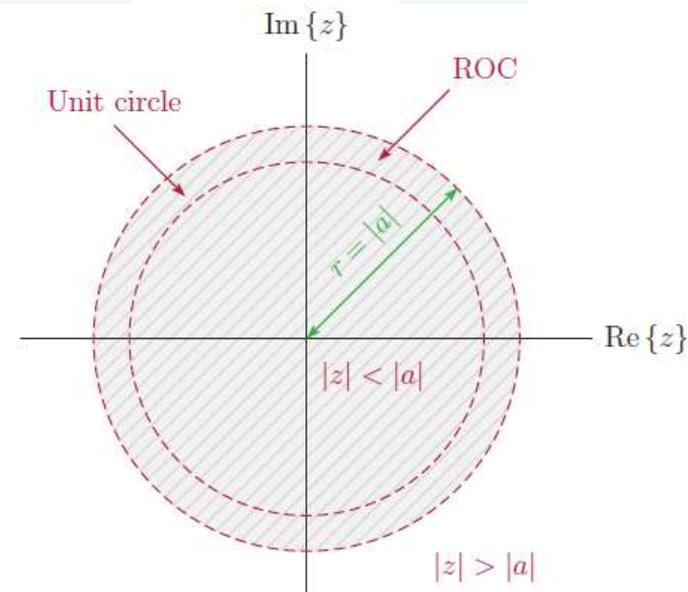
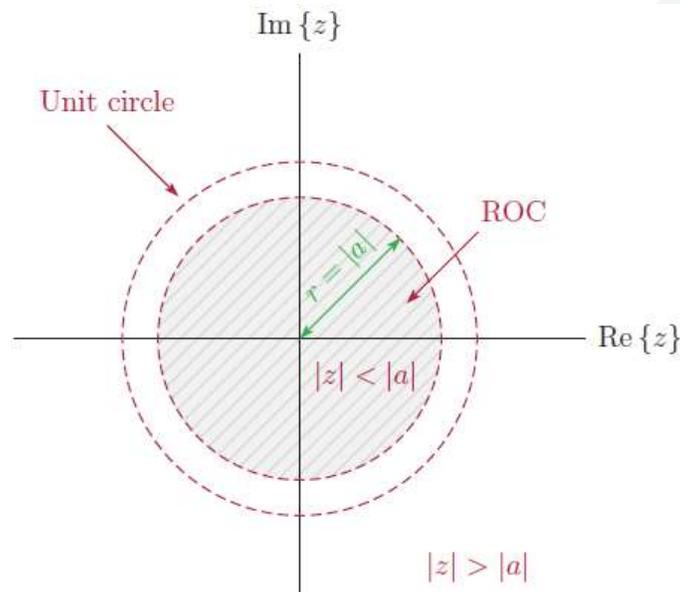
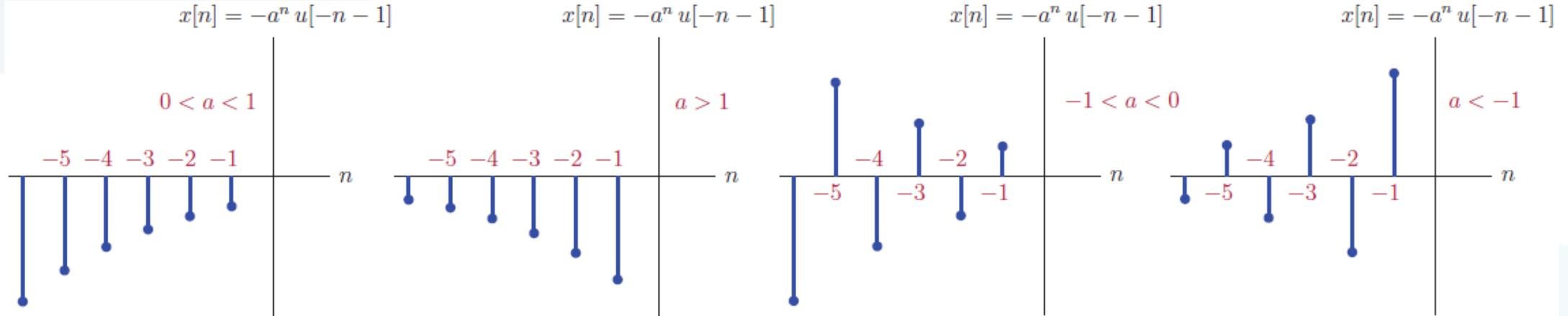




- Example 7:** z -Transform of an anti-causal expo. signal: $x[n] = -a^n u[-n - 1]$

$$\begin{aligned}
 X(z) &= \sum_{n=-\infty}^{\infty} -a^n u[-n - 1] z^{-n} = -\sum_{n=-\infty}^{-1} a^n z^{-n} = -\sum_{m=1}^{\infty} a^{-m} z^m = -\sum_{m=1}^{\infty} (a^{-1} z)^m \\
 &= -a^{-1} z \frac{1}{1 - a^{-1} z} = \frac{z}{z - a}
 \end{aligned}$$

converge if: $|a^{-1} z| < 1 \Rightarrow |z| < |a|$





$$\mathcal{Z}\{a^n u[n]\} = \frac{z}{z-a}, \quad \text{ROC: } |z| > |a|$$

$$\mathcal{Z}\{-a^n u[-n-1]\} = \frac{z}{z-a}, \quad \text{ROC: } |z| < |a|$$

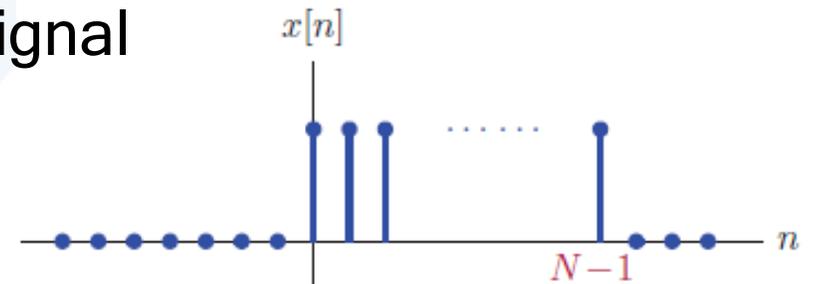
- **Note:** It is possible for 2 \neq signals to have the same transform expression for the z -transform $X(z)$. \Rightarrow the **ROC** must be specified along with the transform.
- In the general case, a rational transform $X(z)$ is expressed in the form:

$$X(z) = K \frac{B(z)}{A(z)} = K \frac{(z - z_1)(z - z_2) \cdots (z - z_M)}{(p - p_1)(p - p_2) \cdots (p - p_N)}$$

Max(M , N) is the order of the transform $X(z)$

- **Example 8:** z -Transform of a discrete-time pulse signal

$$X(z) = \sum_{n=0}^{N-1} (1)z^{-n} = \frac{1 - z^{-N}}{1 - z^{-1}}, \quad \text{ROC: } |z| > 0$$



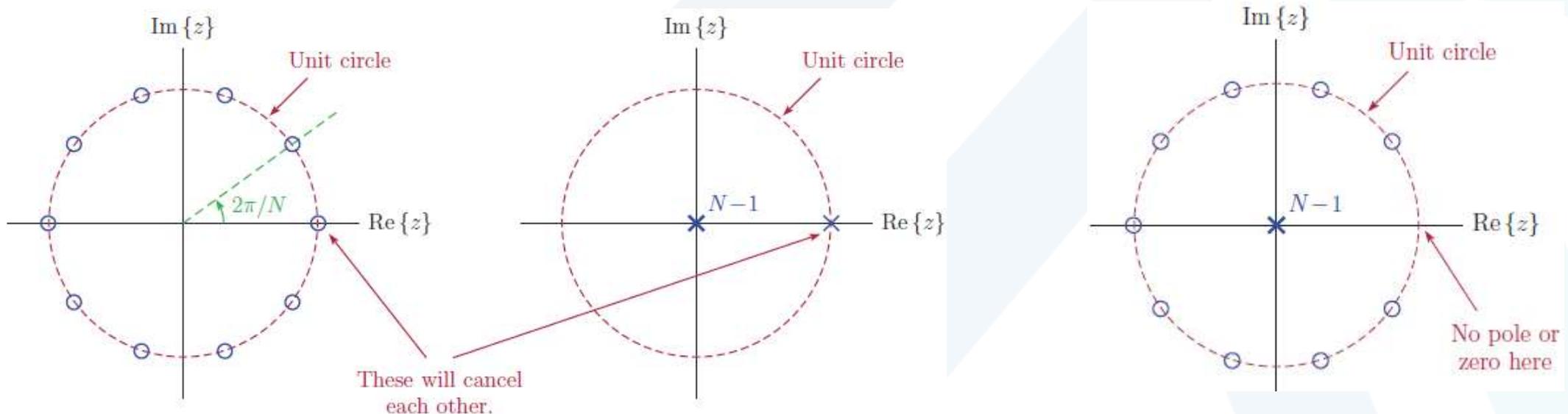


$$X(z) = \frac{z^N - 1}{z^{N-1}(z - 1)}$$

It seems as though $X(z)$ might have a pole at $z = 1$

Zeros: $z_k = e^{j2\pi k/N}$, $k = 1, \dots, N - 1$ Poles: $z = 1$ and $p_k = 0$, $k = 1, \dots, N - 1$

The factors $(z - 1)$ in numerator and denominator polynomials cancel each other, therefore there is neither a zero nor a pole at $z = 1$.





Properties of z -Transform

Property	$x[n]$	$X(z)$	ROC
Linearity	$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	$\supset (R_1 \cap R_2)$
Time shifting	$x[n - k]$	$X(z)z^{-k}$	$R \pm \{0 \text{ or } \infty\}$
Time reversal	$x[-n]$	$X(z^{-1})$	R^{-1}
Multiply by exp.	$x[n]a^n$	$X(z/a)$	$ a R$
Differentiate in z	$nx[n]$	$-z dX(z)/dz$	R
Convolution	$x_1[n] * x_2[n]$	$X_1(z) X_2(z)$	$\supset (R_1 \cap R_2)$
Summation	$\sum_{k=-\infty}^n x[k]$	$\frac{z}{z-1} X(z)$	$\supset (R \cap (z > 1))$

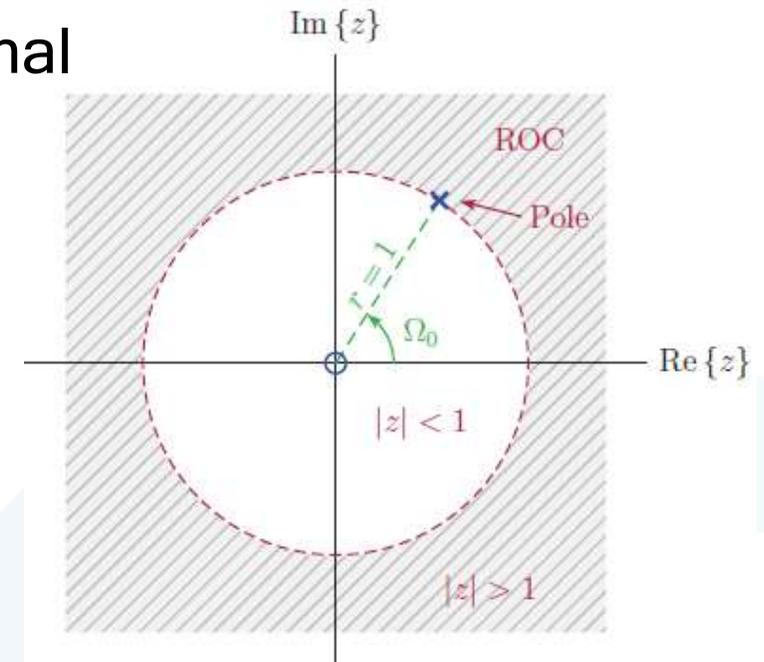


- **Example 9:** z -Transform of complex exponential signal

$$x[n] = e^{j\Omega_0 n} u[n]$$

$$X(z) = \sum_{n=0}^{\infty} e^{j\Omega_0 n} z^{-n} = \sum_{n=0}^{\infty} (e^{j\Omega_0} z^{-1})^n = \frac{1}{1 - e^{j\Omega_0} z^{-1}}$$

$$X(z) = \frac{z}{z - e^{j\Omega_0}} \quad \text{ROC: } |e^{j\Omega_0} z^{-1}| < 1 \Rightarrow |z| > 1$$



- **Example 10:** z -Transform of a cosine and sine signals

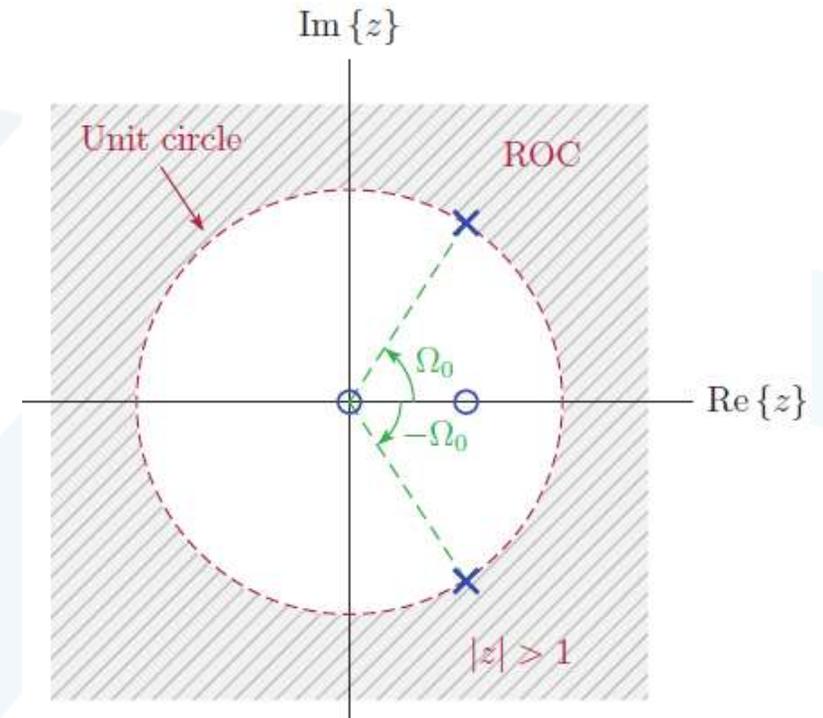
$$x[n] = \cos(\Omega_0 n) u[n] \quad x[n] = \sin(\Omega_0 n) u[n]$$

$$\cos(\Omega_0 n) u[n] = \frac{1}{2} e^{j\Omega_0 n} u[n] + \frac{1}{2} e^{-j\Omega_0 n} u[n] \quad \sin(\Omega_0 n) u[n] = \frac{1}{2j} e^{j\Omega_0 n} u[n] - \frac{1}{2j} e^{-j\Omega_0 n} u[n]$$



$$\begin{aligned}
 \mathcal{Z}\{\cos(\Omega_0 n)u[n]\} &= \frac{1}{2} \mathcal{Z}\{e^{j\Omega_0 n}u[n]\} + \frac{1}{2} \mathcal{Z}\{e^{-j\Omega_0 n}u[n]\} \\
 &= \frac{1/2}{1 - e^{j\Omega_0} z^{-1}} + \frac{1/2}{1 - e^{-j\Omega_0} z^{-1}} = \frac{1 - \cos(\Omega_0)z^{-1}}{1 - 2\cos(\Omega_0)z^{-1} + z^{-2}} \\
 &= \frac{z[z - \cos(\Omega_0)]}{z^2 - 2\cos(\Omega_0)z + 1} \quad \text{ROC is } |z| > 1
 \end{aligned}$$

$$\begin{aligned}
 \mathcal{Z}\{\sin(\Omega_0 n)u[n]\} &= \frac{1}{2j} \mathcal{Z}\{e^{j\Omega_0 n}u[n]\} - \frac{1}{2j} \mathcal{Z}\{e^{-j\Omega_0 n}u[n]\} \\
 &= \frac{1/2j}{1 - e^{j\Omega_0} z^{-1}} - \frac{1/2j}{1 - e^{-j\Omega_0} z^{-1}} = \frac{\sin(\Omega_0)z^{-1}}{1 - 2\cos(\Omega_0)z^{-1} + z^{-2}} \\
 &= \frac{\sin(\Omega_0)z}{z^2 - 2\cos(\Omega_0)z + 1} \quad \text{ROC is } |z| > 1
 \end{aligned}$$





- Example 11:** Multiplication by an exponential signal: $x[n] = a^n \cos(\Omega_0 n) u[n]$

$$x[n] = a^n x_1[n], \quad X_1(z) = \frac{z[1 - \cos(\Omega_0)]}{z^2 - 2\cos(\Omega_0)z + 1}$$

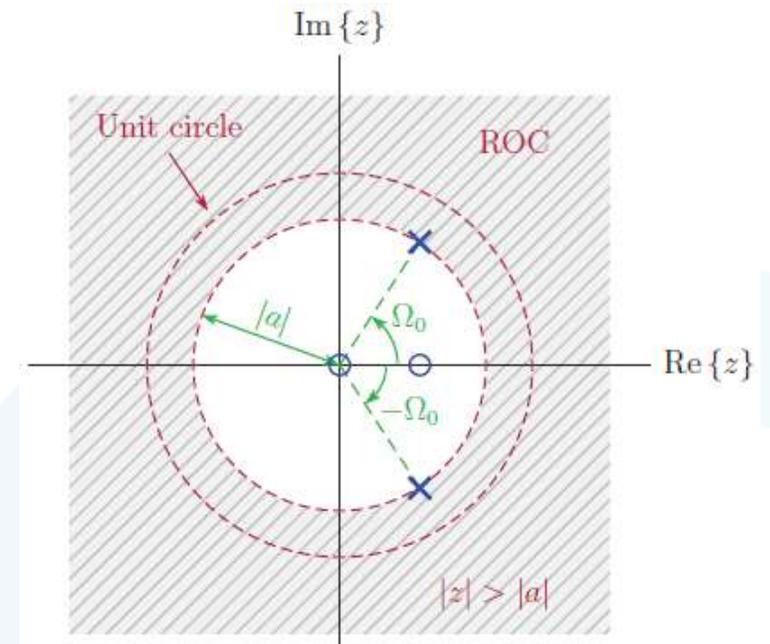
$$X_1(z) = X_1(z/a) = \frac{z[z - a\cos(\Omega_0)]}{z^2 - 2a\cos(\Omega_0)z + a^2}$$

The transform $X(z)$ has two poles at:

$$z = ae^{\pm j\Omega_0} \Rightarrow \text{ROC: } |z| > |a|$$

- Example 12:** Using the differentiation property: $x[n] = na^n u[n]$

$$\mathcal{Z}\{a^n u[n]\} = \frac{z}{z - a}, \quad \text{ROC: } |z| > |a|$$





$$X(z) = (-z) \frac{d}{dz} \frac{z}{z-a} = \frac{az}{(z-a)^2}, \quad \text{ROC: } |z| > |a|$$

z -Transform of a unit-ramp signal $x[n] = nu[n]$

$$\text{Setting } a = 1 \Rightarrow X(z) = \left. \frac{az}{(z-a)^2} \right|_{a=1} = \frac{z}{(z-1)^2}, \quad \text{ROC: } |z| > 1$$

- Example 13:** Using the convolution property $x_1[n] = \{4, 3, 2, 1\}$, $x_2[n] = \{3, 7, 4\}$

Determine $x[n] = x_1[n] * x_2[n]$ using z -transform techniques.

$$X_1(z) = 4 + 3z^{-1} + 2z^{-2} + z^{-3}, \quad X_2(z) = 3 + 7z^{-1} + 4z^{-2}$$

$$X(z) = X_1(z)X_2(z) = 12 + 37z^{-1} + 43z^{-2} + 29z^{-3} + 15z^{-4} + 4z^{-5}$$

$$x[n] = \{12, 37, 43, 29, 15, 4\}$$



- Example 14:** Finding the output signal of a DTLTI system using inverse z -transform: $h[n] = (0.9)^n u[n]$, $x[n] = u[n] - u[n - 7]$

$$H(z) = \mathcal{Z}\{h[n]\} = \frac{z}{z - 0.9}, \quad \text{ROC: } |z| > 0.9$$

$$X(z) = \sum_{n=0}^6 z^{-n} = 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5} + z^{-6} = \frac{z^7 - 1}{z^6(z - 1)}, \quad \text{ROC: } |z| > 0$$

$$\begin{aligned} Y(z) &= X(z)H(z) \\ &= H(z) + z^{-1}H(z) + z^{-2}H(z) + z^{-3}H(z) + z^{-4}H(z) + z^{-5}H(z) + z^{-6}H(z) \end{aligned}$$

$$y[n] = h[n] + h[n - 1] + h[n - 2] + h[n - 3] + h[n - 4] + h[n - 5] + h[n - 6]$$

$$\begin{aligned} y[n] &= (0.9)^n u[n] + (0.9)^{n-1} u[n - 1] + (0.9)^{n-2} u[n - 2] + (0.9)^{n-3} u[n - 3] \\ &\quad + (0.9)^{n-4} u[n - 4] + (0.9)^{n-5} u[n - 5] + (0.9)^{n-6} h[n - 6] \end{aligned}$$



Initial and final value Theorems

Initial and final value properties of the z -transform applies to **causal** signals only.

$$\text{Initial value: } x[0] = \lim_{z \rightarrow \infty} X(z)$$

$$\text{Final value: } \lim_{n \rightarrow \infty} x[n] = \lim_{z \rightarrow 1} (z - 1)X(z)$$

- **Example 15:** Using the initial value property

$$X(z) = \frac{3z^3 + 2z + 5}{2z^3 - 7z^2 + z - 4}$$

Determine the initial value $x[0]$ of the signal.

$$x[0] = \lim_{z \rightarrow \infty} \frac{3z^3 + 2z + 5}{2z^3 - 7z^2 + z - 4} = \frac{3}{2}$$



Inverse Z-Transform

- Recall that the inverse z -transform x of X is given by:

$$x[n] = \frac{1}{2\pi j} \oint_{\Gamma} X(z)z^{n-1}dz$$

where Γ is a counterclockwise closed circular contour centered at the origin and with radius r such that Γ is in the ROC of X .

- Unfortunately, the above contour integration can often be quite tedious to compute. Consequently, we do not usually compute the inverse z -transform directly using the above equation.
- For rational functions, the inverse z -transform can be more easily computed using **partial fraction expansions**.



- **Example 16:** Finding the inverse z -transform using partial fractions

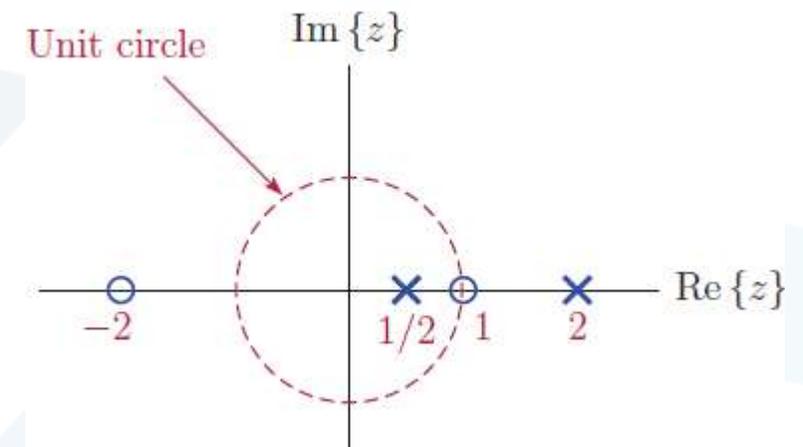
$$X(z) = \frac{(z-1)(z+2)}{(z-1/2)(z-2)}$$

$$\frac{X(z)}{z} = \frac{(z-1)(z+2)}{z(z-1/2)(z-2)} = \frac{-2}{z} + \frac{\frac{5}{3}}{(z-\frac{1}{2})} + \frac{\frac{4}{3}}{(z-2)}$$

$$X(z) = -2 + \frac{\frac{5}{3}z}{(z-\frac{1}{2})} + \frac{\frac{4}{3}z}{(z-2)} = X_1(z) + X_2(z) + X_3(z)$$

$X_1(z)$, is a constant, and its ROC is the entire z -plane. $x_1[n] = \mathcal{Z}^{-1}\{-2\} = -2\delta[n]$

The ROC of $X(z)$ will be determined based on the individual ROCs of the terms $X_2(z)$ and $X_3(z)$. Three possibilities:





Possibility 1: ROC: $|z| < \frac{1}{2}$

$X_2(z)$ and $X_3(z)$ must correspond to anti-causal signals. We need:

ROC for $X_2(z)$: $|z| < \frac{1}{2}$

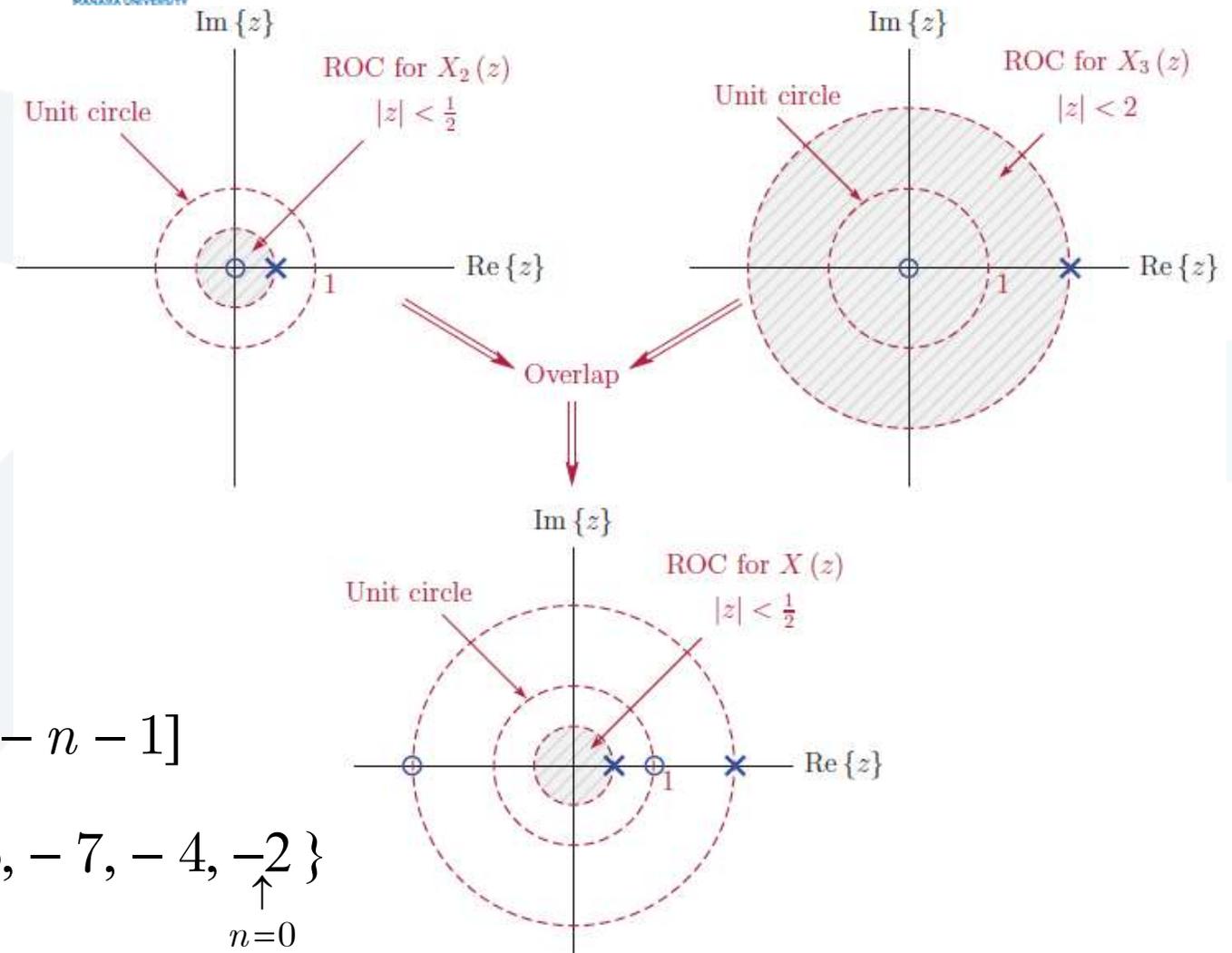
ROC for $X_3(z)$: $|z| < 2$

$$x_2[n] = -\frac{5}{3} \left(\frac{1}{2}\right)^n u[-n-1]$$

$$x_3[n] = -\frac{4}{3} (2)^n u[-n-1]$$

$$x[n] = -2\delta[n] - \left[\frac{5}{3} \left(\frac{1}{2}\right)^n + \frac{4}{3} (2)^n \right] u[-n-1]$$

$$x[n] = \{ \dots, -53.375, -26.75, -13.5, -7, -4, \underset{\substack{\uparrow \\ n=0}}{-2} \}$$



Possibility 2: ROC: $|z| > 2$

$X_2(z)$ and $X_3(z)$ must correspond to causal signals. We need:

ROC for $X_2(z)$: $|z| > \frac{1}{2}$

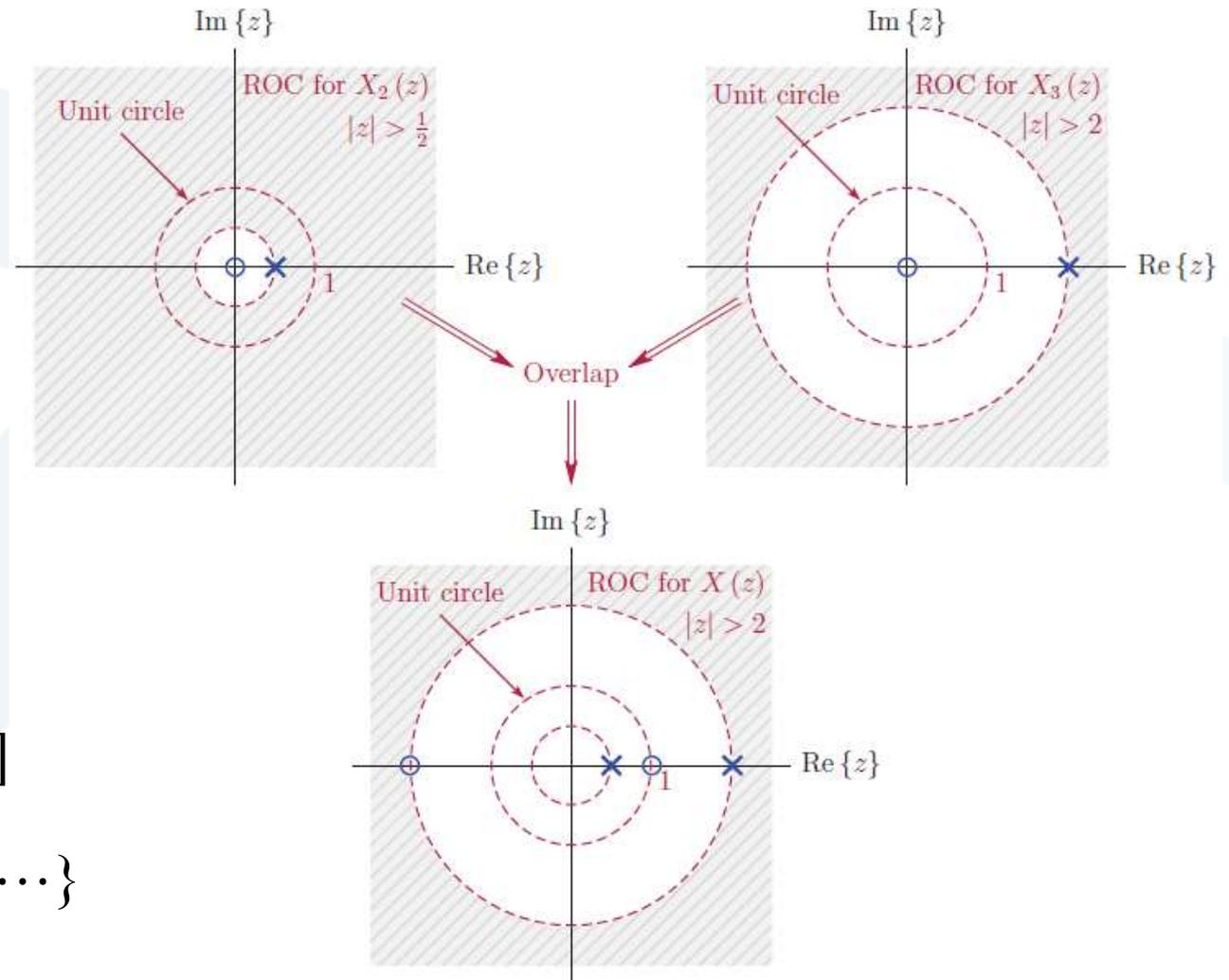
ROC for $X_3(z)$: $|z| > 2$

$$x_2[n] = \frac{5}{3} \left(\frac{1}{2}\right)^n u[n]$$

$$x_3[n] = \frac{4}{3} (2)^n u[n]$$

$$x[n] = -2\delta[n] + \left[\frac{5}{3} \left(\frac{1}{2}\right)^n + \frac{4}{3} (2)^n \right] u[n]$$

$$x[n] = \left\{ \underset{\substack{\uparrow \\ n=0}}{1}, 3.5, 5.75, 8.208, 13.385, \dots \right\}$$





Possibility 3: ROC: $\frac{1}{2} < |z| < 2$

$X_2(z)$ and $X_3(z)$ must correspond to noncausal signals. We need:

ROC for $X_2(z)$: $|z| > \frac{1}{2}$

ROC for $X_3(z)$: $|z| < 2$

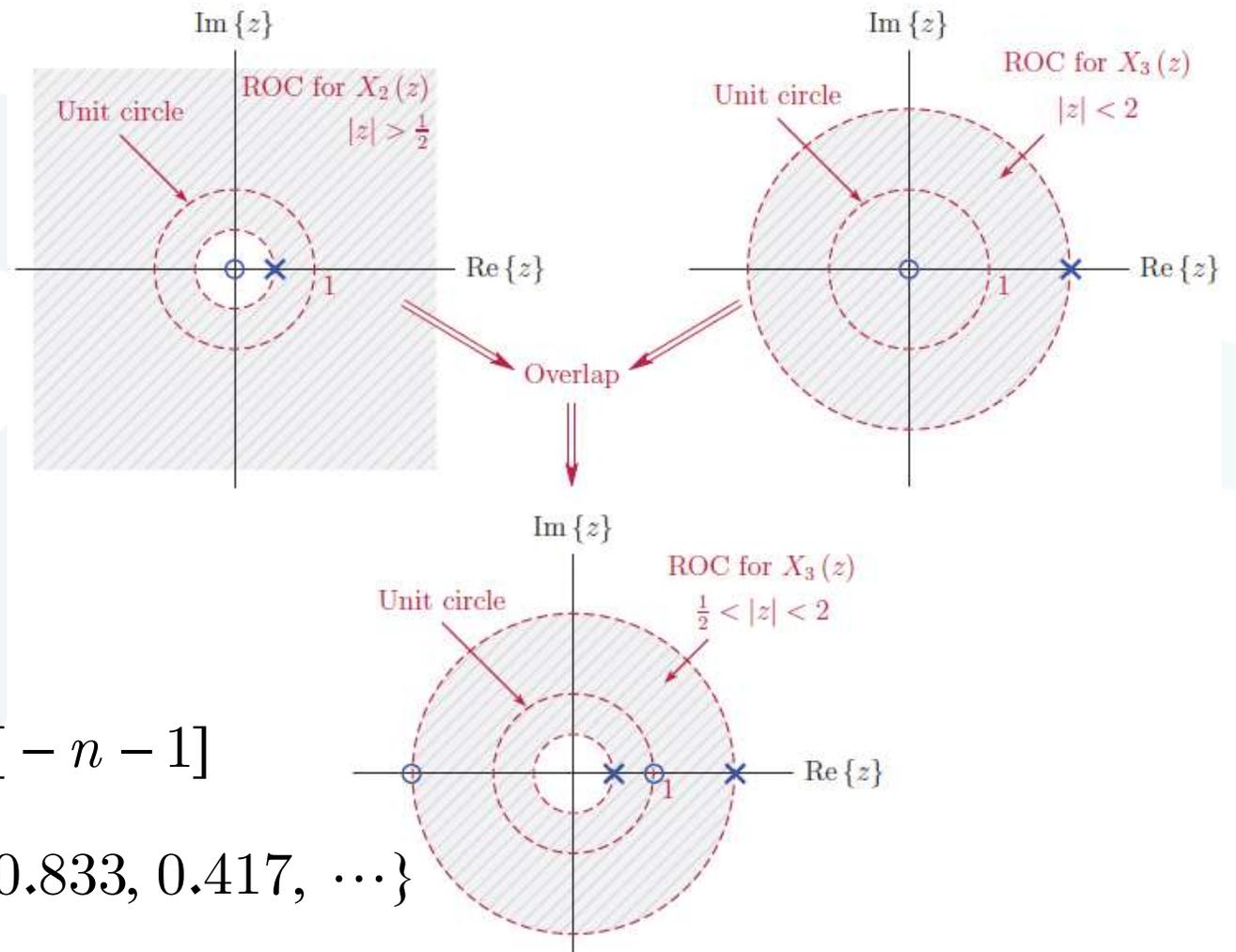
$$x_2[n] = \frac{5}{3} \left(\frac{1}{2}\right)^n u[n]$$

$$x_3[n] = -\frac{4}{3} (2)^n u[-n-1]$$

$$x[n] = -2\delta[n] + \frac{5}{3} \left(\frac{1}{2}\right)^n u[n] - \frac{4}{3} (2)^n u[-n-1]$$

$$x[n] = \{\dots, -0.333, -0.667, -0.333, 0.833, 0.417, \dots\}$$

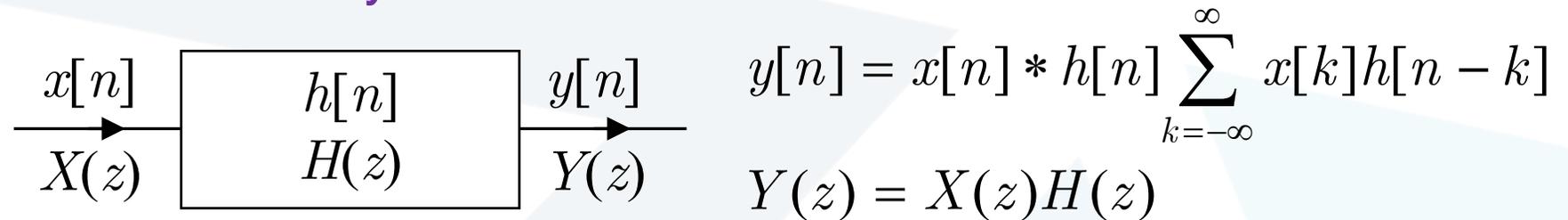
\uparrow
 $n=0$





3. Using the z -Transform with DTLTI Systems

Transfer Function and LTI Systems



Block Diagram Representation

- Since $y[n] = x[n] * h[n]$, the system is characterized in the Laplace domain by $Y(z) = X(z)H(z)$.
- $H(z)$ is the **transfer function** (or **system function**) of the system (i.e., the transfer function is the LT of the impulse response).
- A LTI system is **completely characterized** by its transfer function H .



Relating the transfer function to the difference equation

- Many DTLTI systems of practical interest can be represented using an ***N*th-order linear difference equation with constant coefficients**.
- Consider a system with input x and output y that is characterized by an equation of the form:

$$\sum_{k=0}^N a_k y[n-k] = \sum_{k=0}^M b_k x[n-k]$$

where the a_k and b_k are complex constants and

$$\mathcal{Z} \left\{ \sum_{k=0}^N a_k y[n-k] \right\} = \mathcal{Z} \left\{ \sum_{k=0}^M b_k x[n-k] \right\} \Rightarrow \sum_{k=0}^N \mathcal{Z} \{ a_k y[n-k] \} = \sum_{k=0}^M \mathcal{Z} \{ b_k x[n-k] \}$$

$$\sum_{k=0}^N a_k \mathcal{Z} \{ y[n-k] \} = \sum_{k=0}^M b_k \mathcal{Z} \{ x[n-k] \}$$



$$\sum_{k=0}^N a_k z^{-k} Y(z) = \sum_{k=0}^M b_k z^{-k} X(z) \Rightarrow H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^M b_k z^{-k}}{\sum_{k=0}^N a_k z^{-k}}$$

- The impulse response of the system $h[n] = \mathcal{Z}^{-1}\{H(z)\}$.
- The **convolution** operation is only applicable to problems involving **LTI systems**.
- Therefore it follows that the **transfer function** concept is meaningful only for systems that are both **linear and time invariant**.
- In determining the transfer function from the difference equation, **all initial conditions must be assumed to be zero**.



- **Example 17:** Finding the transfer function from the DE

A DTLTI system is defined by means of the difference equation:

$$y[n] - 0.4y[n - 1] + 0.89y[n - 2] = x[n] - x[n - 1]$$

$$Y(z) - 0.4z^{-1}Y(z) + 0.89z^{-2}Y(z) = X(z) - z^{-1}X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 - z^{-1}}{1 - 0.4z^{-1} + 0.89z^{-2}} = \frac{z(z - 1)}{z^2 - 0.4z + 0.89}$$

Transfer function and causality

- For a DTLTI system to be **causal**, its impulse response $h[n]$ needs to be equal to zero for $n < 0$.

$$H(z) = \sum_{k=-\infty}^{\infty} h[k]z^{-k} = \sum_{k=0}^{\infty} h[k]z^{-k}$$



- The ROC for the transfer function of a causal system is the outside of a circle in the z -plane. Consequently, the transfer function must also converge at $|z| \rightarrow \infty$. Consider a transfer function in the form:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_M z^M + b_{M-1} z^{M-1} + \dots + b_1 z + b_0}{a_N z^N + a_{N-1} z^{N-1} + \dots + a_1 z + a_0}$$

For the system described by $H(z)$ to be causal we need:

$$\lim_{z \rightarrow \infty} H(z) = \lim_{z \rightarrow \infty} \frac{b_M}{a_N} z^{M-N} < \infty \Leftrightarrow M - N \leq 0 \Rightarrow M \leq N$$

- **Note:** this condition is **necessary** for a system to be **causal**, but it is not **sufficient**. It is also possible for a **non-causal** system to have a system function with $M \leq N$.



Transfer function and stability:

- For a DTLTI system to be stable its impulse response must be absolute integrable.

$$\sum_{k=-\infty}^{\infty} |h[n]| z^{-n} < \infty$$

- Fourier transform of a signal exists if the signal is absolute integrable.

$$H(\Omega) = H(z) \Big|_{z=e^{j\Omega}}$$

Stability condition:

- For a DTLTI system to be stable, the ROC of its z -domain transfer function must include the **unit circle**.
- For a **causal** system to be stable, the transfer function must not have any poles **on** or **outside** the unit circle of the z -plane.



- For a **anticausal** system to be stable, the transfer function must not have any poles **on** or **inside** the unit circle of the z -plane.
- For a **noncausal** system the ROC for the TF, if it exists, is the region between two circles with radii r_1 and r_2 , $r_1 < |z| < r_2$. The poles of the TF may be either:
 - a. On or inside the circle with radius r_1
 - b. On or outside the circle with radius r_2and the ROC must include the unit circle.
- **Example 18:** Impulse response of a stable system
Determine the impulse response of a stable system characterized by:

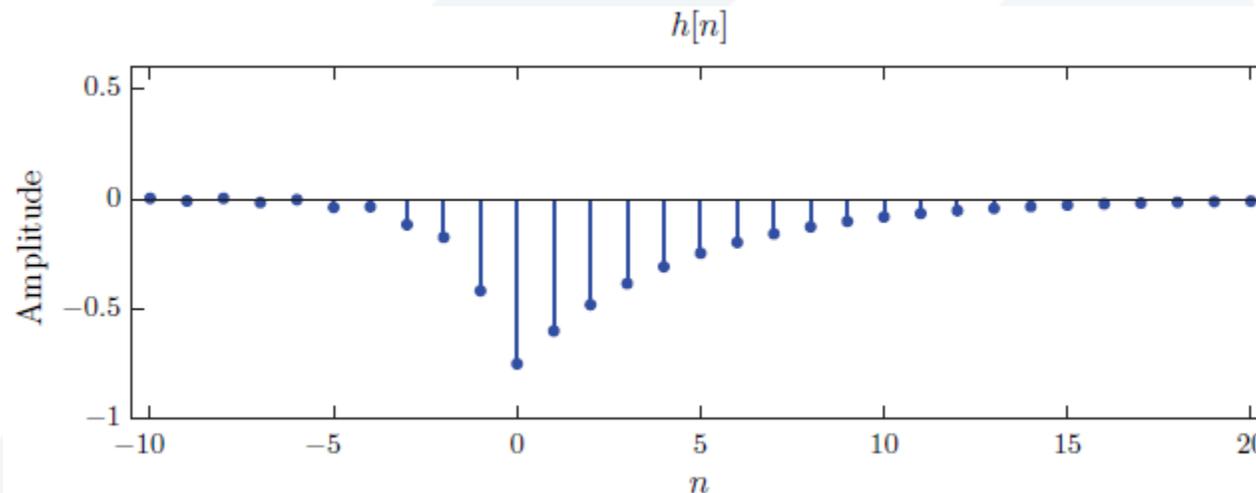
$$H(z) = \frac{z(z + 1)}{(z - 0.8)(z + 1.2)(z - 2)}$$



The poles of the system are at $p = -1.2, 0.8, 2$. Since the system is known to be stable, its ROC must include the unit circle. The only possible choice is $0.8 < |z| < 1.2$.

$$H(z) = \frac{z(z+1)}{(z-0.8)(z+1.2)(z-2)} = -\frac{0.75z}{z-0.8} - \frac{0.0312z}{z+1.2} + \frac{0.7813z}{z-2}$$

$$h[n] = -0.75(0.8)^n u[n] + 0.0312(-1.2)^n u[-n-1] - 0.7813(2)^n u[-n-1]$$





3. Unilateral z -Transform

The **unilateral z -transform** of the signal x is defined as:

$$X_u(z) = \mathcal{Z}_u\{x[n]\} = \sum_{n=0}^{\infty} x[n]z^{-n}$$

- If $x[n]$ is a causal signal, then the unilateral transform $X_u(z)$ becomes identical to the bilateral transform $X(z)$.
- The unilateral ZT is related to the bilateral z -transform as follows:

$$\mathcal{Z}_u\{x[n]\} = \mathcal{Z}\{x[n]u[n]\} = \sum_{n=-\infty}^{\infty} x[n]u[n]z^{-n}$$

- One property of the unilateral z -transform that differs from its counterpart for the bilateral z -transform is the time-shifting property.



$$\mathcal{Z}\{x[n-1]\} = z^{-1}\mathcal{Z}\{x[n]\} = z^{-1}X(z)$$

$$\mathcal{Z}_u\{x[n-1]\} = \sum_{n=0}^{\infty} x[n-1]z^{-n} = x[-1] + \sum_{n=1}^{\infty} x[n-1]z^{-n} = x[-1] + z^{-1} \sum_{n=0}^{\infty} x[n]z^{-n}$$

$$\mathcal{Z}_u\{x[n-1]\} = x[-1] + z^{-1}X_u(z)$$

$$\mathcal{Z}_u\{x[n-k]\} = \sum_{n=-k}^{-1} x[n]z^{-n-k} + z^{-k}X_u(z), \quad k > 0$$

$$\mathcal{Z}_u\{x[n+k]\} = z^{-k}X_u(z) - \sum_{n=0}^{k-1} x[n]z^{k-n}, \quad k > 0$$

- The unilateral z -transform is useful in the use of z -transform techniques for solving difference equations with specified initial conditions.



- **Example 19:** Finding the natural response of a system through z -transform

$$y[n] - \frac{5}{6}y[n-1] + \frac{1}{6}y[n-2] = 0$$

Using z -transform techniques, determine the natural response of the system for the initial conditions: $y[-1] = 19$, $y[-2] = 53$.

$$Z_u\{y[n-1]\} = y[-1] + z^{-1}Y_u(z) = 19 + z^{-1}Y_u(z)$$

$$Z_u\{y[n-2]\} = y[-1] + y[-2]z^{-1} + z^{-2}Y_u(z) = 53 + 19z^{-1} + z^{-2}Y_u(z)$$

$$Y_u(z) - \frac{5}{6}[19 + z^{-1}Y_u(z)] + \frac{1}{6}[53 + 19z^{-1} + z^{-2}Y_u(z)] = 0$$

$$Y_u(z) = \frac{z(7z - \frac{19}{6})}{z^2 - \frac{5}{6}z + \frac{1}{6}} = \frac{z(7z - \frac{19}{6})}{(z - \frac{1}{2})(z - \frac{1}{3})} = \frac{2z}{z - \frac{1}{2}} + \frac{5z}{z - \frac{1}{3}}$$

$$y_h[n] = 2\left(\frac{1}{2}\right)^n u[n] + 5\left(\frac{1}{3}\right)^n u[n]$$



- **Example 20:** Finding the forced response of a system through z -transform

Consider a system defined by means of the difference equation:

$$y[n] = 0.9y[n - 1] + 0.1x[n]$$

Determine the response of this system for the input signal $x[n] = 20 \cos(0.2\pi n)$ if the initial value of the output is $y[-1] = 2.5$.

$$\mathcal{Z}\{\cos(\Omega_0 n)u[n]\} = \frac{z[z - \cos(\Omega_0)]}{z^2 - 2\cos(\Omega_0)z + 1}$$

$$\mathcal{Z}_u\{20\cos(0.2\pi n)\} = \frac{20z[z - \cos(0.2\pi)]}{z^2 - 2\cos(0.2\pi)z + 1}$$

$$\mathcal{Z}_u\{y[n - 1]\} = y[-1] + z^{-1}Y_u(z) = 2.5 + z^{-1}Y_u(z)$$

$$\mathcal{Z}_u\{y[n]\} = 0.9\mathcal{Z}_u\{y[n - 1]\} + 0.1\mathcal{Z}_u\{x[n]\}$$



$$Y_u(z) = 0.9[2.5 + z^{-1}Y_u(z)] + 0.1 \frac{20z[z - \cos(0.2\pi)]}{z^2 - 2\cos(0.2\pi)z + 1}$$

$$Y_u(z) = 0.9z^{-1}Y_u(z) + 2.25 + \frac{2z[z - \cos(0.2\pi)]}{z^2 - 2\cos(0.2\pi)z + 1}$$

$$Y_u(z) = \frac{2z^2[z - \cos(0.2\pi)] + 2.25(z - e^{j0.2\pi})(z - e^{-j0.2\pi})}{(z - 0.9)(z - e^{j0.2\pi})(z - e^{-j0.2\pi})}$$

$$Y_u(z) = \frac{2.7129}{z - 0.9} + \frac{0.7685 - j1.4953}{z - e^{j0.2\pi}} + \frac{0.7685 + j1.4953}{z - e^{-j0.2\pi}}$$

The forced response of the system is:

$$y[n] = 2.7129 (0.9)^n u[n] + 1.5371 \cos(0.2\pi n) u[n] + 2.9907 \sin(0.2\pi n) u[n]$$