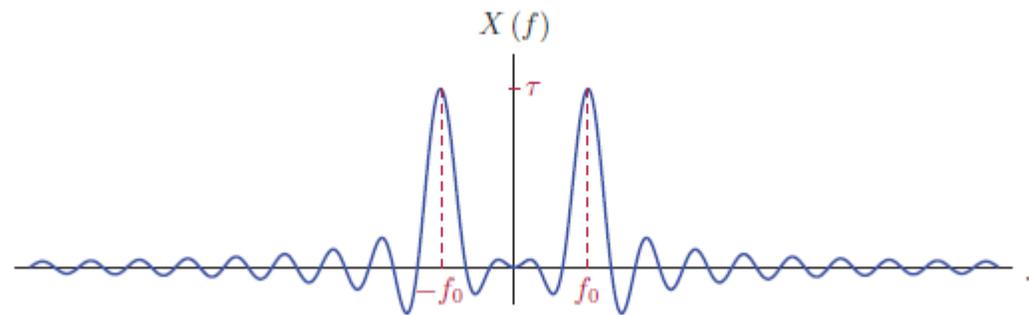




CECC507: Signals and Systems

Lecture Notes 8 & 9: Laplace Transform for Continuous-Time Signals and Systems



Ramez Koudsieh, Ph.D.
Faculty of Engineering
Department of Mechatronics
Manara University



Chapter 7

Laplace Transform for Continuous-Time Signals and Systems

1. Laplace Transform
2. Laplace Transform with CTLTI Systems
3. Simulation Structures for CTLTI Systems
4. Unilateral Laplace Transform



Introduction

- The Laplace transform (LT) can be viewed as a **generalization of the (classical) Fourier transform**.
- Certain characteristics of continuous-time (CT) systems can only be studied via the Laplace transform. Such is the case of **stability**, **transient** and steady-state **responses**.

1. Laplace Transform

- The Laplace transform of a continuous-time signal $x(t)$ is defined as:

$$L\{x(t)\} = X(s) = \int x(t)e^{-st} dt$$

where $s = \sigma + j\omega$, the independent variable of the transform. σ : damping factor, ω : frequency variable.



- There are two important variants:

Unilateral (or one-sided): $X(s) = \mathcal{L}_u\{x(t)\} = \int_{0^-}^{\infty} x(t)e^{-st} dt;$

Bilateral (or two sided): $X(s) = \mathcal{L}\{x(t)\} = \int_{-\infty}^{\infty} x(t)e^{-st} dt;$

Relationship Between LT and Continuous-Time FT

$$X(\sigma + j\omega) = \int_{-\infty}^{\infty} x(t)e^{-(\sigma + j\omega)t} dt = \int_{-\infty}^{\infty} [x(t)e^{-\sigma t}]e^{-j\omega t} dt = \mathcal{F}\{e^{-\sigma t}x(t)\}$$

$$X(j\omega) = \left[\int_{-\infty}^{\infty} x(t)e^{-st} dt \right] \Big|_{s=j\omega} = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt = \mathcal{F}\{x(t)\}$$

- Example 1:** Laplace transform of the unit impulse

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st} dt = \int_{-\infty}^{\infty} \delta(t)e^{-st} dt = 1$$



- **Example 2:** Laplace transform of the unit-step signal

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st} dt = \int_{-\infty}^{\infty} u(t)e^{-st} dt = \int_0^{\infty} e^{-st} dt = \frac{1}{s}, \quad \text{Re}\{s\} > 0$$

Regions of Convergence

- All points in the s -plane for which the Laplace transform converges is called the **region of convergence** (ROC). For the LT $X(s)$ of $x(t)$ to exist we need that:

$$\left| \int_{-\infty}^{\infty} x(t)e^{-\sigma t} dt \right| = \left| \int_{-\infty}^{\infty} x(t)e^{-\sigma t} e^{-j\omega t} dt \right| \leq \int_{-\infty}^{\infty} |x(t)e^{-\sigma t}| dt < \infty$$

- **Note:** The frequency does not affect the ROC.

Poles and Zeros and the Region of Convergence

- Typically, $X(s)$ is rational, $X(s) = N(s)/D(s)$.



- For the Laplace The roots of $N(s)$ are called **zeros**, and the roots of $D(s)$ are called **poles**. The ROC is related to the poles of the transform.
- If $\{\sigma_i\}$ are the real parts of the poles of $X(s)$, the ROC corresponding to different types of signals is determined from its poles as follows:
- For a **causal signal** $x(t)$, the region of convergence of its Laplace transform $X(s)$ is a plane to the right of the poles, $R_c = \{(\sigma, \omega): \sigma > \max\{\sigma_i\}, -\infty < \omega < \infty\}$
- For a **anticausal signal** $x(t)$, the ROC of its Laplace transform $X(s)$ is a plane to the left of the poles, $R_{ac} = \{(\sigma, \omega): \sigma < \min\{\sigma_i\}, -\infty < \omega < \infty\}$
- For a **noncausal signal** $x(t)$, the region of convergence of its Laplace transform $X(s)$ is the intersection of the ROC corresponding to the causal component, R_c , and R_{ac} corresponding to the anticausal component, $R_c \cap R_{ac}$

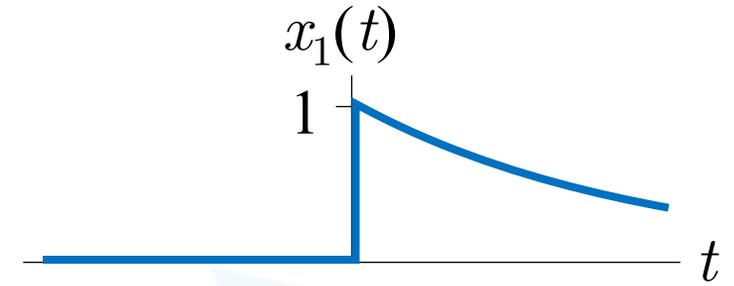


- **Example 3:** Find the Laplace transform of $x_1(t)$

$$x_1(t) = \begin{cases} e^{-t} & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

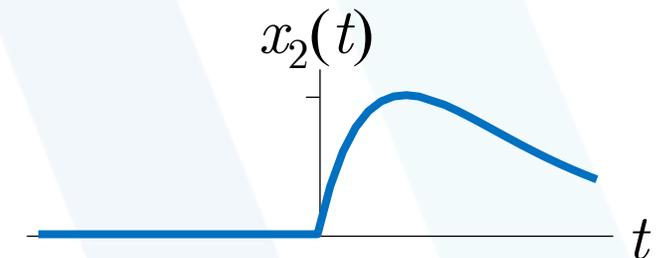
$$X_1(s) = \int_{-\infty}^{\infty} x_1(t) e^{-st} dt = \int_0^{\infty} e^{-t} e^{-st} dt = \frac{e^{-(s+1)t}}{-(s+1)} \Big|_0^{\infty} = \frac{1}{s+1},$$

$$\text{Re}\{s\} > -1$$



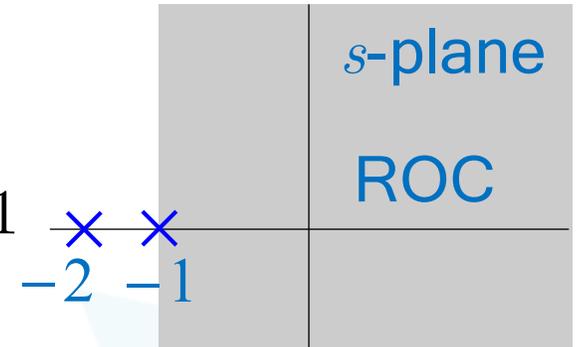
- **Example 4:** Find the Laplace transform of $x_2(t)$

$$x_2(t) = \begin{cases} e^{-t} - e^{-2t} & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$





$$\begin{aligned}
 X_2(s) &= \int_0^{\infty} (e^{-t} - e^{-2t})e^{-st} dt \\
 &= \int_0^{\infty} e^{-t}e^{-st} dt - \int_0^{\infty} e^{-2t}e^{-st} dt = \frac{1}{(s+1)(s+2)}, \operatorname{Re}\{s\} > -1
 \end{aligned}$$

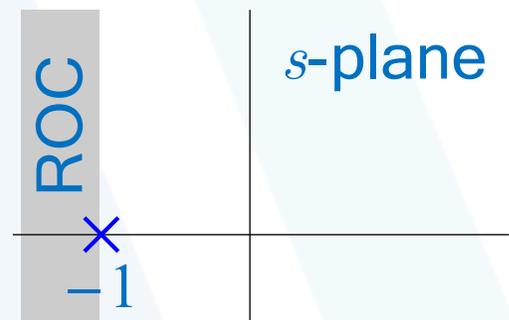
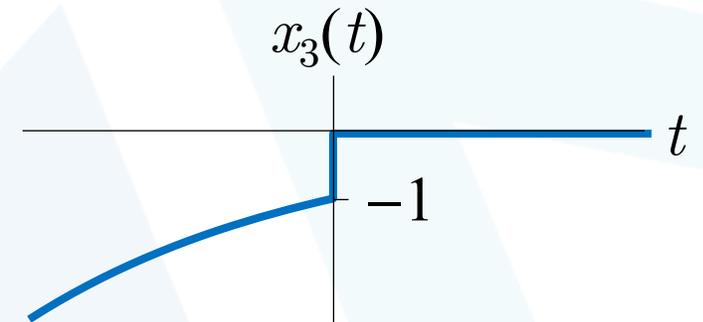


- **Example 5:** LT of an anti-causal exponential signal

$$x_3(t) = \begin{cases} -e^{-t} & \text{if } t \leq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$X_3(s) = \int_{-\infty}^{\infty} x_3(t)e^{-st} dt = \int_{-\infty}^0 -e^{-t}e^{-st} dt = \frac{-e^{-(s+1)t}}{-(s+1)} \Big|_{-\infty}^0 = \frac{1}{s+1},$$

$$\operatorname{Re}\{s\} < -1$$



- It is possible for two different signals to have the same transform expression for $X(s)$.



$$L\{e^{-t}u(t)\} = \frac{1}{s+1}, \quad \text{ROC: } \text{Re}\{s\} > -1$$

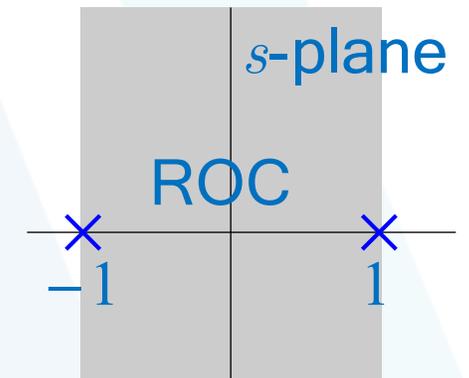
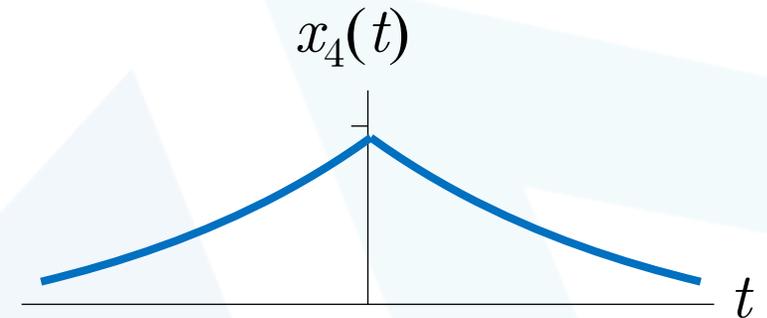
$$L\{-e^{-t}u(-t)\} = \frac{1}{s+1}, \quad \text{ROC: } \text{Re}\{s\} < -1$$

⇒ the ROC must be **specified** along with the transform

- **Example 6:** Find the Laplace transform of $x_4(t)$

$$x_4(t) = e^{-|t|}$$

$$\begin{aligned} X_4(s) &= \int_{-\infty}^{\infty} e^{-|t|} e^{-st} dt = \int_{-\infty}^0 e^{(1-s)t} dt + \int_0^{\infty} e^{-(1+s)t} dt \\ &= \frac{e^{(1-s)t}}{(1-s)} \Big|_{-\infty}^0 + \frac{-e^{-(1+s)t}}{-(1+s)} \Big|_0^{\infty} = \frac{1}{1-s} + \frac{1}{1+s} = \frac{2}{1-s^2}, \quad -1 < \text{Re}\{s\} < 1 \end{aligned}$$



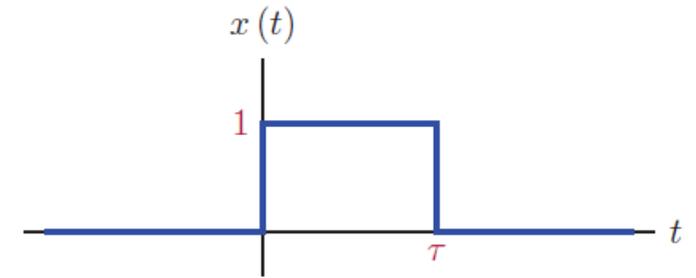


- **Example 7:** Laplace transform of a pulse signal

$$x(t) = \Pi\left(\frac{t - \tau/2}{\tau}\right)$$

$$X(s) = \int_0^{\tau} (1)e^{-st} dt = \frac{e^{-st}}{-s} \Big|_0^{\tau} = \frac{1 - e^{-s\tau}}{s}$$

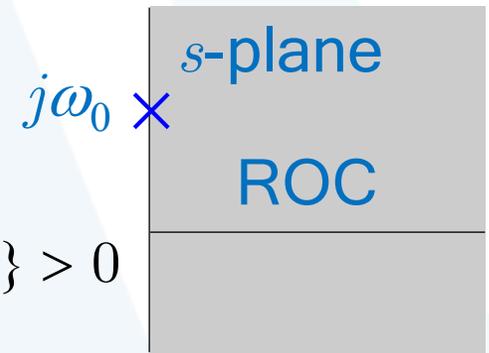
$$X(s) \Big|_{s=0} = \frac{\tau e^{-s\tau}}{1} \Big|_{s=0} = \tau \Rightarrow X(s) \text{ converge at } s = 0$$



- **Example 8:** Laplace transform of complex exponential signal

$$x(t) = e^{j\omega_0 t} u(t)$$

$$X(s) = \int_{-\infty}^{\infty} e^{j\omega_0 t} u(t) e^{-st} dt = \int_0^{\infty} e^{(j\omega_0 - s)t} dt = \frac{e^{(j\omega_0 - s)t}}{j\omega_0 - s} \Big|_0^{\infty} = \frac{1}{s - j\omega_0}, \text{Re}\{s\} > 0$$





Properties of Laplace Transform

Property	$x(t)$	$X(s)$	ROC
Linearity	$ax_1(t) + bx_2(t)$	$aX_1(s) + bX_2(s)$	$\supset (R_1 \cap R_2)$
Delay by T	$x(t - T)$	$X(s)e^{-sT}$	R
Multiply by t	$tx(t)$	$-dX(s)/ds$	R
Multiply by e^{-at}	$x(t)e^{-at}$	$X(s + \alpha)$	Shift R by $-\alpha$
Scaling in t	$x(at)$	$\frac{1}{ a } X\left(\frac{s}{a}\right)$	aR
Differentiate in t	$dx(t)/dt$	$sX(s)$	$\supset R$
Integrate in t	$\int_{-\infty}^t x(\tau) d\tau$	$\frac{X(s)}{s}$	$\supset (R \cap (\text{Re}(s) > 0))$
Convolve in t	$x_1 * x_2(t)$	$X_1(s) X_2(s)$	$\supset (R_1 \cap R_2)$



Laplace Transform Pairs

1	$\delta(t)$	1	All s
2	$u(t)$	$1/s$	$\text{Re}\{s\} > 0$
3	$-u(-t)$	$1/s$	$\text{Re}\{s\} < 0$
4	$t^n u(t)$	$\frac{n!}{s^{n+1}}$	$\text{Re}\{s\} > 0$
5	$-t^n u(-t)$	$\frac{n!}{s^{n+1}}$	$\text{Re}\{s\} < 0$
6	$e^{-at} u(t)$	$\frac{1}{s+a}$	$\text{Re}\{s\} > -a$
7	$-e^{-at} u(-t)$	$\frac{1}{s+a}$	$\text{Re}\{s\} < -a$



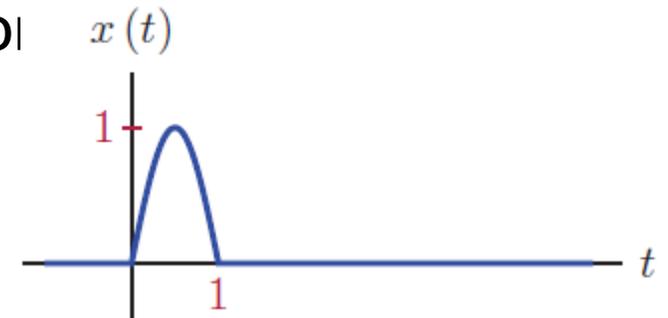
8	$t^n e^{-at} u(t)$	$\frac{n!}{(s+a)^{n+1}}$	$\text{Re}\{s\} > -a$
9	$-t^n e^{-at} u(-t)$	$\frac{n!}{(s+a)^{n+1}}$	$\text{Re}\{s\} < -a$
10	$[\cos \omega_0 t] u(t)$	$\frac{s}{s^2 + \omega_0^2}$	$\text{Re}\{s\} > 0$
11	$[\sin \omega_0 t] u(t)$	$\frac{\omega_0}{s^2 + \omega_0^2}$	$\text{Re}\{s\} > 0$
12	$[e^{-at} \cos \omega_0 t] u(t)$	$\frac{s+a}{(s+a)^2 + \omega_0^2}$	$\text{Re}\{s\} > -a$
13	$[e^{-at} \sin \omega_0 t] u(t)$	$\frac{\omega_0}{(s+a)^2 + \omega_0^2}$	$\text{Re}\{s\} > -a$



- **Example 9:** Laplace transform of a truncated sine function

$$x(t) = \begin{cases} \sin(\pi t), & 0 < t < 1 \\ 0, & \text{otherwise} \end{cases}$$

$$X(s) = \int_0^1 \sin(\pi t) e^{-st} dt = \frac{1}{2j} \int_0^1 (e^{j\pi t} - e^{-j\pi t}) e^{-st} dt = \frac{\pi(1 + e^{-s})}{s^2 + \pi^2}$$



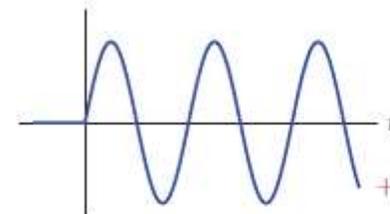
Another method

$$x(t) = \sin(\pi t)u(t) + \sin(\pi[t-1])u(t-1)$$

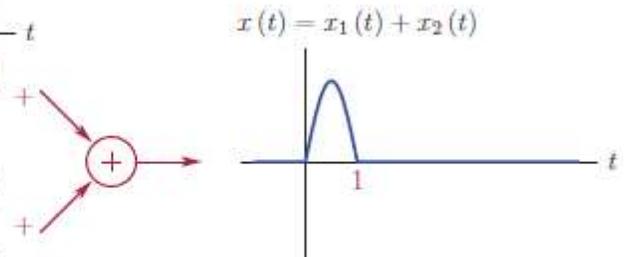
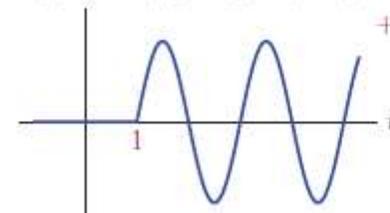
$$X(s) = \frac{\pi}{s^2 + \pi^2} + \frac{\pi}{s^2 + \pi^2} e^{-s} = \frac{\pi(1 + e^{-s})}{s^2 + \pi^2}$$

ROC: entire s -plane except points where $\text{Re}\{s\} \rightarrow -\infty$

$$x_1(t) = \sin(\pi t)u(t)$$



$$x_2(t) = \sin(\pi[t-1])u(t-1)$$





- **Example 10:** Using the convolution property of the Laplace transform

$$x_1(t) = e^{-t}u(t), \quad x_2(t) = \delta(t) - e^{-2t}u(t)$$

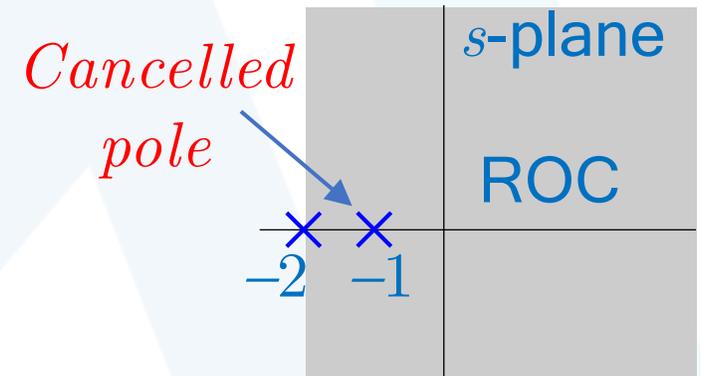
Determine $x(t) = x_1(t) * x_2(t)$ using Laplace transform techniques.

$$X_1(s) = \frac{1}{s+1}, \quad \text{ROC: } \text{Re}\{s\} > -1$$

$$X_2(s) = 1 - \frac{1}{s+2} = \frac{s+1}{s+2}, \quad \text{ROC: } \text{Re}\{s\} > -2$$

$$X(s) = X_1(s)X_2(s) = \frac{1}{s+2}, \quad \text{ROC: } \text{Re}\{s\} > -2$$

$$x(t) = \mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} = e^{-2t}u(t)$$





Initial Value Theorem

For a function x with Laplace transform X , if x is **causal** and contains **no impulses or higher order singularities at the origin**, then:

$$x(0^+) = \lim_{s \rightarrow \infty} sX(s)$$

- When X is known but x is not, the initial value theorem eliminates the need to explicitly find x in order to evaluate $x(0^+)$.
- Example 11:** Calculate the initial value of the function $x(t)$, whose LT is:

$$X(s) = \frac{2(s+1)}{(s+1)^2 + 5^2}$$

$$x(0^+) = \lim_{t \rightarrow 0^+} x(t) = \lim_{s \rightarrow \infty} sX(s) = \lim_{s \rightarrow \infty} \frac{2s(s+1)}{(s+1)^2 + 5^2} = 2$$

Verification: $x(t) = 2e^{-t} \cos(5t) u(t)$



Final Value Theorem

For a function x with Laplace transform X , if x is **causal** and $x(t)$ has a **finite limit** as $t \rightarrow \infty$, then:

$$\lim_{t \rightarrow \infty} x(t) = \lim_{s \rightarrow 0} sX(s)$$

- When X is known but x is not, the final value theorem eliminates the need to explicitly find x in order to evaluate limit $\lim_{t \rightarrow \infty} x(t)$.
- Example 12:** Calculate the final value of the function $x(t)$, whose Laplace transform is:

$$X(s) = \frac{s + 3}{s(s + 1)}$$

$$\lim_{t \rightarrow \infty} x(t) = \lim_{s \rightarrow 0} sX(s) = \lim_{s \rightarrow 0} \frac{s(s + 3)}{s(s + 1)} = \lim_{s \rightarrow 0} \frac{s + 3}{s + 1} = 3$$

Verification: $x(t) = (3 - 2e^{-t})u(t)$



Inverse Laplace Transform

The inverse LT x of X is given by $L^{-1}\{X(s)\} = x(t) = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st} ds$, where $\text{Re}(s) = \sigma$ is in the ROC of X .

- For rational functions, the inverse Laplace transform can be more easily computed using **partial fraction expansions (PFE)**.
- Example 13:** Calculate the inverse LT of the function $H(s) = 1/(s + a)$

$h(t) = e^{-at}u(t)$	with ROC: $\text{Re}\{s\} > -a$
$h(t) = -e^{-at}u(-t)$	with ROC: $\text{Re}\{s\} < -a$
- Example 14:** The Laplace transform of a signal $x(t)$ is $X(s) = \frac{s + 1}{s(s^2 + 9)}$ with the ROC specified as $\text{Re}\{s\} > 0$. Determine $x(t)$.



$$X(s) = \frac{k_1}{s} + \frac{k_2}{s + j3} + \frac{k_3}{s - j3}$$

$$k_1 = \frac{1}{9}, \quad k_2 = -\frac{1}{18} + j\frac{1}{6}, \quad k_3 = \frac{1}{18} - j\frac{1}{6}$$

$$x(t) = \frac{1}{9} u(t) - \frac{1}{9 \cdot 18} [e^{-j3t} + e^{j3t}] u(t) + j\frac{1}{6} [e^{-j3t} - e^{j3t}] u(t)$$

$$x(t) = \frac{1}{9} u(t) - \frac{1}{9} \cos(3t) u(t) + \frac{1}{3} \sin(3t) u(t)$$

Based on the specified ROC,
the signal $x(t)$ is causal

- **Example 15:** Multiple-order poles

A causal signal $x(t)$ has the Laplace transform $X(s) = \frac{s(s+1)}{(s+1)^3(s+2)}$

$$X(s) = \frac{s(s+1)}{(s+1)^3(s+2)} = \frac{-3}{s+1} + \frac{3}{(s+1)^2} - \frac{2}{(s+1)^3} + \frac{3}{s+2}$$



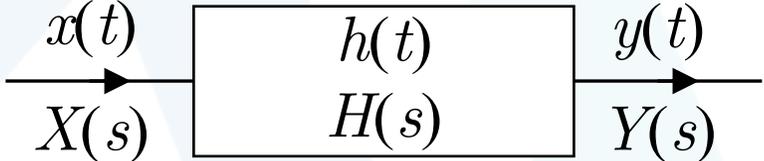
$$L\{e^{-t}u(t)\} = \frac{1}{s+1}, \quad L\{te^{-t}u(t)\} = -\frac{d}{ds} \left[\frac{1}{s+1} \right] = \frac{1}{(s+1)^2}$$

$$L\{t^2e^{-t}u(t)\} = -\frac{d}{ds} \left[\frac{1}{(s+1)^2} \right] = \frac{2}{(s+1)^3}$$

$$x(t) = -3e^{-t}u(t) + 3te^{-t}u(t) - t^2e^{-3t}u(t) + 3e^{-2t}u(t)$$

2. Laplace Transform with CTLTI Systems

Transfer Function and LTI Systems

- Since $y(t) = x(t) * h(t)$, the system is characterized in the Laplace domain by $Y(s) = X(s)H(s)$.
 
- $H(s)$ is the **transfer function** (or **system function**) of the system.



- A LTI system is **completely characterized** by its transfer function $H(s)$.

Relating the transfer function to the differential equation

- Many LTI systems of practical interest can be represented using an **N th-order linear differential equation with constant coefficients**.
- Consider a system with input x and output y that is characterized by an equation of the form:

$$\sum_{k=0}^N a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^M b_k \frac{d^k x(t)}{dt^k}$$

where the a_k and b_k are complex constants and

$$\mathcal{L} \left\{ \sum_{k=0}^N a_k \frac{d^k y(t)}{dt^k} \right\} = \mathcal{L} \left\{ \sum_{k=0}^M b_k \frac{d^k x(t)}{dt^k} \right\} \Rightarrow \sum_{k=0}^N \mathcal{L} \left\{ a_k \frac{d^k y(t)}{dt^k} \right\} = \sum_{k=0}^M \mathcal{L} \left\{ b_k \frac{d^k x(t)}{dt^k} \right\}$$



$$\sum_{k=0}^N a_k \mathcal{L} \left\{ \frac{d^k y(t)}{dt^k} \right\} = \sum_{k=0}^M b_k \mathcal{L} \left\{ \frac{d^k x(t)}{dt^k} \right\}$$

$$\sum_{k=0}^N a_k s^k Y(s) = \sum_{k=0}^M b_k s^k X(s) \Rightarrow H(s) = \frac{Y(s)}{X(s)} = \frac{\sum_{k=0}^M b_k s^k}{\sum_{k=0}^N a_k s^k}$$

- The transfer function is always **rational**.
- The impulse response of the system $h(t) = \mathcal{L}^{-1}\{H(s)\}$.
- The **convolution** operation is only applicable to problems involving **LTI systems**.
- Therefore it follows that the **transfer function** concept is meaningful only for systems that are both **linear and time invariant**.
- In determining the transfer function from the differential equation, **all initial conditions must be assumed to be zero**.



- **Example 16:** Finding the transfer function from the DE

A CTLTI system is defined by means of the differential equation:

$$\frac{d^3 y(t)}{dt^3} + 5 \frac{d^2 y(t)}{dt^2} + 17 \frac{dy(t)}{dt} + 13y(t) = \frac{d^2 x(t)}{dt^2} + x(t)$$

$$s^3 Y(s) + 5s^2 Y(s) + 17s Y(s) + 13Y(s) = s^2 X(s) + X(s)$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{s^2 + 1}{s^3 + 5s^2 + 17s + 13}$$

Transfer function and causality

- **Theorem:** For a LTI system with a **rational** transfer function H , **causality** of the system is **equivalent** to the ROC of H being the **right sided to the right of the rightmost pole** or, if H has no poles, the entire complex plane.



- For a CTLTI system to be causal, its impulse response $h(t)$ needs to be equal to zero for $t < 0$.

$$H(s) = \int_{-\infty}^{\infty} h(t)e^{-st} dt = \int_0^{\infty} h(t)e^{-st} dt$$

- Consider a transfer function in the form:

$$H(s) = \frac{Y(s)}{X(s)} = \frac{b_M s^M + b_{M-1} s^{M-1} + \dots + b_1 s + b_0}{a_N s^N + a_{N-1} s^{N-1} + \dots + a_1 s + a_0}$$

For the system described by $H(s)$ to be causal we need:

$$\lim_{s \rightarrow \infty} H(s) = \lim_{s \rightarrow \infty} \frac{b_M}{a_N} s^{M-N} < \infty \Leftrightarrow M - N \leq 0 \Rightarrow M \leq N$$

Causality condition:

- In the transfer function of a causal CTLTI system the order of the numerator must **not be greater** than the order of the denominator.



Transfer function and stability:

- For a CTLTI system to be stable its impulse response must be absolute integrable.

$$\int_{-\infty}^{\infty} |h(t)| dt < \infty$$

Stability condition:

- For a CTLTI system to be **stable**, the ROC of its s -domain transfer function must include the **imaginary axis**.
- For a **causal** system to be stable, the transfer function must not have any poles on the imaginary axis or in the right half s -plane.
- For a **anticausal** system to be stable, the transfer function must not have any poles on the imaginary axis or in the right half s -plane.



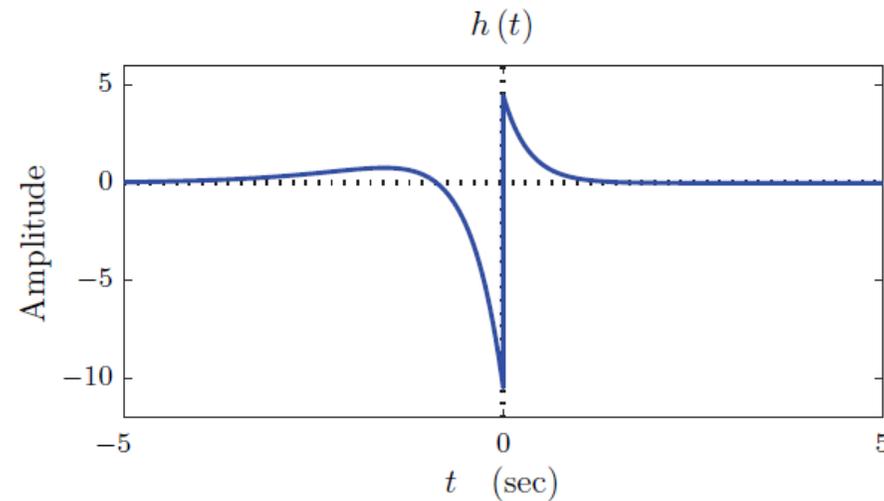
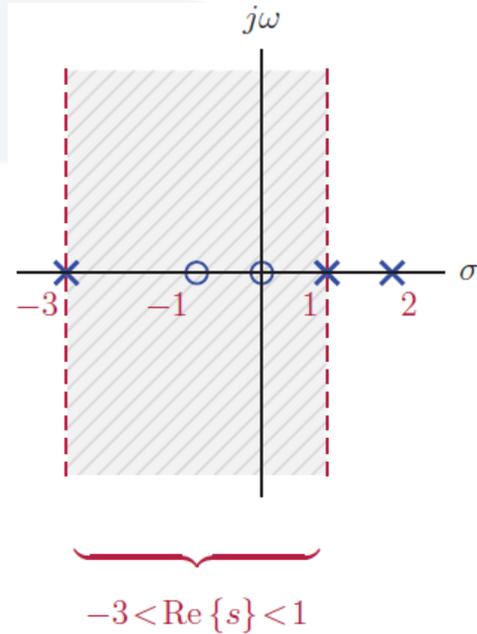
- For a **noncausal** system the ROC for the TF, if it exists, is the region in the form $\sigma_1 < \text{Re} \{s\} < \sigma_2$. For **stability** we need $\sigma_1 < 0$ and $\sigma_2 > 0$. The poles of the TF may be either:
 - On or to the left of the vertical line $\sigma = \sigma_1$
 - On or to the right of the vertical line $\sigma = \sigma_2$

- Example 17:** Impulse response of a stable system $H(s) = \frac{15s(s+1)}{(s+3)(s-1)(s-2)}$

Determine the ROC of the TF. Afterwards find the impulse response.

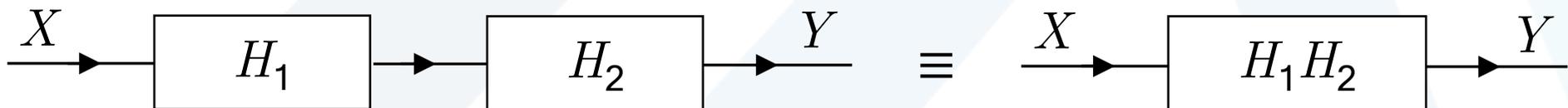
The 3 poles are at $s = -3, 1, 2$. Since the system is known to be stable, its ROC must include the $j\omega$ axis. The only possible choice is $-3 < \text{Re} \{s\} < 1$.

$$H(s) = \frac{4.5}{s+3} - \frac{7.5}{s-1} + \frac{18}{s-2} \quad \Rightarrow \quad h(t) = 4.5e^{-3t}u(t) + 7.5e^tu(-t) - 18e^{2t}u(-t)$$

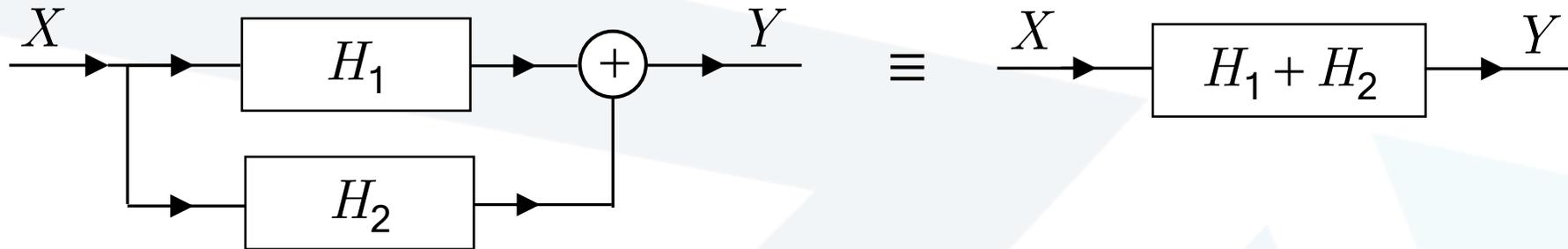


Interconnection of LTI Systems

- The **series** interconnection of the LTI systems with TFs H_1 and H_2 .

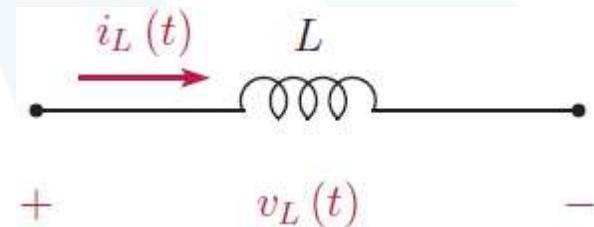
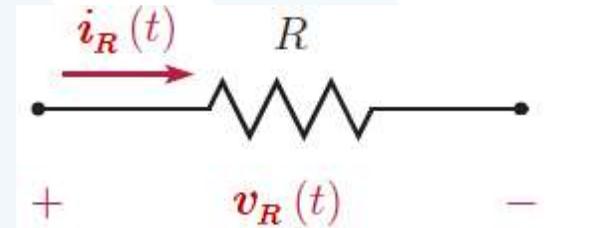


- The **parallel** interconnection of the LTI systems with TFs H_1 and H_2 .



Application: Circuit Analysis: Electronic Circuits

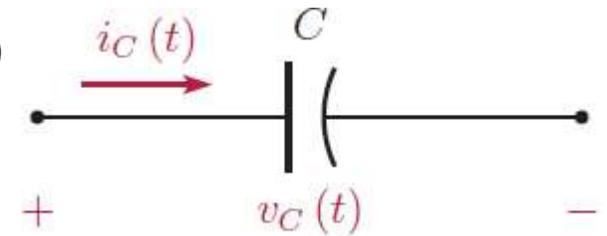
- A resistor $v_R(t) = Ri_R(t)$ or $i_R(t) = \frac{1}{R}v_R(t)$
 $V_R(s) = RI_R(s)$ or $I_R(s) = \frac{1}{R}V_R(s)$
- An inductor $v_L(t) = L \frac{d}{dt}i_L(t)$ or $i_L(t) = \frac{1}{L} \int_{-\infty}^t v_L(\tau) d\tau$
 $V_L(s) = sLI_L(s)$ or $I_L(s) = \frac{1}{sL}V_L(s)$





- A capacitor $v_C(t) = \frac{1}{C} \int_{-\infty}^t i_C(\tau) d\tau$ or $i_C(t) = C \frac{d}{dt} v_C(t)$

$$V_C(s) = \frac{1}{sC} I_C(s) \quad \text{or} \quad I_C(s) = sC V_C(s)$$



Application: Design and Analysis of Control Systems

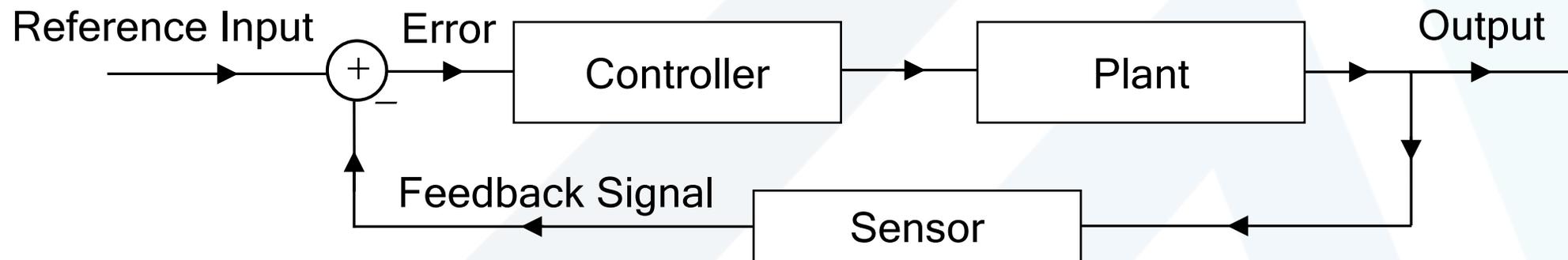
Control Systems

- The **desired** values of the quantities being controlled are collectively viewed as the **input** of the control system.
- The **actual** values of the quantities being controlled are collectively viewed as the **output** of the control system.
- A control system whose behavior is not influenced by the actual values of the quantities being controlled is called an **open loop** (or **non-feedback**) system.



- A control system whose behavior is influenced by the actual values of the quantities being controlled is called a **closed loop** (or **feedback**) system.
- An example of a simple control system would be a **thermostat** system, which controls the **temperature** in a room or building.

Feedback Control Systems



- **input**: **desired value** of the quantity to be controlled.
- **output**: **actual value** of the quantity to be controlled.



- **error**: difference between the desired and actual values.
- **plant**: system to be controlled.
- **controller**: device that monitors the error and changes the input of the plant. with the goal of forcing the error to zero.
- **sensor**: device used to measure the actual output.

A control system includes two very important components:

- **Transducer**: Since it is possible that the output signal $y(t)$ and the reference signal $x(t)$ might not be of the same type, a transducer is used to change $y(t)$ so it is compatible with the reference input $x(t)$.
- **Actuator**: A device that makes possible the execution of the control action on the plant, so that the output of the plant follows the reference input.

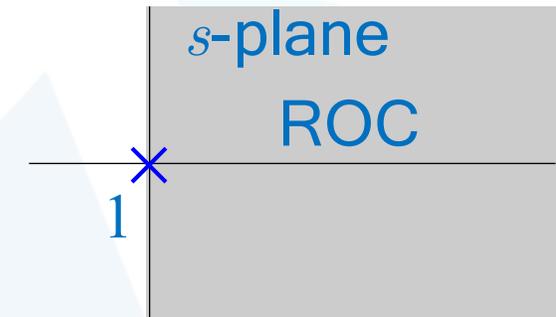
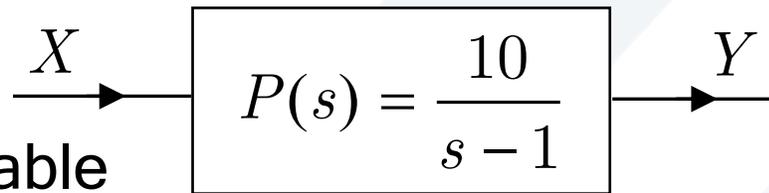


Stability Analysis of Feedback Systems

- Often, we want to ensure that a system is BIBO **stable**. BIBO stability property is more easily characterized in the **Laplace domain** than in the time domain.

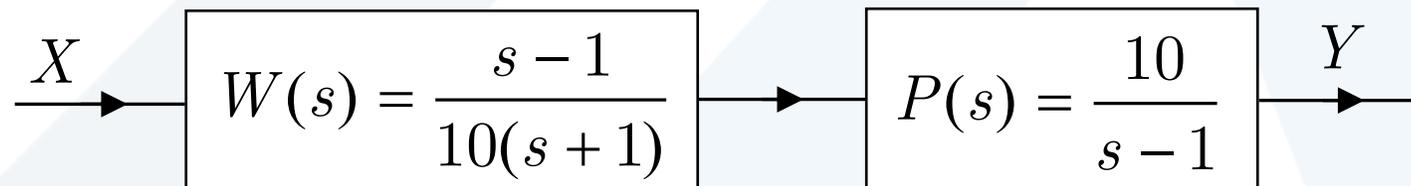
Example 18: Stabilization Example: Unstable Plant

- Causal LTI plant
- System is not BIBO stable



Example 19: Stabilization Example: Using Pole-Zero Cancellation

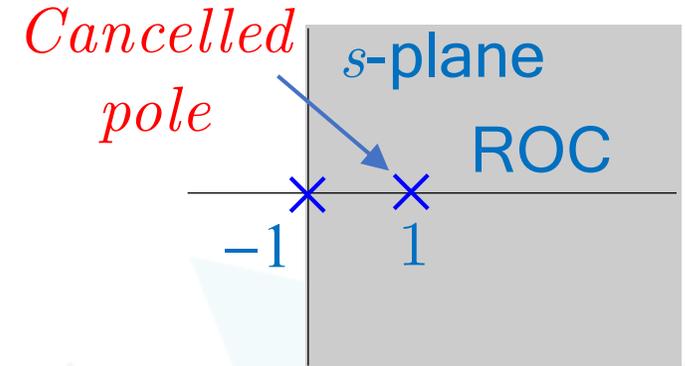
- System formed by series interconnection of plant and causal LTI compensator:





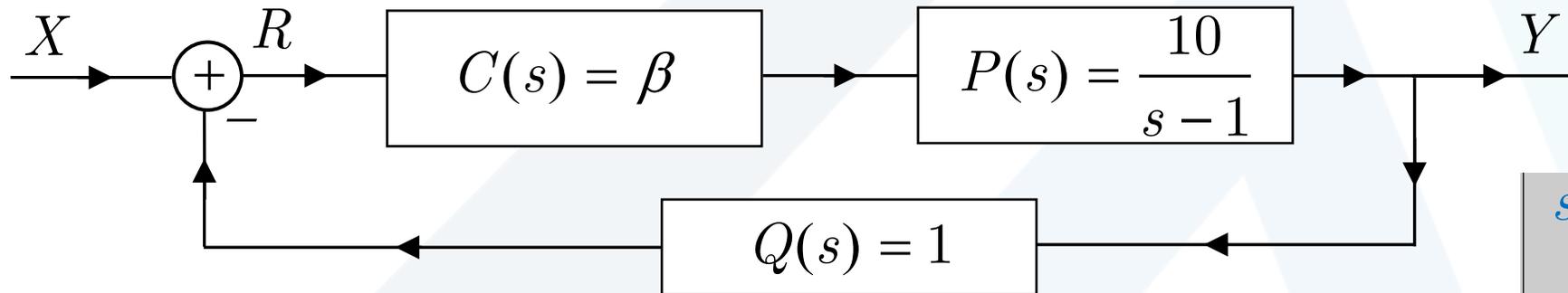
- Transfer function H of overall system (BIBO stable):

$$H(s) = W(s)P(s) = \frac{s-1}{10(s+1)} \frac{10}{s-1} = \frac{1}{s+1}$$



Example 20: Stabilization Example: Using Feedback

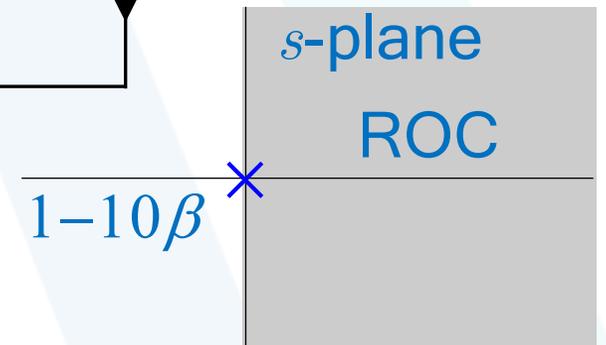
- Feedback system (with causal LTI compensator and sensor):



$$H(s) = \frac{C(s)P(s)}{1 + C(s)P(s)Q(s)} = \frac{10\beta}{s - (1 - 10\beta)}$$

BIBO stable iff

$$1 - 10\beta < 0$$





3. Simulation Structures for CTLTI Systems

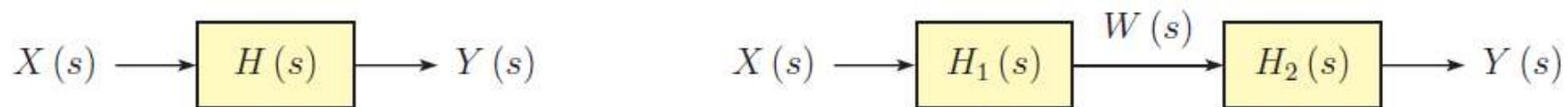
Direct-form implementation

- Consider a third-order CTLTI system described by a TF $H(s)$:

$$H(s) = \frac{Y(s)}{X(s)} = \frac{b_2s^2 + b_1s + b_0}{s^3 + a_2s^2 + a_1s + a_0}$$

Let us use an intermediate function $W(s)$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{W(s)}{X(s)} \frac{Y(s)}{W(s)} = \frac{b_2s^{-1} + b_1s^{-2} + b_0s^{-3}}{1 + a_2s^{-1} + a_1s^{-2} + a_0s^{-3}}$$

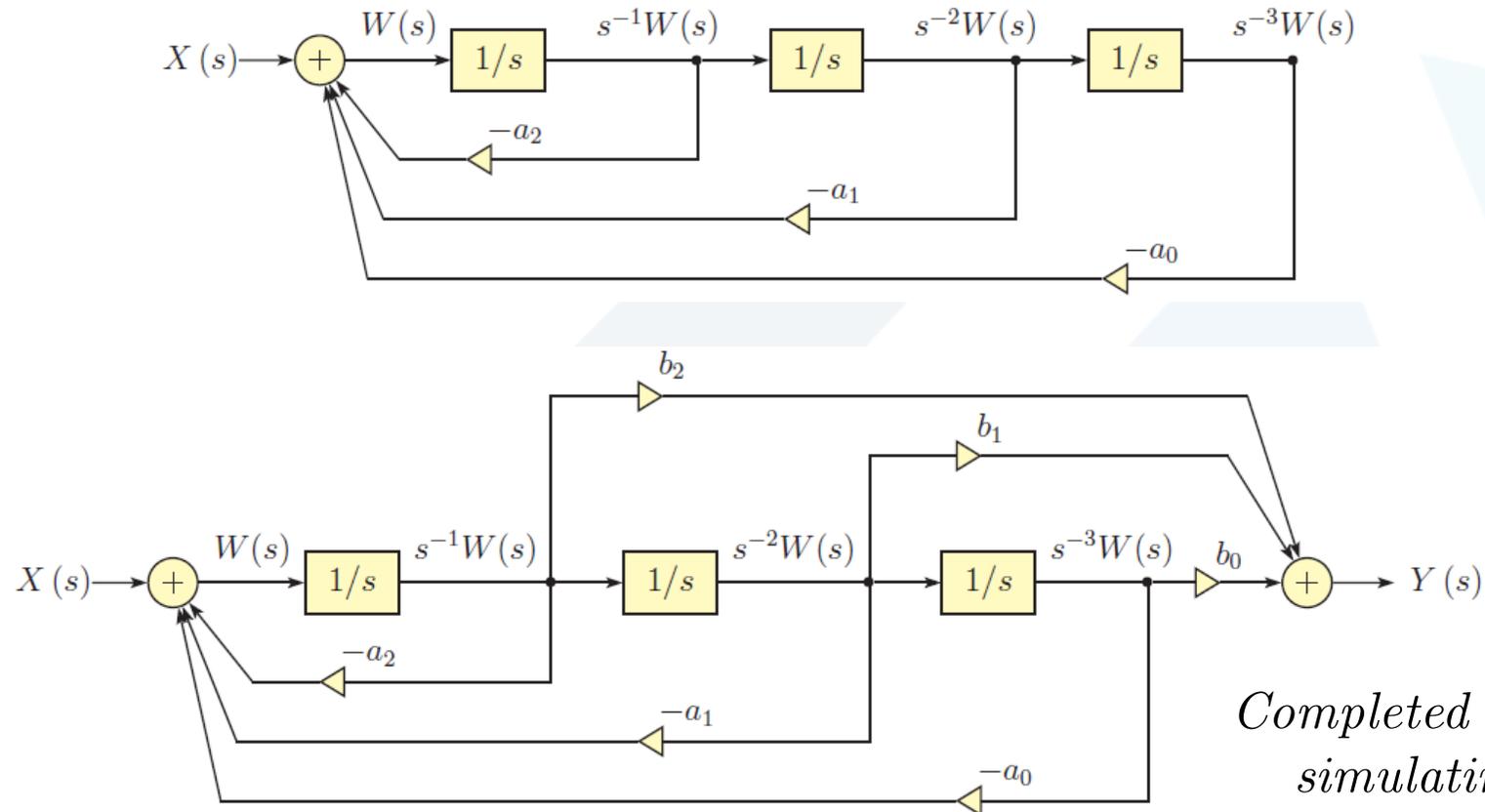


$$H_1(s) = \frac{W(s)}{X(s)} = \frac{1}{1 + a_2s^{-1} + a_1s^{-2} + a_0s^{-3}}, \quad H_2(s) = \frac{Y(s)}{W(s)} = b_2s^{-1} + b_1s^{-2} + b_0s^{-3}$$

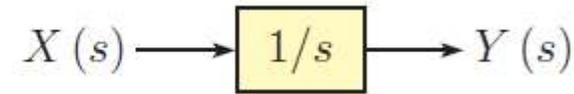
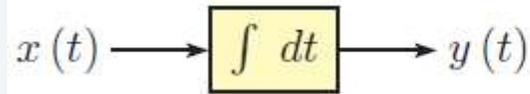


$$W(s) = X(s) - a_2s^{-1}W(s) - a_1s^{-2}W(s) - a_0s^{-3}W(s)$$

$$Y(s) = b_2s^{-1}W(s) + b_1s^{-2}W(s) + b_0s^{-3}W(s)$$

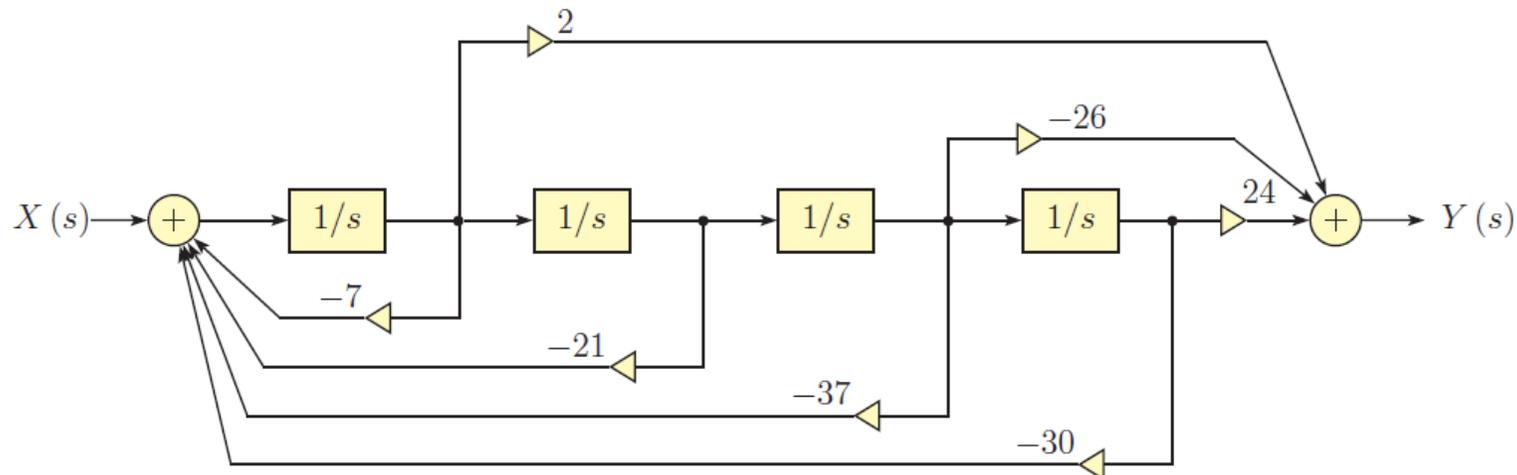


Completed block diagram for simulating the TF $H(s)$



- Example 21:** Obtaining a block diagram (BD) from transfer function
 A CTLTI system is described through the transfer function:

$$H(s) = \frac{Y(s)}{X(s)} = \frac{2s^3 - 26s + 24}{s^4 + 7s^3 + 21s^2 + 37s + 30}$$

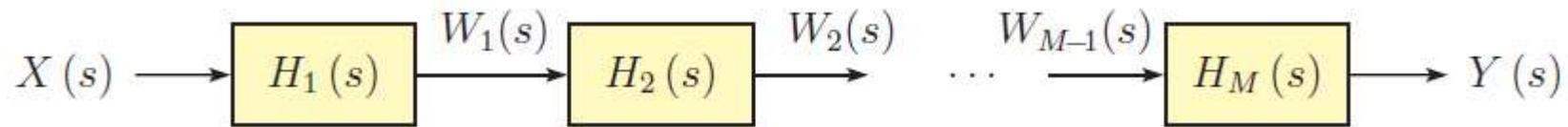




Cascade and parallel forms

Cascade form

$$H(s) = H_1(s)H_2(s)\cdots H_M(s) = \frac{W_1(s)}{X(s)} \frac{W_2(s)}{W_1(s)} \cdots \frac{Y(s)}{W_{M-1}(s)}$$



- **Example 22:** Obtaining a block diagram from transfer function

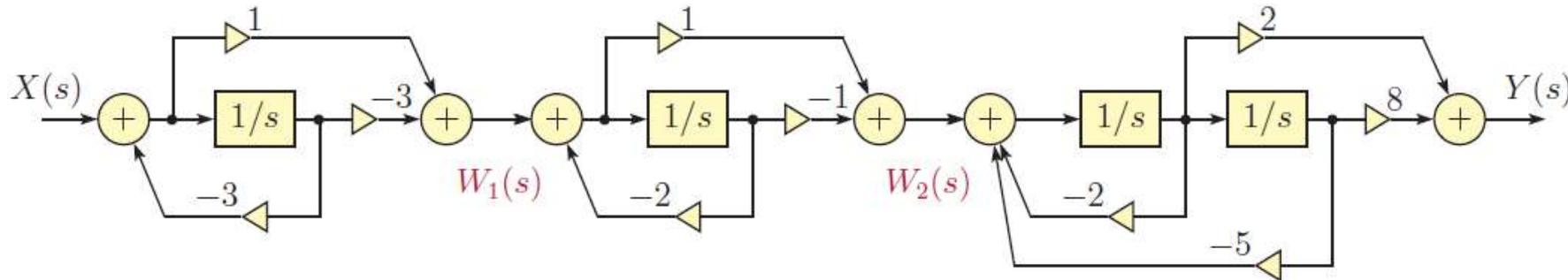
Develop a cascade form block diagram for simulating the system used in example 21.

$$H(s) = \frac{Y(s)}{X(s)} = \frac{2(s+4)(s-3)(s-1)}{(s+1-j2)(s+1+j2)(s+3)(s+2)}$$

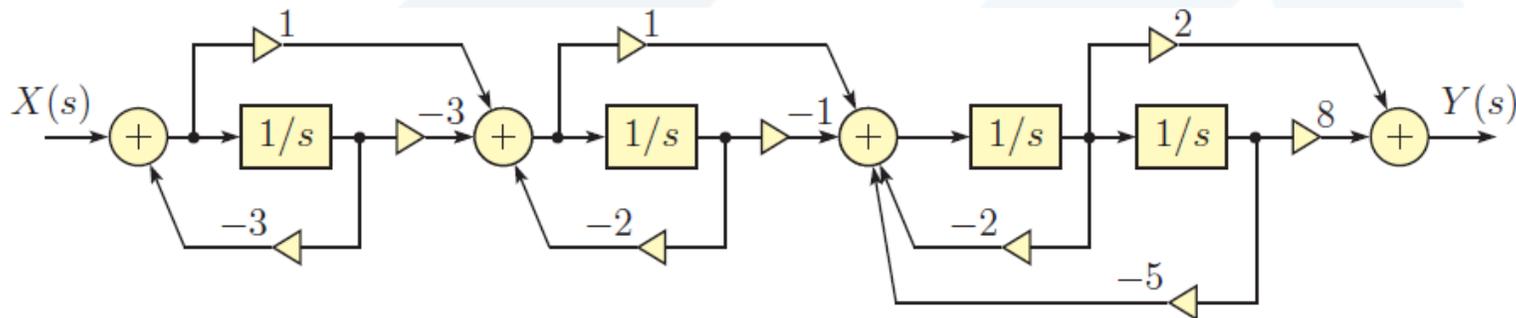


$$H(s) = H_1(s)H_2(s)H_3(s)$$

$$H_1(s) = \frac{2(s+4)}{(s+1-j2)(s+1+j2)} = \frac{2s+8}{s^2+2s+5}, \quad H_2(s) = \frac{s-3}{s+3}, \quad H_3(s) = \frac{s-1}{s+2}$$

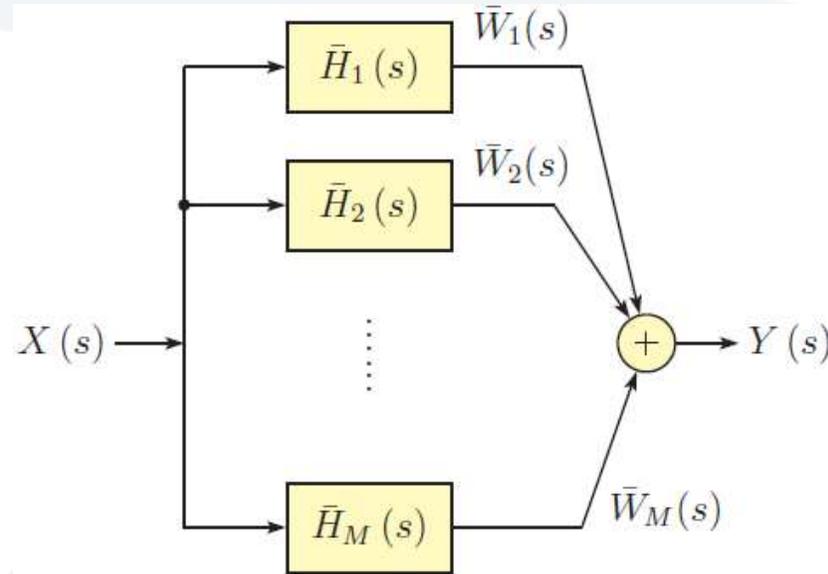


Further simplified cascade form block diagram





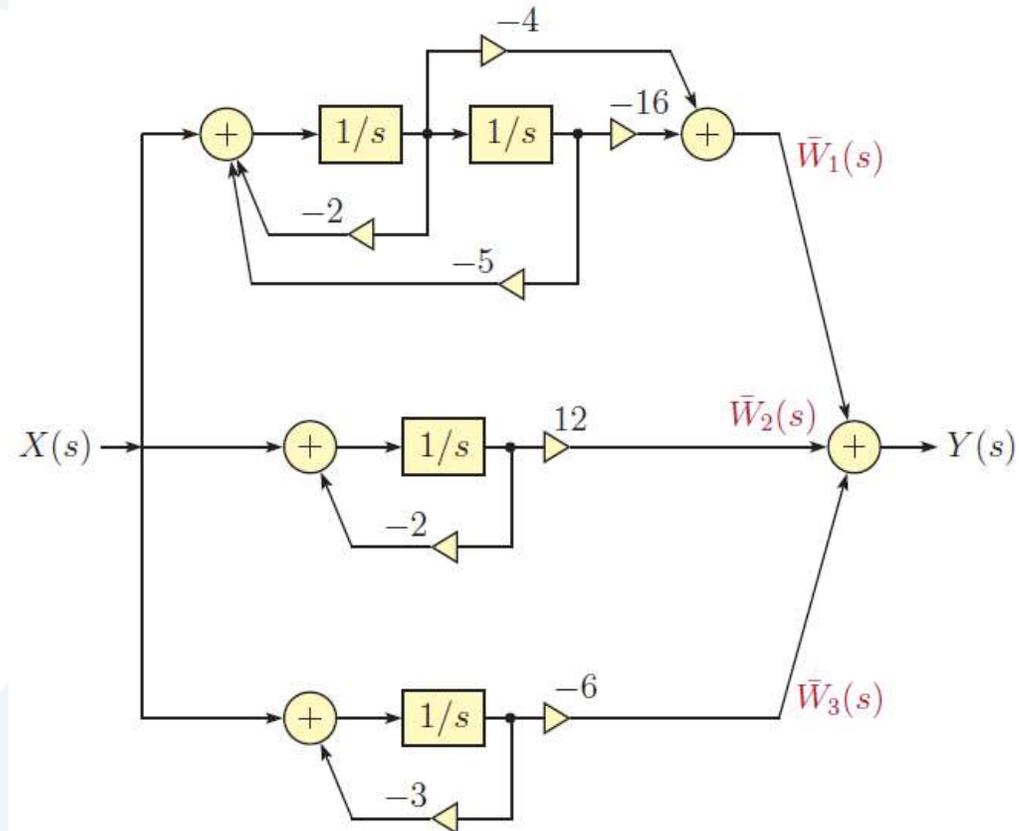
Parallel form
$$H(s) = \bar{H}_1(s) + \bar{H}_2(s) + \dots + \bar{H}_M(s) = \frac{\bar{W}_1(s)}{X(s)} + \frac{\bar{W}_2(s)}{X(s)} + \dots + \frac{\bar{W}_M(s)}{X(s)}$$



- **Example 23:** Obtaining a block diagram from TF
 Develop a parallel form BD for simulating the system used in example 21.



$$H(s) = \frac{2s + 8}{s^2 + 2s + 5} + \frac{12}{s + 2} + \frac{-6}{s + 3}$$





4. Unilateral Laplace Transform

The **unilateral Laplace transform** of the function x is defined as:

$$\mathcal{L}_u\{x(t)\} = X(s) = \int_{0^-}^{\infty} x(t)e^{-st} dt$$

- The unilateral LT is related to the bilateral Laplace transform as follows:

$$\mathcal{L}_u\{x(t)\} = \int_{0^-}^{\infty} x(t)e^{-st} dt = \int_{-\infty}^{\infty} x(t)u(t)e^{-st} dt = \mathcal{L}\{x(t)u(t)\}$$

- With the unilateral LT, the same inverse transform equation is used as in the bilateral case.
- The unilateral LT is **only invertible for causal functions**.
- For a noncausal function x , we can only recover $x(t)$ for $t \geq 0$.



Unilateral Versus Bilateral Laplace Transform

In the unilateral case:

- The time-domain convolution property has the additional requirement that the functions being convolved must be **causal**.
- The time/Laplace-domain scaling property has the additional constraint that the scaling factor must be **positive**.
- The time-domain differentiation property has an **extra term** in the expression of $\mathcal{L}_u(dx(t)/dt)$, namely $-x(0^-)$.
- The time-domain integration property has a **different lower limit** in the time-domain integral (0^- instead of $-\infty$);
- The time-domain shifting property **does not hold** (except in special cases).



Properties of the Unilateral Laplace Transform

Property	$x(t)$	$X(s)$	ROC
Linearity	$ax_1(t) + bx_2(t)$	$aX_1(s) + bX_2(s)$	$\supset (R_1 \cap R_2)$
Multiply by t	$tx(t)$	$-dX(s)/ds$	R
Multiply by $e^{-\alpha t}$	$x(t)e^{-\alpha t}$	$X(s + \alpha)$	Shift R by $-\alpha$
Scaling in t	$x(at), a > 0$	$\frac{1}{a} X\left(\frac{s}{a}\right)$	aR
Differentiate in t	$dx(t)/dt$	$sX(s) - x(0^-)$	$\supset R$
Integrate in t	$\int_{0^-}^t x(\tau) d\tau$	$\frac{X(s)}{s}$	$\supset (R \cap (\text{Re}(s) > 0))$
Convolve in t	$x_1 * x_2(t)$	$X_1(s) X_2(s)$	$\supset (R_1 \cap R_2)$



Unilateral Laplace Transform Pairs

Pair	$x(t); t \geq 0$	$X(s)$	Pair	$x(t); t \geq 0$	$X(s)$
1	$\delta(t)$	1	6	$\cos \omega_0 t$	$\frac{s}{s^2 + \omega_0^2}$
2	1	$\frac{1}{s}$	7	$\sin \omega_0 t$	$\frac{\omega_0}{s^2 + \omega_0^2}$
3	t^n	$\frac{n!}{s^{n+1}}$	8	$e^{-at} \cos \omega_0 t$	$\frac{s + a}{(s + a)^2 + \omega_0^2}$
4	e^{-at}	$\frac{1}{s + a}$	9	$e^{-at} \sin \omega_0 t$	$\frac{\omega_0}{(s + a)^2 + \omega_0^2}$
5	$t^n e^{-at}$	$\frac{n!}{(s + a)^{n+1}}$			



■ **Example 24:** Response of a second-order system (RLC circuit)

A voltage $x(t) = 10e^{-3t}u(t)$ is applied at the input of the RLC circuit. Find the output voltage $v_C(t) = y(t)$ for $t \geq 0$ if the initial inductor current is $i_L(0^-) = 0$, and the initial capacitor voltage $v_C(0^-) = 5$ V. Use $R = 3 \Omega$, $L = 1$ H and $C = 1/2$ F.

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = 2x(t) \quad y(0^-) = 5, \quad \frac{dy}{dt}(0^-) = \frac{i(0^-)}{C} = 0$$

$$s^2Y(s) - 5s - 0 + 3(sY(s) - 5) + 2Y(s) = 2X(s) = \frac{20}{s + 3}$$

$$(s^2 + 3s + 2)Y(s) = 5s + 15 + \frac{20}{s + 3}$$

$$Y(s) = \frac{5s^2 + 30s + 65}{(s + 3)(s^2 + 3s + 2)}$$



$$y(t) = \underbrace{20e^{-t} - 25e^{-2t} + 10e^{-3t}}_{y_t(t)}, t \geq 0$$

$$y(t) = \underbrace{20e^{-t} - 25e^{-2t}}_{y_n(t)} + \underbrace{10e^{-3t}}_{y_\phi(t)}, t \geq 0$$

