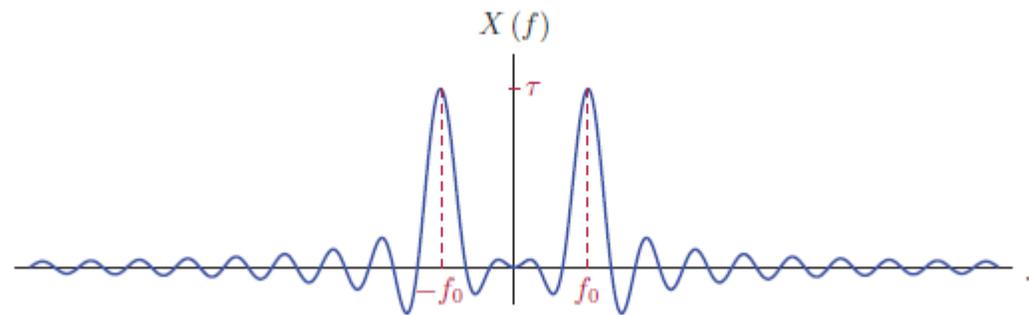




# CECC507: Signals and Systems

## Lecture Notes 7: Sampling and Reconstruction



Ramez Koudsieh, Ph.D.  
Faculty of Engineering  
Department of Mechatronics  
Manara University



## Chapter 6

# Sampling and Reconstruction

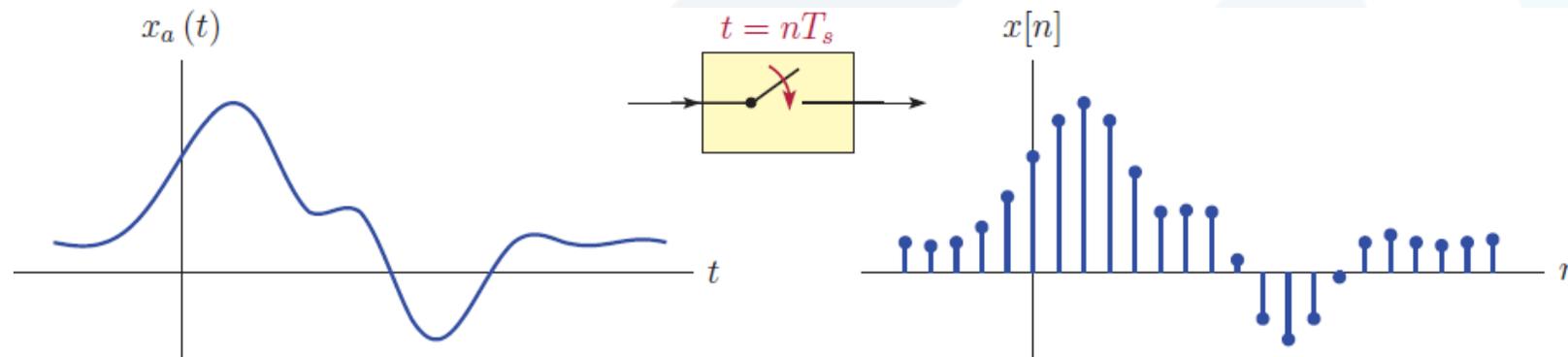
1. Sampling of a Continuous-Time Signal
2. Reconstruction of a Signal from Its Sampled Version
3. Resampling Discrete-Time Signals



## Introduction

$$x[n] = x_a(t)|_{t=nT_s} = x_a(nT_s)$$

where  $T_s$  is the **sampling interval**, that is, the time interval between consecutive samples. It is also referred to as the **sampling period**. The reciprocal of the sampling interval is called the **sampling rate** or the **sampling frequency**:  $f_s = 1/T_s$



- Does the measurements (samples) taken at intervals of  $T_s$  will be sufficient to reconstruct the continuous-time signal  $x_a(t)$ ?

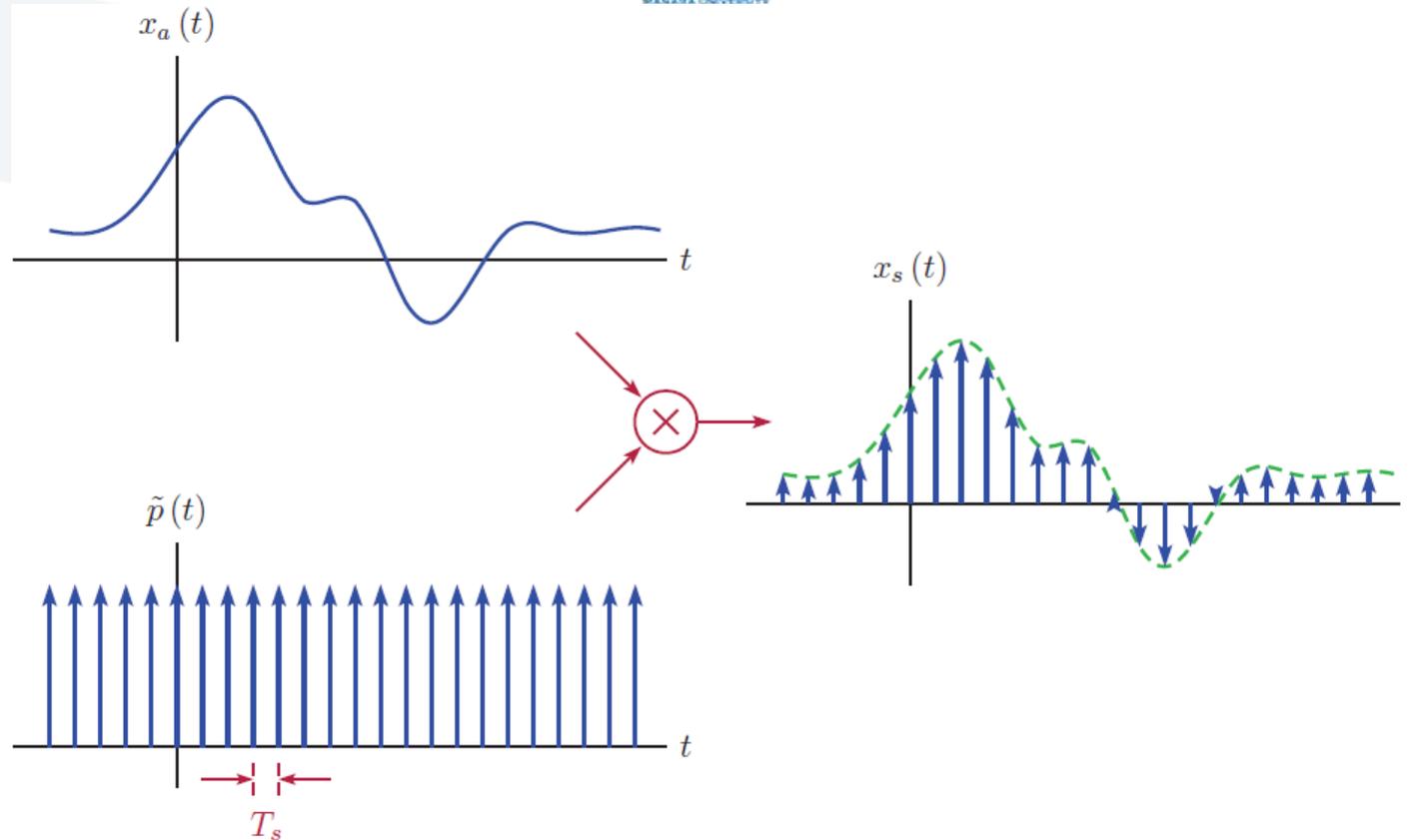


## 1. Sampling of a Continuous-Time Signal

$$\tilde{p}(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

$$x_s(t) = x_a(t)\tilde{p}(t) = x_a(t) \sum_{n=-\infty}^{\infty} \delta(t - nT_s) = \sum_{n=-\infty}^{\infty} x_a(nT_s)\delta(t - nT_s)$$

- We will refer to the signal  $x_s(t)$  as the **impulse-sampled** version of  $x_a(t)$ .
- The impulse-sampled signal  $x_s(t)$  is still a continuous time signal.
- How dense must the impulse train  $\tilde{p}(t)$  be so that the impulse-sampled signal  $x_s(t)$  is an accurate and complete representation of the original signal  $x_a(t)$ ?

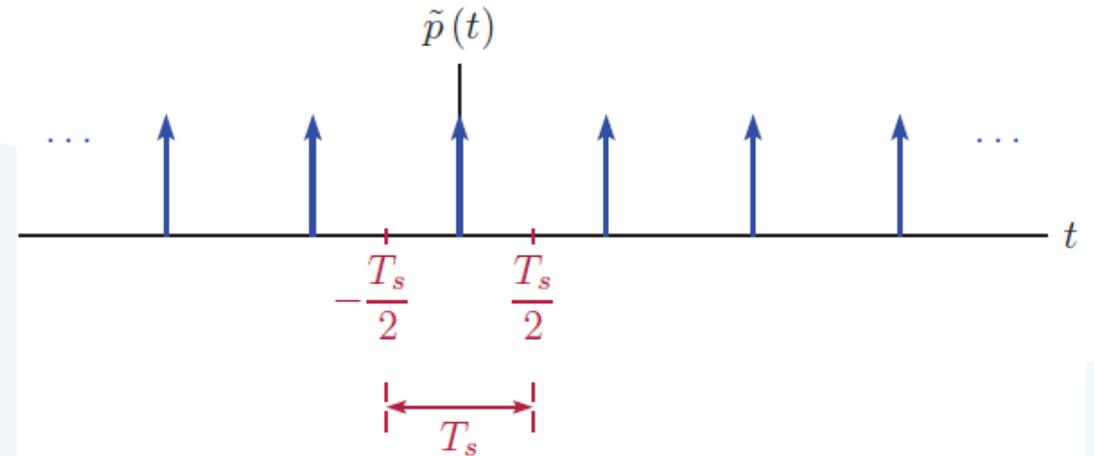


- how the frequency spectrum of the impulse-sampled signal  $x_s(t)$  relates to the spectrum of the original signal  $x_a(t)$ ?

$$\tilde{p}(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_s t}$$

$$c_k = \frac{1}{T_s} \int_{-T_s/2}^{T_s/2} \tilde{p}(t) e^{-jk\omega_s t} dt$$

$$= \frac{1}{T_s} \int_{-T_s/2}^{T_s/2} \delta(t) e^{-jk\omega_s t} dt = \frac{1}{T_s}$$



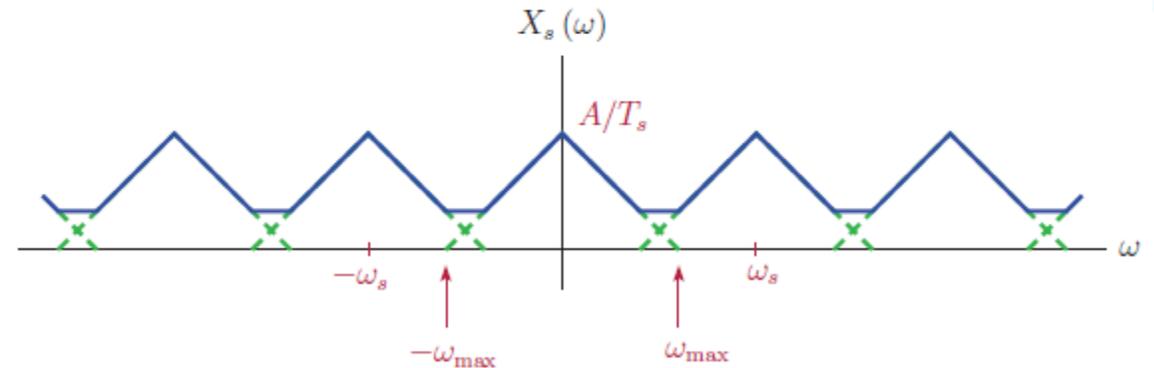
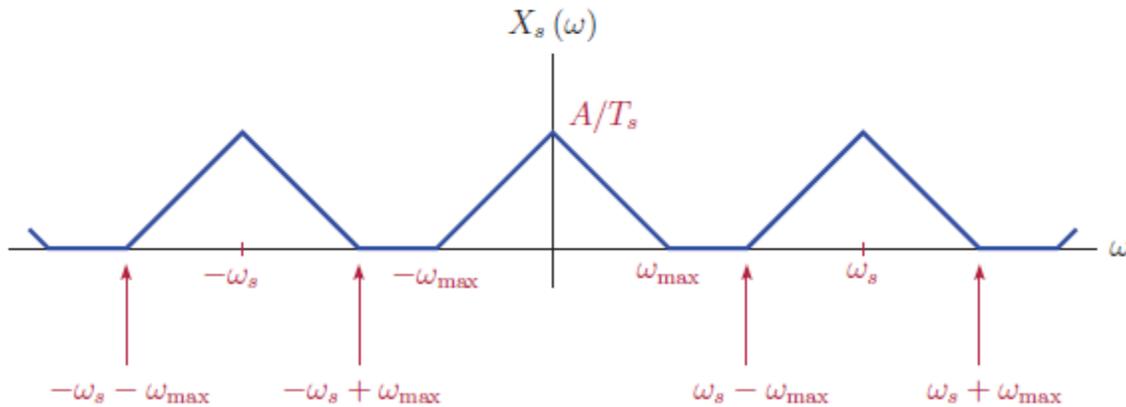
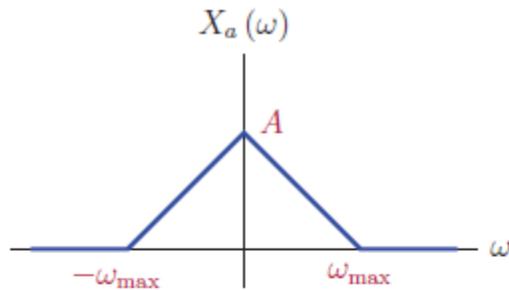
where  $\omega_s$  is both the sampling rate in rad/s and the fundamental frequency of the impulse train. It is computed as  $\omega_s = 2\pi f_s = 2\pi/T_s$ .

$$\tilde{p}(t) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} e^{jk\omega_s t} \Rightarrow x_s(t) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} x_a(t) e^{jk\omega_s t}$$

$$F\{x_s(t)\} = F\left\{\frac{1}{T_s} \sum_{k=-\infty}^{\infty} x_a(t) e^{jk\omega_s t}\right\} = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} F\{x_a(t) e^{jk\omega_s t}\}$$



$$X_s(\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X_a(\omega - k\omega_s), \quad X_s(f) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X_a(f - kf_s)$$



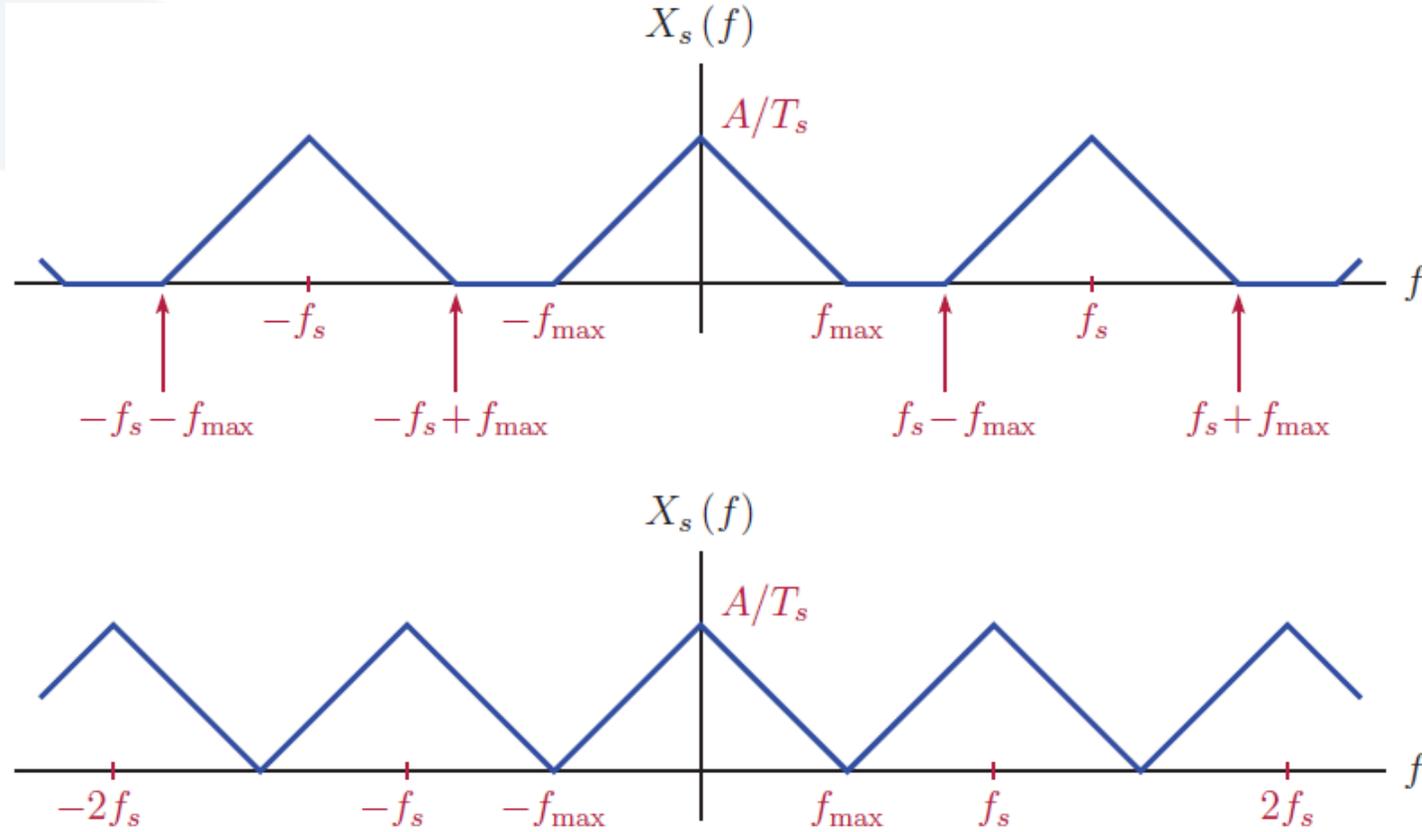


## Nyquist sampling criterion

- If the range of frequencies in the signal  $x_a(t)$  is not limited, then the periodic repetition of spectral components creates overlapped regions.
- This effect is known as **aliasing**, and it results in the shape of the spectrum  $X_s(f)$  being different than the original spectrum  $X_a(f)$ .
- Once the spectrum is aliased, the original signal is no longer recoverable from its sampled version.
- For the impulse-sampled signal to form an accurate representation of the original signal, the sampling rate  $f_s$  must be at least twice the highest frequency in the spectrum of the original signal  $f_{max}$  ( $f_s \geq 2f_{max}$ ). This is known as the **Nyquist sampling** criterion.

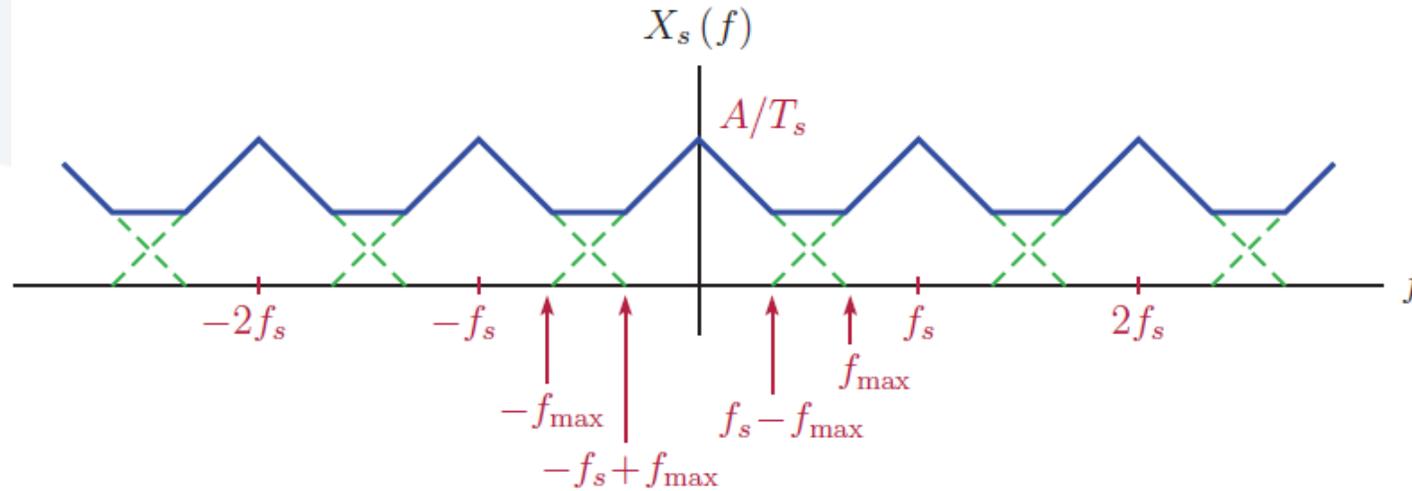


جامعة  
المنارة  
MANARA UNIVERSITY

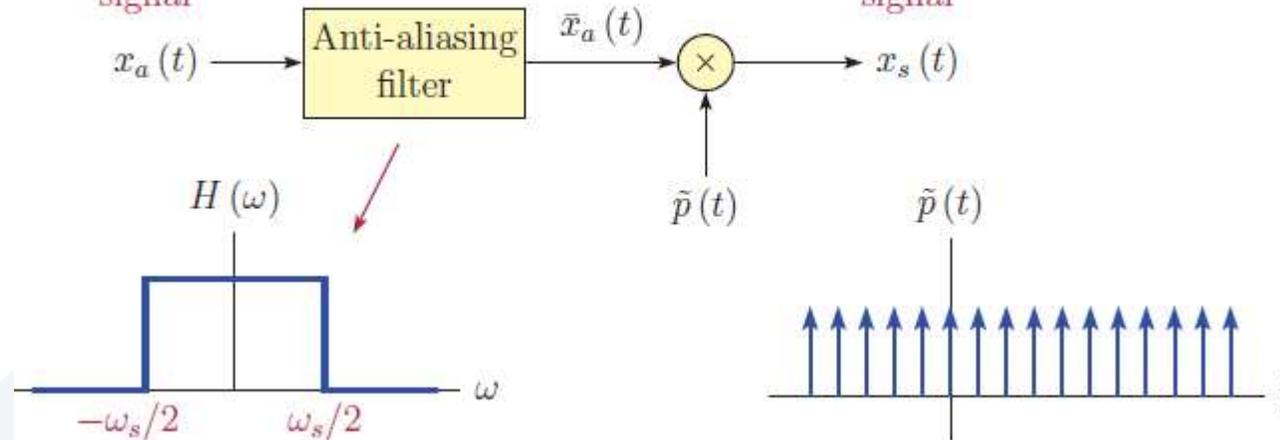




جامعة  
المنارة  
MANARA UNIVERSITY



Analog signal  $x_a(t)$       Impulse-sampled signal  $x_s(t)$





- Use of an **antialiasing** filter to ensure compliance with the requirements of Nyquist sampling criterion.

## DTFT of sampled signal

$$x_a(t) \Rightarrow X_a(\omega) = \int_{-\infty}^{\infty} x_a(t) e^{-j\omega t} dt \quad \text{CTFT}$$

$$x[n] = x_a(nT_s) \Rightarrow X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n} \quad \text{DTFT}$$

$$x_s(t) = \sum_{n=-\infty}^{\infty} x_a(nT_s) \delta(t - nT_s)$$

$$X_s(\omega) = \int_{-\infty}^{\infty} x_s(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} \left[ \sum_{n=-\infty}^{\infty} x_a(nT_s) \delta(t - nT_s) \right] e^{-j\omega t} dt$$



$$X_s(\omega) = \sum_{n=-\infty}^{\infty} x_a(nT_s) \left[ \int_{-\infty}^{\infty} \delta(t - nT_s) e^{-j\omega t} dt \right] = \sum_{n=-\infty}^{\infty} x_a(nT_s) e^{-j\omega nT_s}$$

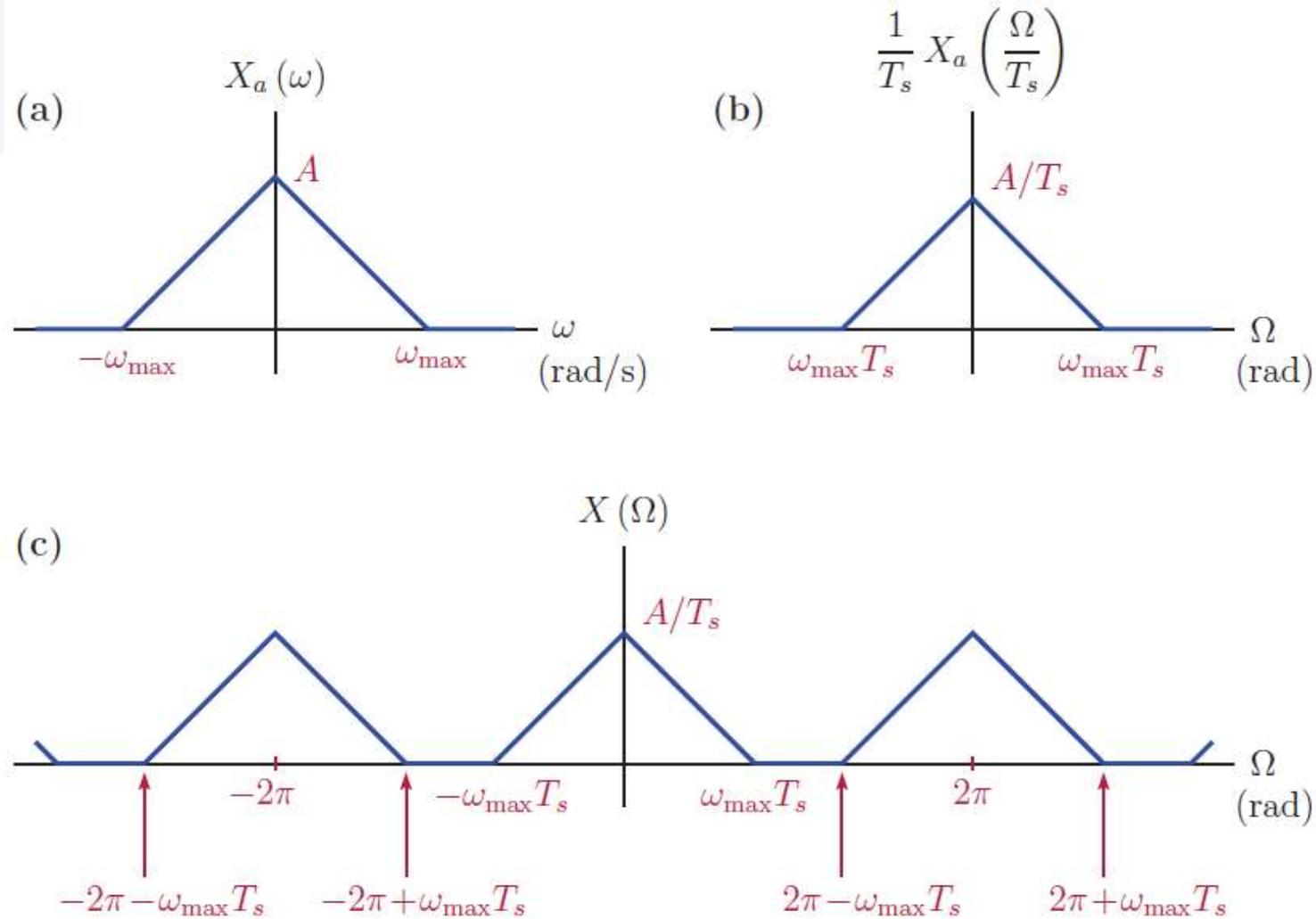
$$\Omega = \omega T_s \Rightarrow X(\Omega) = X_s\left(\frac{\Omega}{T_s}\right)$$

- The relationship between the spectrum of the original continuous-time signal and the DTFT of the discrete-time signal obtained by sampling it:

$$X(\Omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X_a\left(\frac{\Omega - 2\pi k}{T_s}\right)$$

- In order to avoid overlaps between repetitions of the segments of the spectrum, we need:

$$\omega_{\max} T_s \leq \pi \Rightarrow f_{\max} \leq \frac{f_s}{2}$$



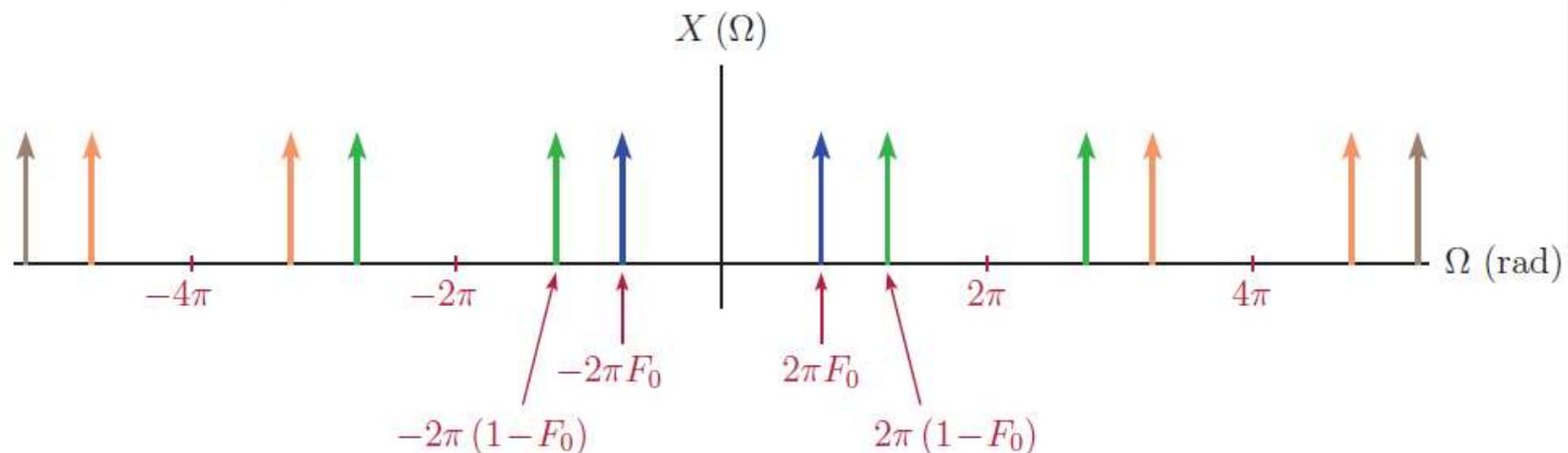


## Sampling of sinusoidal signals

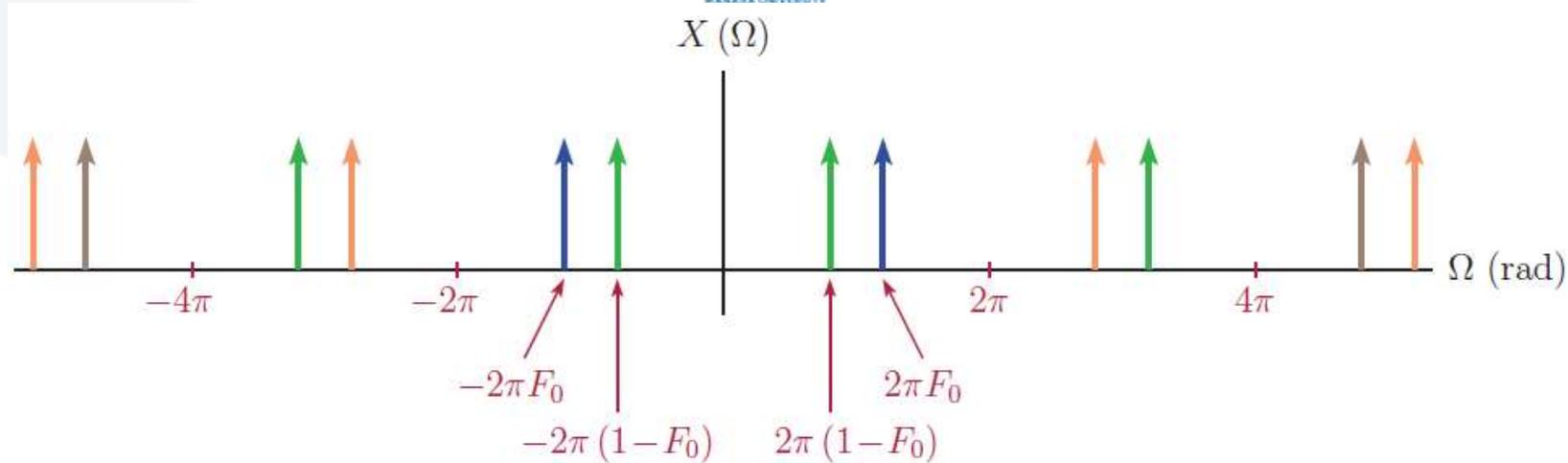
$$x_a(t) = \cos(2\pi f_0 t) \Rightarrow x[n] = x_a(nT_s) = \cos(2\pi f_0 nT_s)$$

Using the normalized frequency  $F_0 = f_0 T_s = f_0 / f_s$ ,  $x[n]$  becomes  $x[n] = x_a(nT_s) = \cos(2\pi F_0 n)$ .

$$X(\Omega) = \sum_{k=-\infty}^{\infty} [\pi\delta(\Omega + 2\pi F_0 - 2\pi k) + \pi\delta(\Omega - 2\pi F_0 - 2\pi k)]$$



*proper sampling rate,  $|F_0| < 0.5$*



*improper sampling rate,  $|F_0| > 0.5$*

- **Example 1:** Sampling a sinusoidal signal

The three sinusoidal signals are sampled using the sampling rate  $f_s = 16$  Hz and  $T_s = 1/f_s = 0.0625$ .

$$x_{1a}(t) = \cos(12\pi t), \quad x_{2a}(t) = \cos(20\pi t), \quad x_{3a}(t) = \cos(44\pi t)$$

$$x_1[n] = x_{1a}(0.0625n), \quad x_2[n] = x_{2a}(0.0625n), \quad x_3[n] = x_{3a}(0.0625n)$$



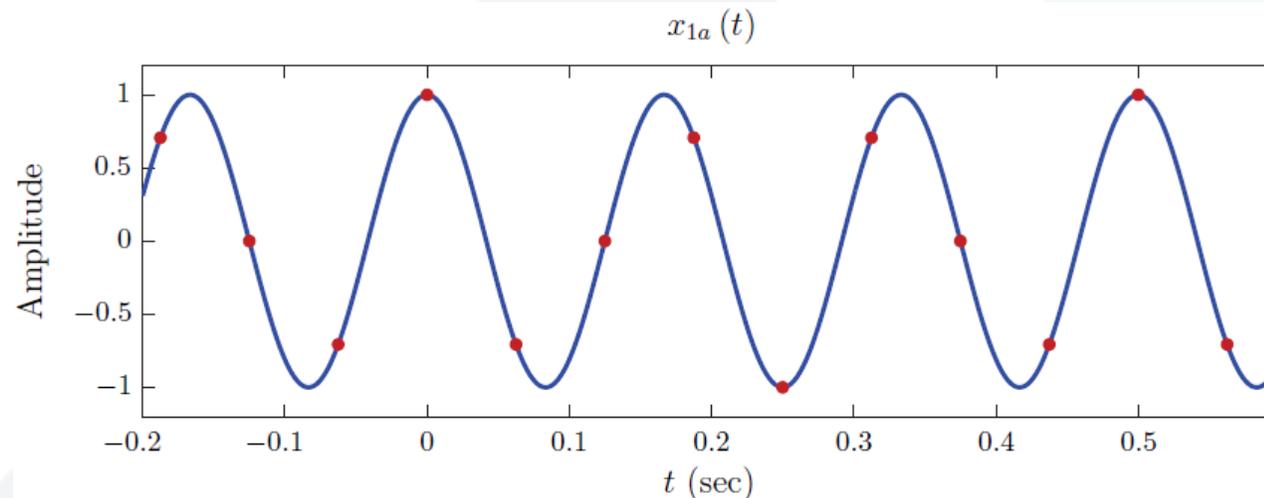
Show that the three discrete-time signals are identical, that is,  $x_1[n] = x_2[n] = x_3[n]$ , for all  $n$ .

$$x_{1a}(t) = \cos(12\pi t), \quad x_{2a}(t) = \cos(20\pi t), \quad x_{3a}(t) = \cos(44\pi t)$$

$$x_1[n] = \cos(0.75\pi n), \quad x_2[n] = \cos(1.25\pi n), \quad x_3[n] = \cos(2.75\pi n)$$

$$x_2[n] = \cos(1.25\pi n - 2\pi n) = \cos(-0.75\pi n) = \cos(0.75\pi n) = x_1[n]$$

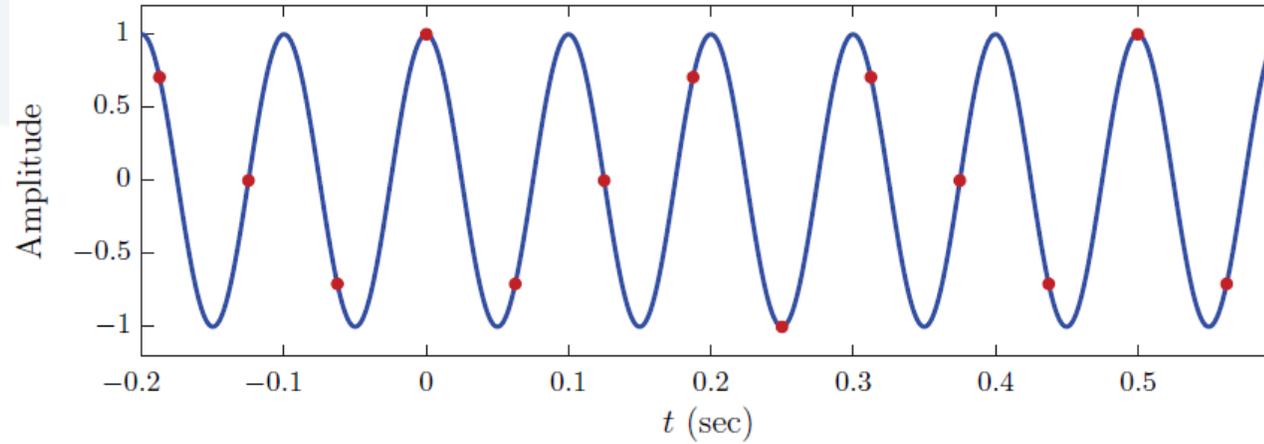
$$x_3[n] = \cos(2.75\pi n - 2\pi n) = \cos(0.75\pi n) = x_1[n]$$



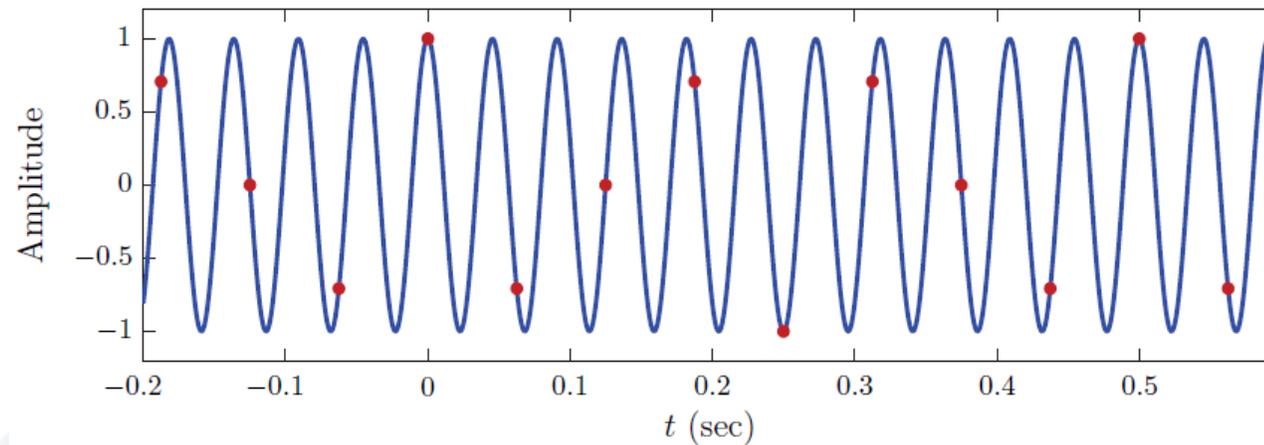


جامعة  
المنارة  
MANARA UNIVERSITY

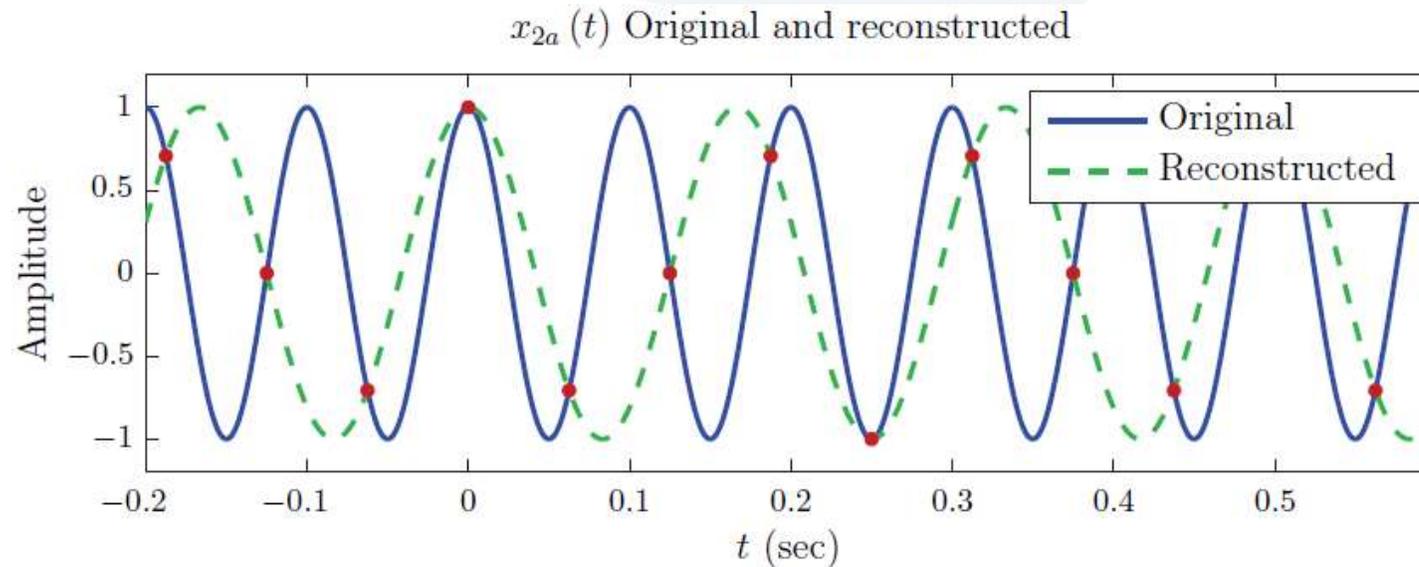
$$x_{2a}(t)$$



$$x_{3a}(t)$$



- Is the **reconstruction** of the original signal from its sampled version?
- An infinite number of different sinusoidal signals can be passed through the points shown with red dots in the figure below.



- Let us determine the actual and the normalized frequencies for the three signals:



$$f_1 = 6 \text{ Hz}, \quad F_1 = 6/16 = 0.375$$

$$f_2 = 10 \text{ Hz}, \quad F_2 = 10/16 = 0.625$$

$$f_3 = 22 \text{ Hz}, \quad F_3 = 22/16 = 1.375$$

- Of the three normalized frequencies only  $F_1$  satisfies the condition  $|F| \leq 0.5$ , and the other two violate it.
- In terms of the Nyquist sampling theorem, the signal  $x_{1a}(t)$  is sampled using a proper sampling rate, that is,  $f_s > 2f_1$ . The other two signals are sampled improperly since  $f_s < 2f_2$  and  $f_s < 2f_3$ .
- **Example 2:** Spectral relationships in sampling a sinusoidal signal  
Find the frequency spectrum for each of the 3 signals of example 1, and then use it in obtaining the DTFT spectrum of the sampled signal.



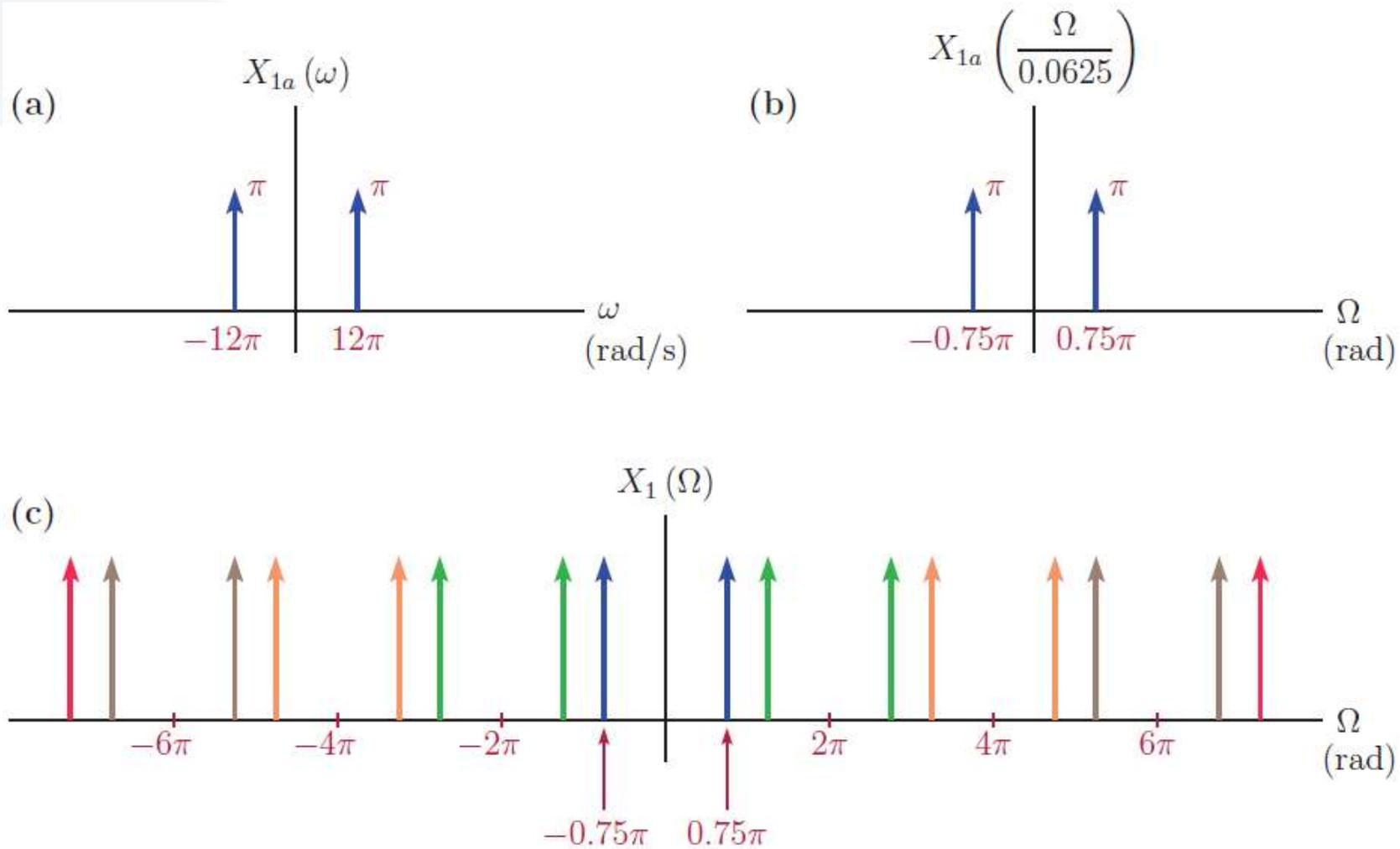
The Fourier transform of  $x_{1a}(t)$  is:

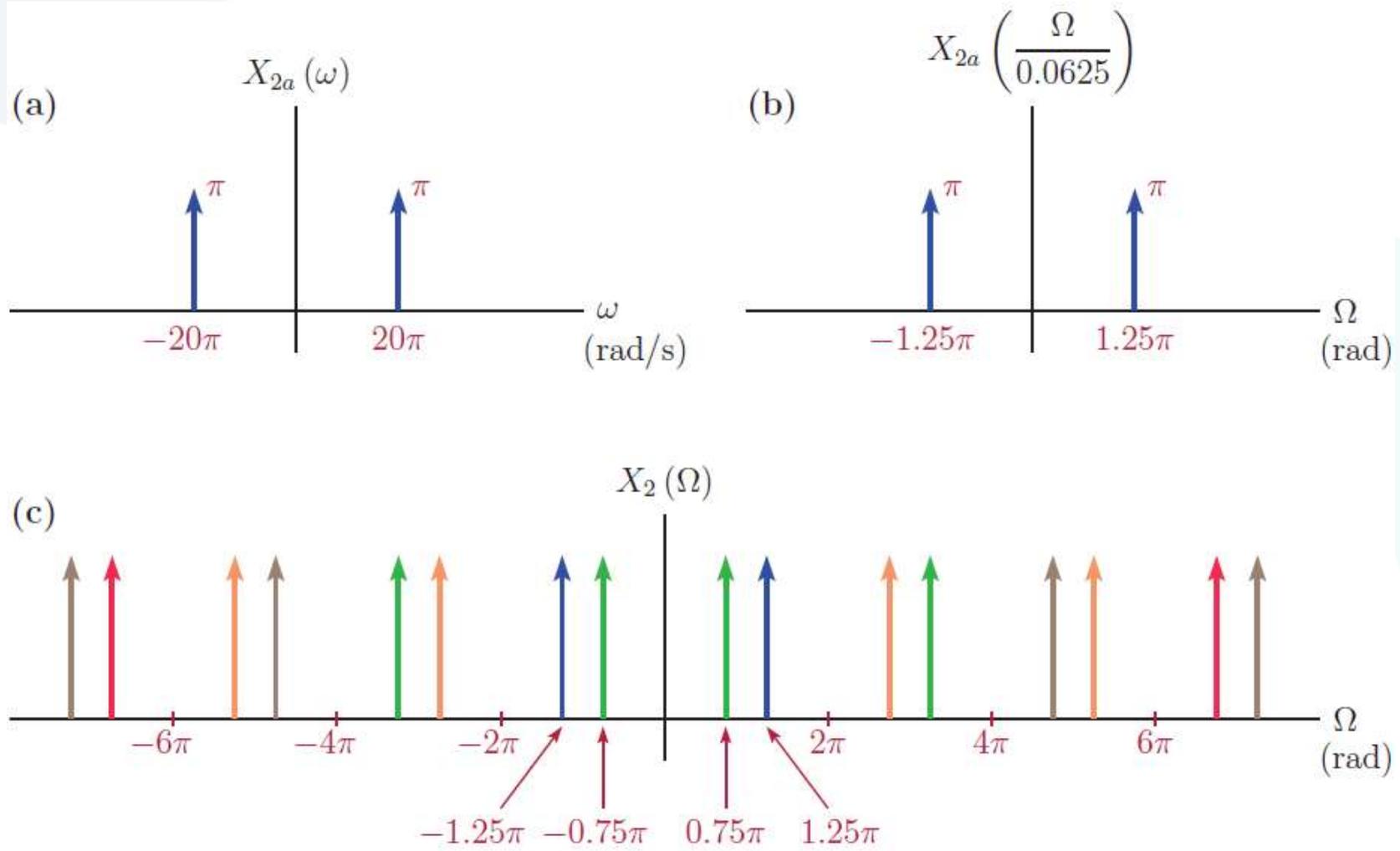
$$X_{1a}(\omega) = \pi\delta(\omega + 12\pi) + \pi\delta(\omega - 12\pi)$$

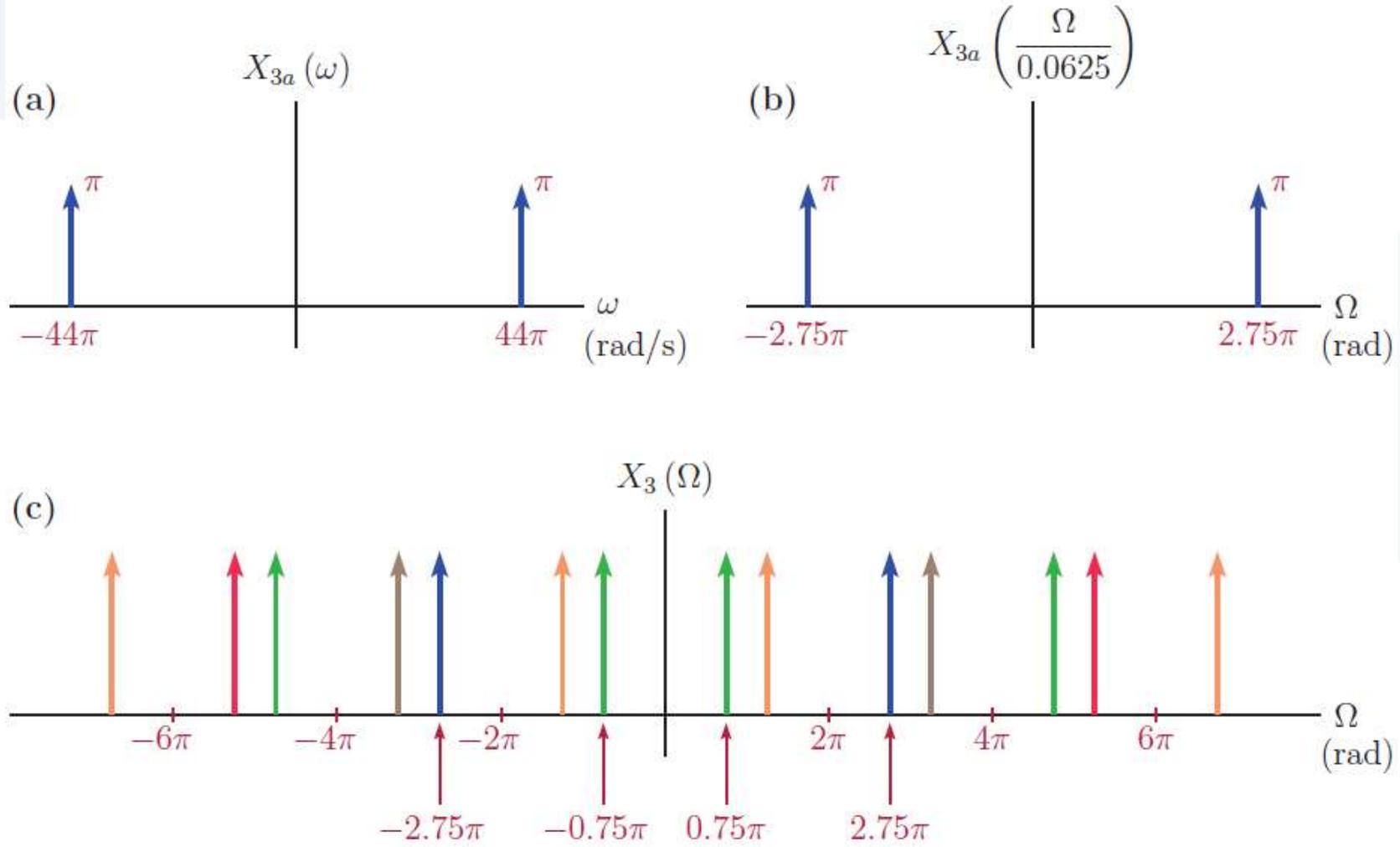
$$X_{1a}\left(\frac{\Omega}{T_s}\right) = \pi\delta\left(\frac{\Omega + 0.75\pi}{0.0625}\right) + \pi\delta\left(\frac{\Omega - 0.75\pi}{0.0625}\right)$$

$$\begin{aligned} X_1(\Omega) &= \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X_{1a}\left(\frac{\Omega - 2\pi k}{T_s}\right) \\ &= 16 \sum_{k=-\infty}^{\infty} \left[ \pi\delta\left(\frac{\Omega + 0.75 - 2\pi k}{0.0625}\right) + \pi\delta\left(\frac{\Omega - 0.75 - 2\pi k}{0.0625}\right) \right] \end{aligned}$$

The term for  $k = 0$  is shown in **blue** whereas the terms for  $k = \mp 1$ ,  $k = \mp 2$ ,  $k = \mp 3$  are shown in **green**, **orange** and **brown** respectively.





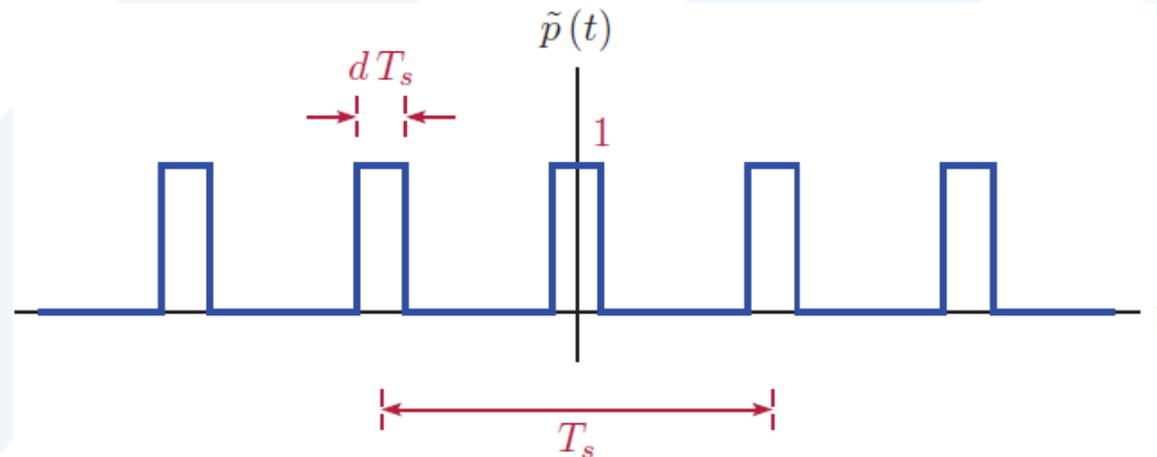


## Practical issues in sampling

- Pulses are used instead of impulses, there are two variations of the sampling operation that can be used, namely **natural sampling** and **zero-order hold sampling**.

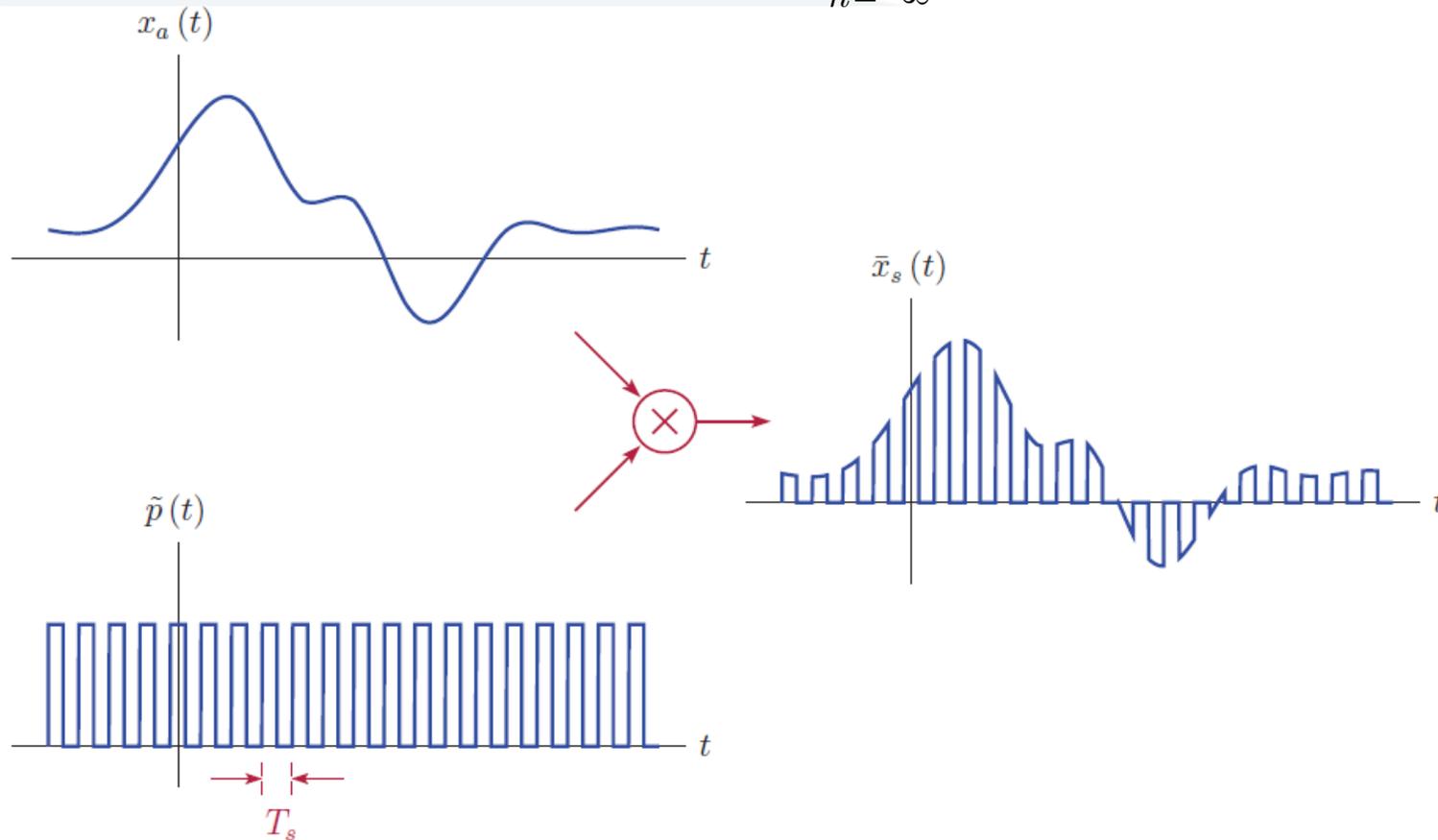
### Natural sampling

$$\tilde{p}(t) = \sum_{n=-\infty}^{\infty} \Pi\left(\frac{t - nT_s}{dT_s}\right)$$





$$\bar{x}_s(t) = x_a(t)\tilde{p}(t) = x_a(t) \sum_{n=-\infty}^{\infty} \Pi\left(\frac{t - nT_s}{dT_s}\right)$$





- The EFS coefficients for a pulse train with duty cycle  $d$  is:  $c_k = d \operatorname{sinc}(kd)$ .

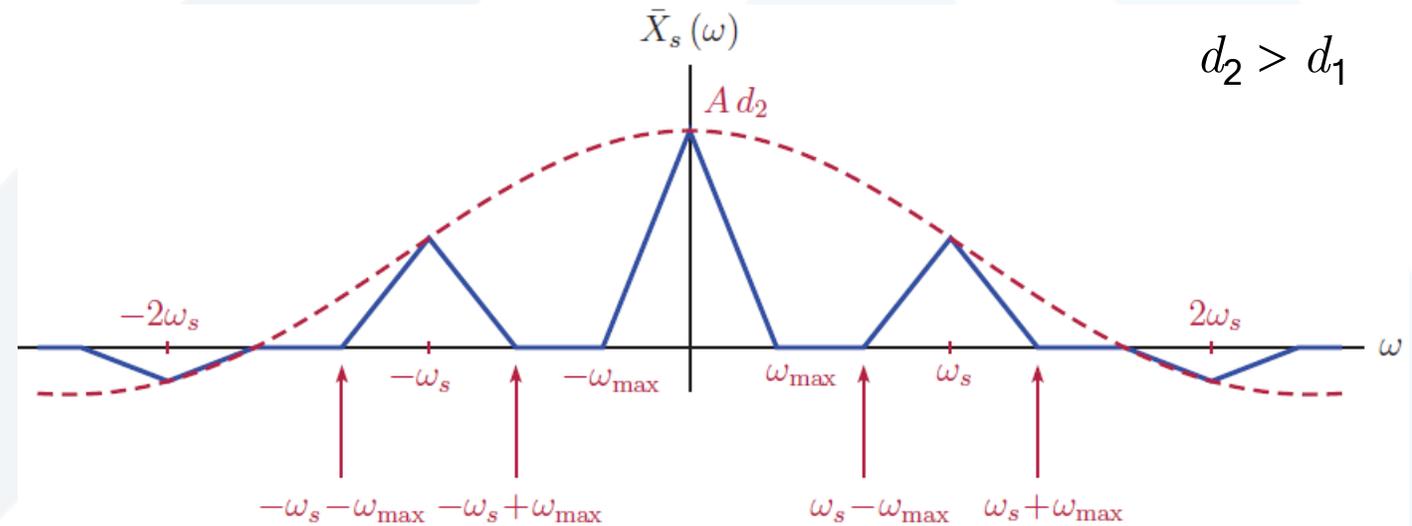
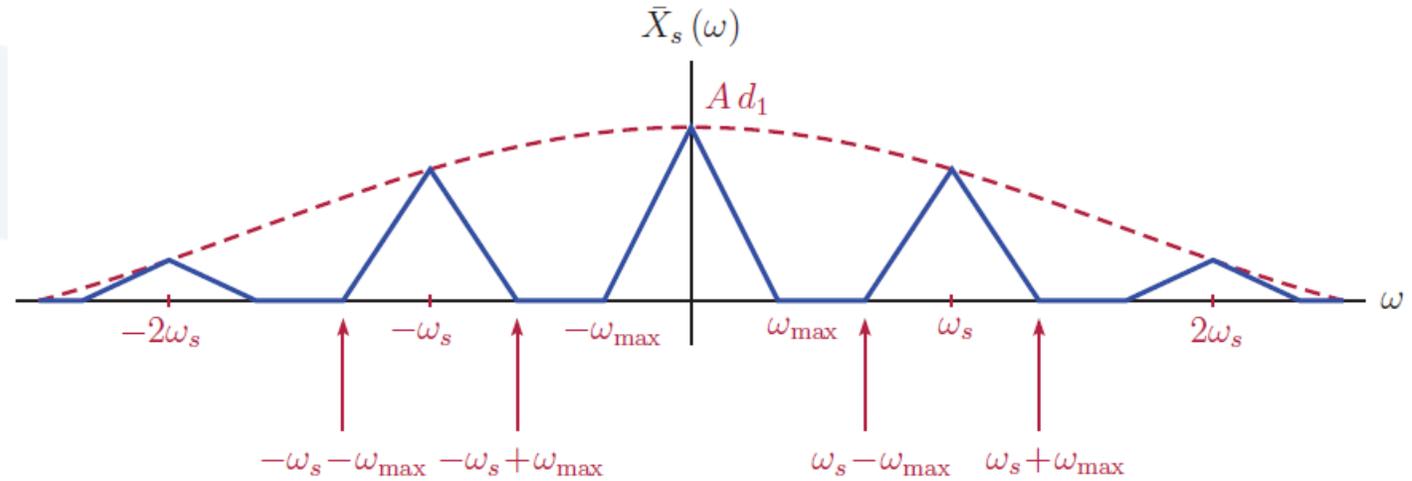
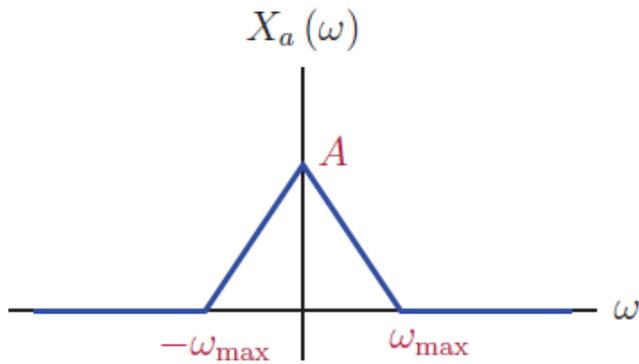
$$\tilde{p}(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_s t} = \sum_{k=-\infty}^{\infty} d \operatorname{sinc}(kd) e^{jk\omega_s t}$$

Fundamental frequency is the same as the sampling rate  $\omega_s = 2\pi/T_s$ .

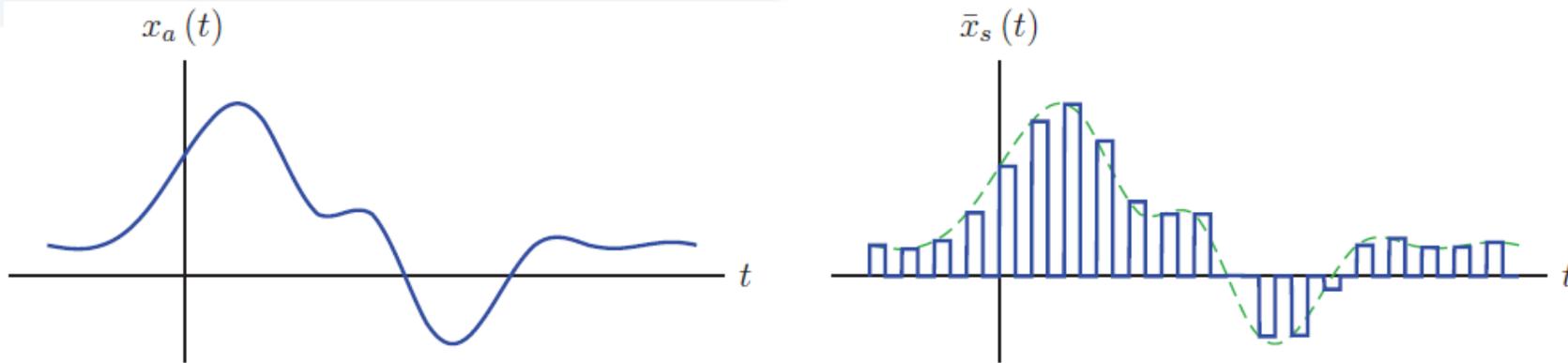
$$\bar{x}_s(t) = x_a(t) \sum_{k=-\infty}^{\infty} d \operatorname{sinc}(kd) e^{jk\omega_s t} \Rightarrow \bar{X}_s(\omega) = F\{\bar{x}_s(t)\} = \int_{-\infty}^{\infty} \bar{x}_s(t) e^{-j\omega t} dt$$

$$\bar{X}_s(\omega) = F\{\bar{x}_s(t)\} = \int_{-\infty}^{\infty} x_a(t) \sum_{k=-\infty}^{\infty} d \operatorname{sinc}(kd) e^{jk\omega_s t} e^{-j\omega t} dt$$

$$\bar{X}_s(\omega) = d \sum_{k=-\infty}^{\infty} \operatorname{sinc}(kd) \left[ \int_{-\infty}^{\infty} x_a(t) e^{-j(\omega - k\omega_s)t} dt \right] = d \sum_{k=-\infty}^{\infty} \operatorname{sinc}(kd) X_a(\omega - k\omega_s)$$



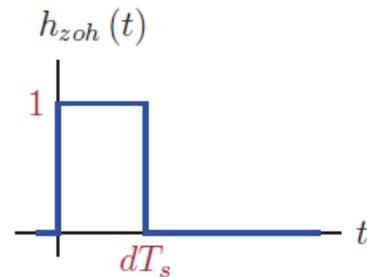
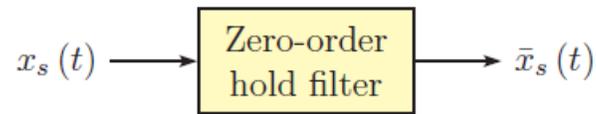
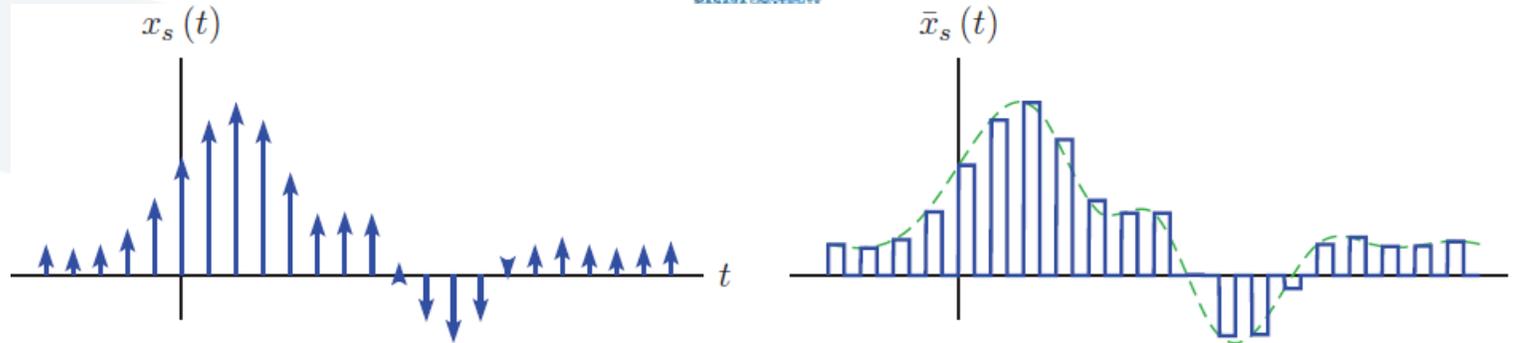
## Zero-order hold sampling



- The signal  $\bar{x}_s(t)$  can be modeled as the convolution of the impulse sampled signal  $x_s(t)$  and a rectangular pulse with unit amplitude and a duration of  $dT_s$ .

$$h_{zoh}(t) = \Pi\left(\frac{t - 0.5dT_s}{dT_s}\right) = u(t) - u(t - dT_s)$$

$$\bar{x}_s(t) = h_{zoh}(t) * x_s(t) \Rightarrow \bar{X}_s(\omega) = H_{zoh}(\omega)X_s(\omega)$$



$$H_{zoh}(\omega) = dT_s \text{sinc}(\omega dT_s / 2\pi) e^{-j\omega dT_s / 2}$$

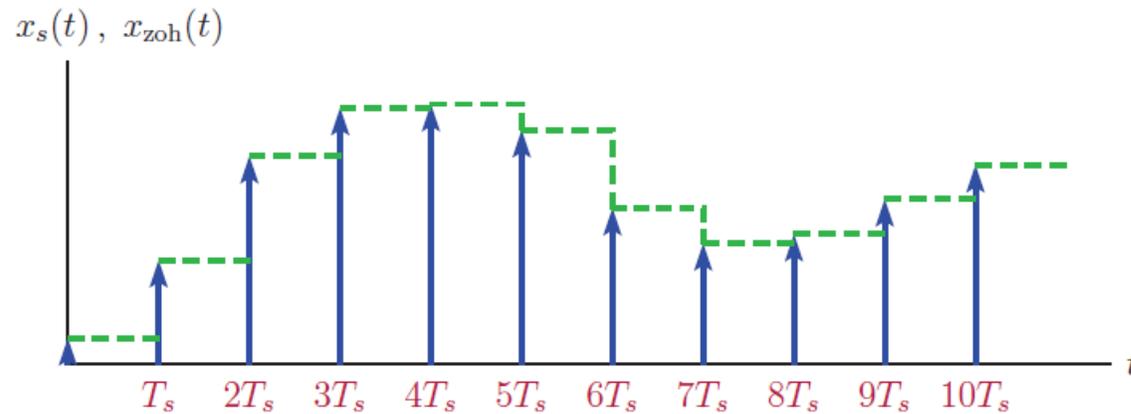
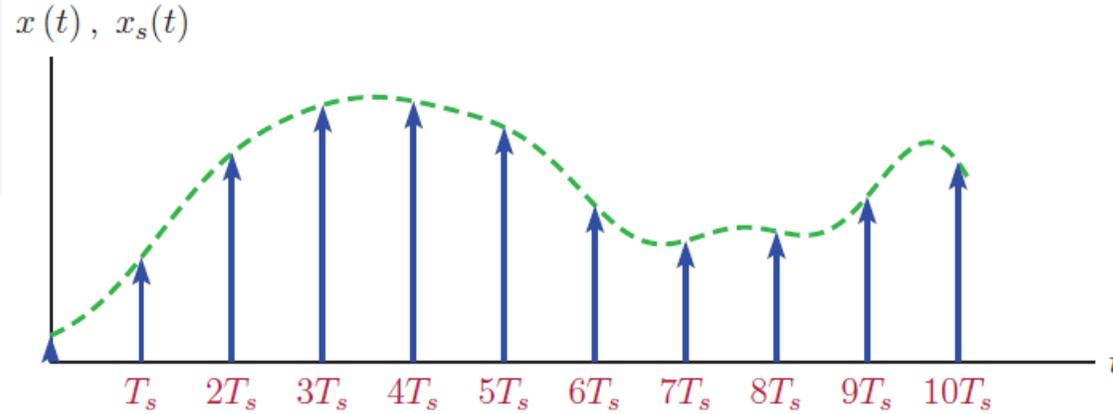
$$\bar{X}_s(\omega) = d \text{sinc}(\omega dT_s / 2\pi) e^{-j\omega dT_s / 2} \sum_{k=-\infty}^{\infty} X_a(\omega - k\omega_s)$$



## 2. Reconstruction of a Signal from Its Sampled Version

- Often the purpose of sampling an analog signal is to **store**, **process** and/or **transmit** it digitally, and to later convert it back to analog format.
- How can the original analog signal be reconstructed from its sampled version?
- Let us first consider the possibility of obtaining a signal similar to  $x_a(t)$  using simple methods. One such method would be to start with the impulse sampled signal  $x_s(t)$ .

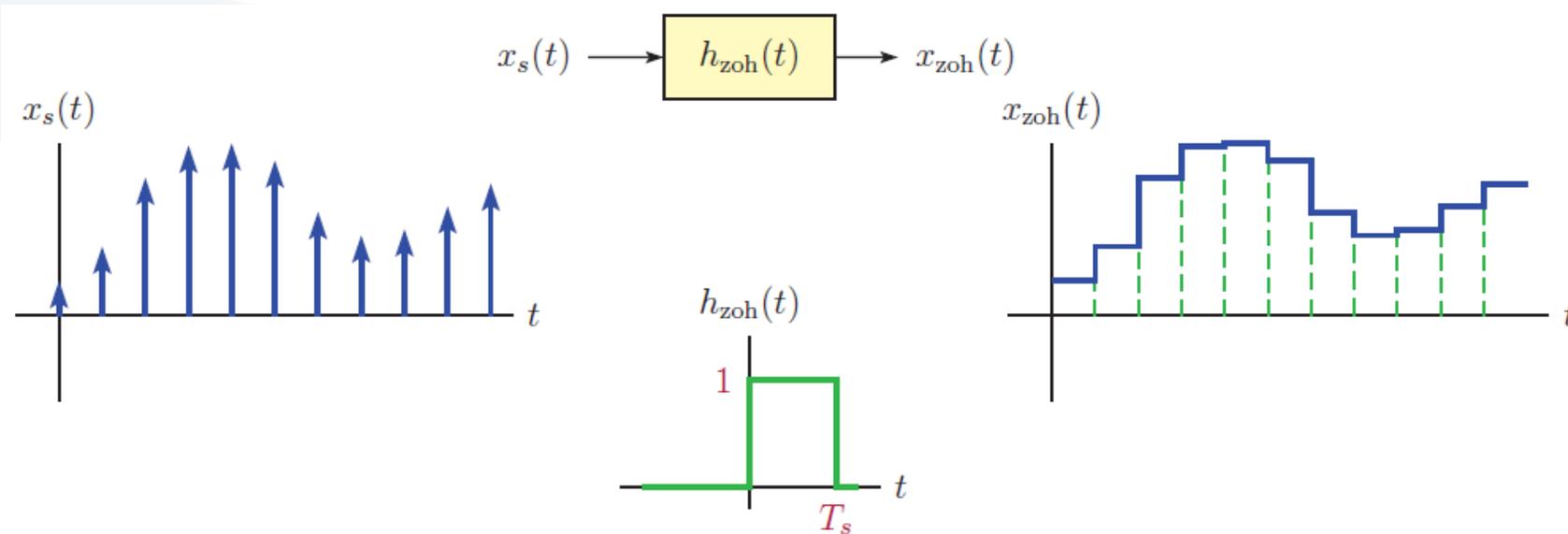
$$x_s(t) = \sum_{n=-\infty}^{\infty} x_a(nT_s)\delta(t - nT_s)$$



**Zero-order hold reconstruction filter:**  $h_{zoh}(t) = \Pi\left(\frac{t - T_s/2}{T_s}\right)$



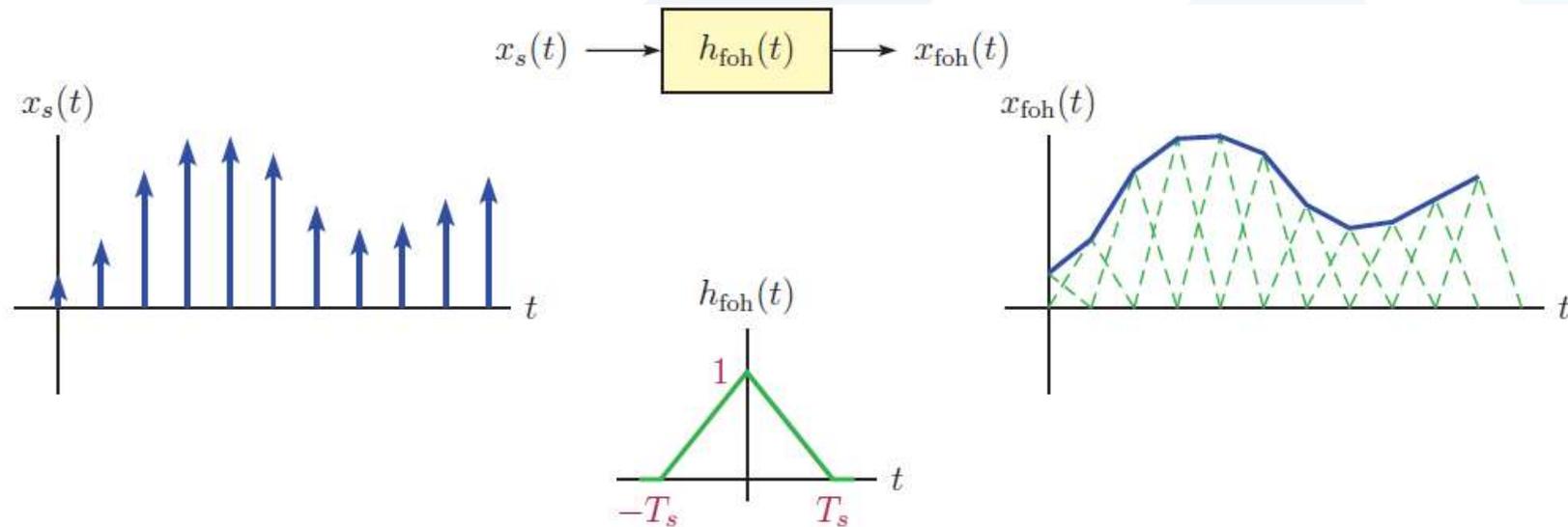
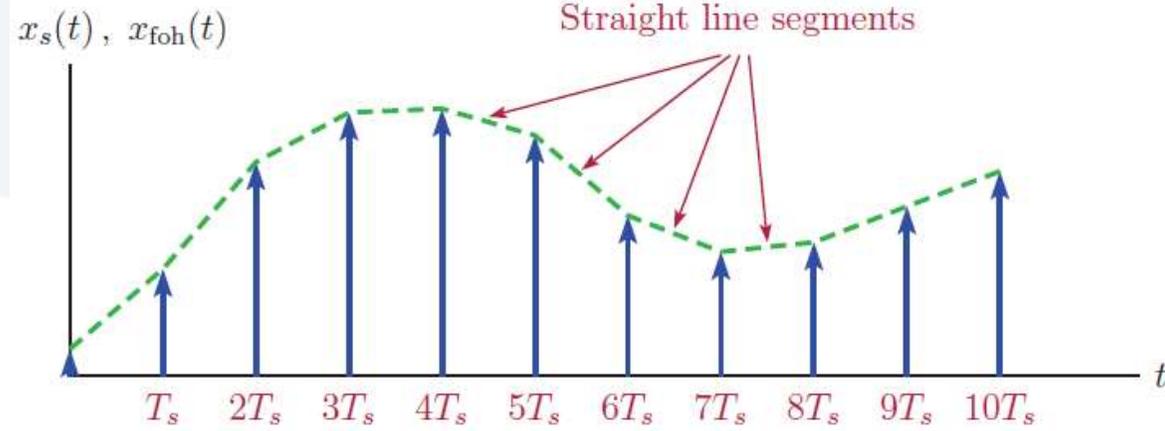
جامعة  
المنارة  
MANARA UNIVERSITY



$$X_{zoh}(\omega) = \text{sinc}(\omega T_s/2\pi) e^{-j\omega T_s/2} \sum_{k=-\infty}^{\infty} X_a(\omega - k\omega_s)$$

**First-order hold reconstruction filter:**

$$h_{foh}(t) = \begin{cases} 1 + t/T_s, & -T_s < t < 0 \\ 1 - t/T_s, & 0 < t < T_s \\ 0, & |t| \geq T_s \end{cases}$$





- The impulse response  $h_{foh}(t)$  of the first-order hold interpolation filter is **non-causal** since it starts at  $t = -T_s$ .

$$\bar{h}_{foh}(t) = h_{foh}(t - T_s) \quad (\text{to achieve causality})$$

$$H_{zoh}(f) = T_s \text{sinc}(fT_s) e^{-j\pi T_s f}$$

- The spectrum of the analog signal constructed using the zero-order hold filter is:

$$X_{zoh}(f) = H_{zoh}(f)X_s(f)$$

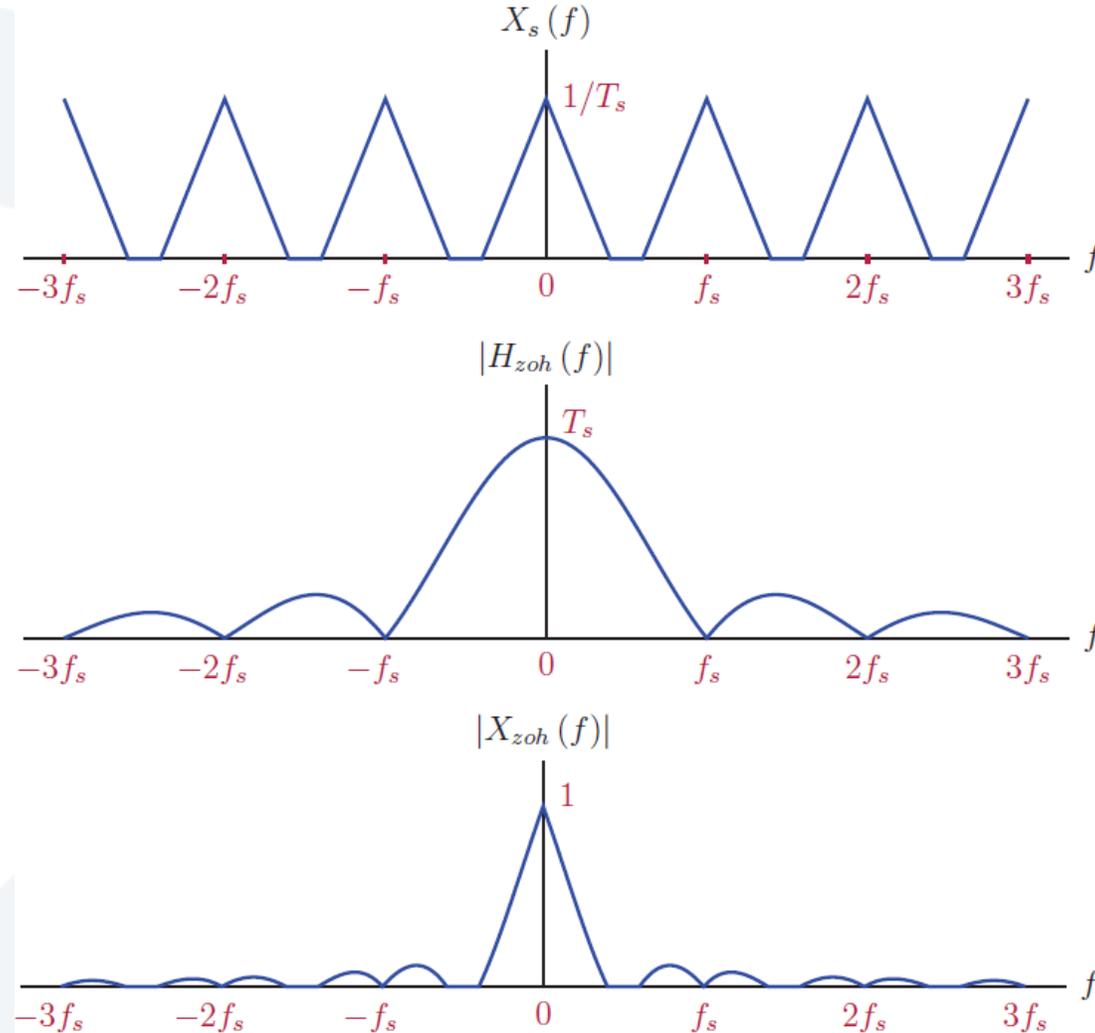
$$H_{foh}(f) = T_s \text{sinc}^2(fT_s)$$

- The spectrum of the analog signal constructed using the zero-order hold filter is:

$$X_{foh}(f) = H_{foh}(f)X_s(f)$$



جامعة  
المنارة  
MANARA UNIVERSITY





- What kind of interpolation is needed for perfect reconstruction of the analog signal from its impulse-sampled version?
- The answer must be found through the frequency spectrum of the sampled signal.

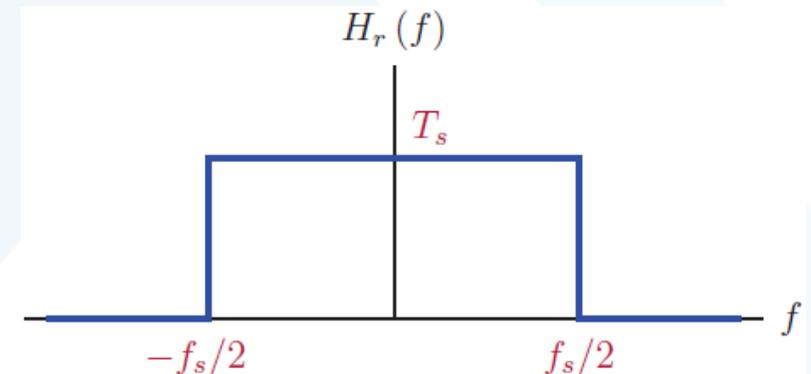
$$X_s(f) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X_a(f - kf_s)$$

- An **ideal lowpass** reconstruction filter with cutoff frequency set equal to  $f_s/2$  is needed.

$$H_r(f) = T_s \Pi(f/f_s)$$

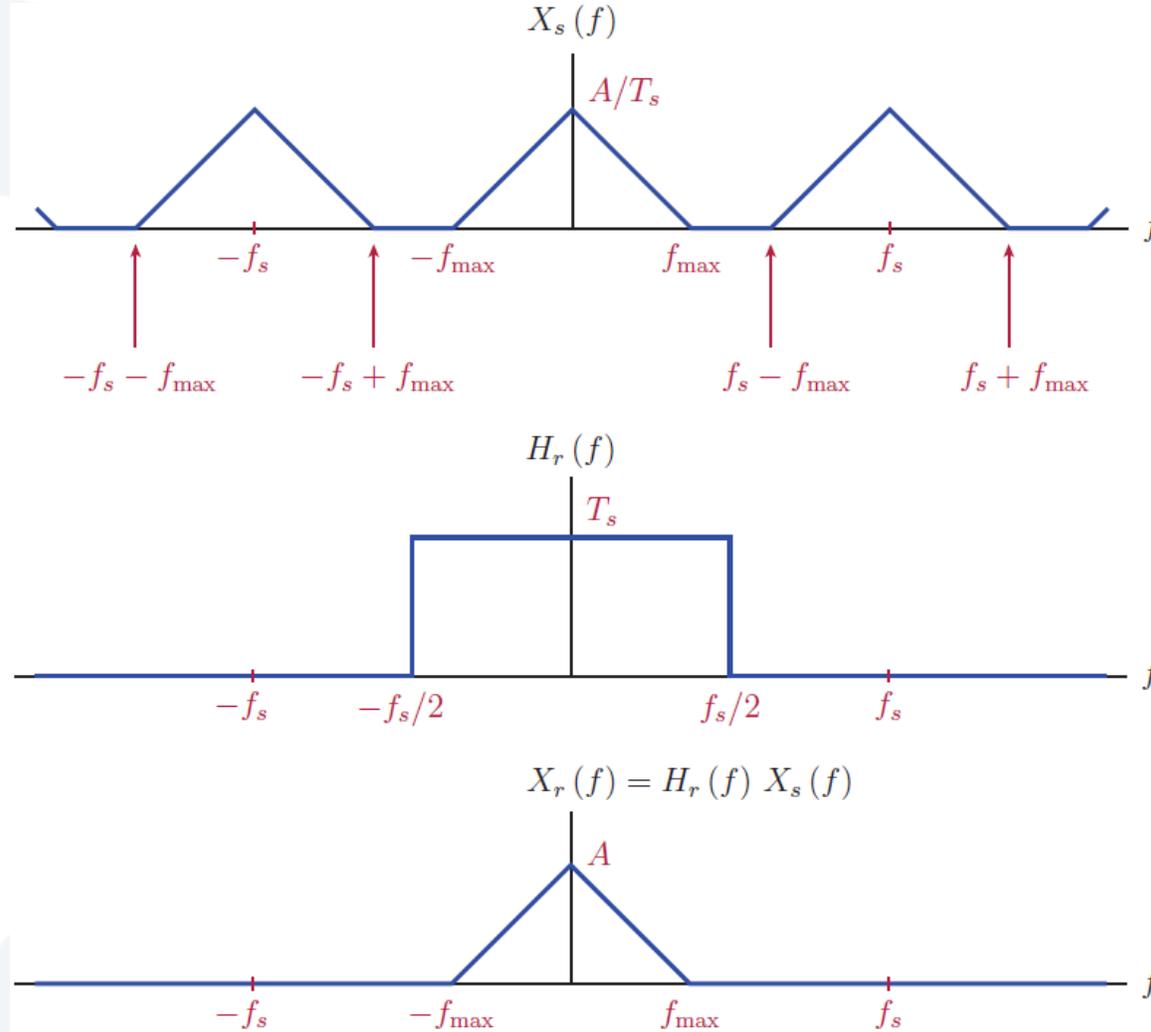
$$X_r(f) = H_r(f)X_s(f)$$

$$= T_s \Pi(f/f_s) \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X_a(f - kf_s) = X_a(f)$$





جامعة  
المنارة  
MANARA UNIVERSITY





- Since  $X_r(f) = X_a(f)$  we have  $x_r(t) = x_a(t)$

$$h_r(t) = \text{sinc}(tf_s) = \text{sinc}(t/T_s)$$

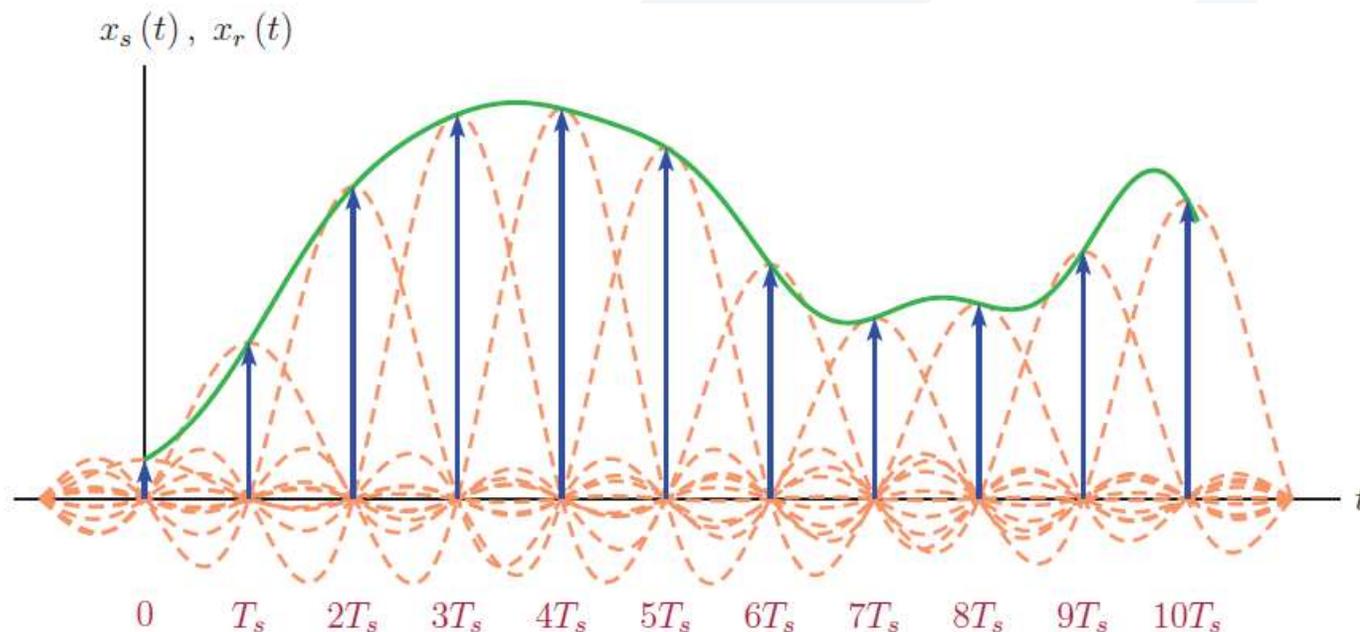
$$x_r(t) = h_r(t) * x_s(t) = \sum_{n=-\infty}^{\infty} x_a(nT_s) \text{sinc}\left(\frac{t - nT_s}{T_s}\right)$$

- Let us consider the output of the filter at one of the sampling instants, say  $t = kT_s$ .

$$x_r(kT_s) = \sum_{n=-\infty}^{\infty} x_a(nT_s) \text{sinc}\left(\frac{kT_s - nT_s}{T_s}\right) = \sum_{n=-\infty}^{\infty} x_a(nT_s) \text{sinc}(k - n)$$

$$\text{sinc}(k - n) = \begin{cases} 1, & n = k \\ 0, & n \neq k \end{cases} \Rightarrow x_r(kT_s) = x_a(kT_s)$$

1. The output  $x_r(t)$  of the ideal lowpass reconstruction filter is equal to the sampled signal at each sampling instant.
2. Between sampling instants,  $x_r(t)$  is obtained by interpolation through the use of sinc functions. This is referred to as **bandlimited interpolation**.





- All 3 methods (**zero-order hold**, **first-order hold**, and **bandlimited interpolation**) result in reconstructed signals that have the correct amplitude values at the sampling instants, and interpolated amplitude values between them.
- What makes the signal obtained by bandlimited interpolation more accurate than the other two?
- The signal obtained by bandlimited interpolation is the only signal among the three that is limited to a bandwidth of  $f_s/2$ .
- The bandwidth of each of the other two signals is greater than  $f_s/2$ , therefore, neither of them could have been the signal that produced a properly sampled  $x_s(t)$ .