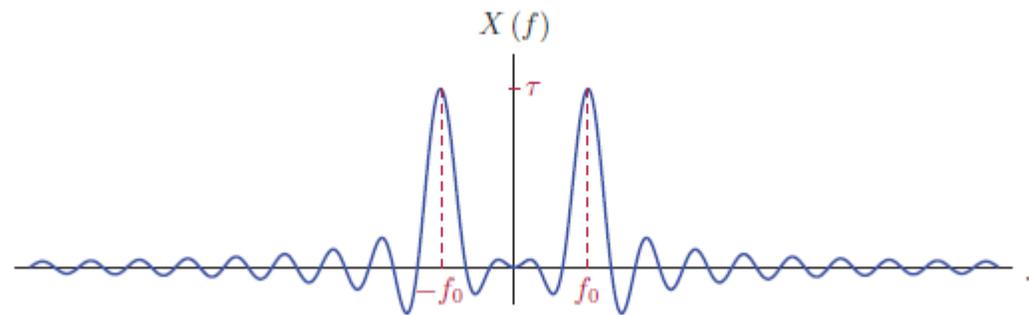




# CECC507: Signals and Systems

## Lecture Notes 6: Fourier Analysis for Discrete Time Signals and Systems



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## Chapter 5

# Fourier Analysis for Discrete Time Signals and Systems

1. Analysis of Non-Periodic Discrete-Time Signals
2. Energy and Power in the Frequency Domain
3. Transfer Function Concept
4. DTLTI Systems with Non Periodic Input Signals
5. Discrete Fourier Transform



## Introduction

- DTLTI system can be represented by means of a **constant coefficient linear difference equation**, or alternatively by means of an **impulse response**.
- The output signal of a DTLTI system can be determined by solving the corresponding difference equation or by using the convolution operation.

### 1. Analysis of Non-Periodic Discrete-Time Signals

#### Discrete-time Fourier transform (DTFT)

1. Synthesis equation: (Inverse transform) 
$$x[n] = \mathcal{F}^{-1}\{X(\Omega)\} = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\Omega) e^{j\Omega n} d\Omega$$

2. Analysis equation: (Forward transform) 
$$X(\Omega) = \mathcal{F}\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}$$



## Existence of the DTFT

- A **sufficient** condition for the **convergence** of DTFT is that the signal  $x[n]$  be absolute summable,

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

- Alternatively, it is also sufficient for  $x[n]$  to be square-summable:  $\sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty$

## DTFT of some signals

- Example 1:** DTFT of unit-impulse signal

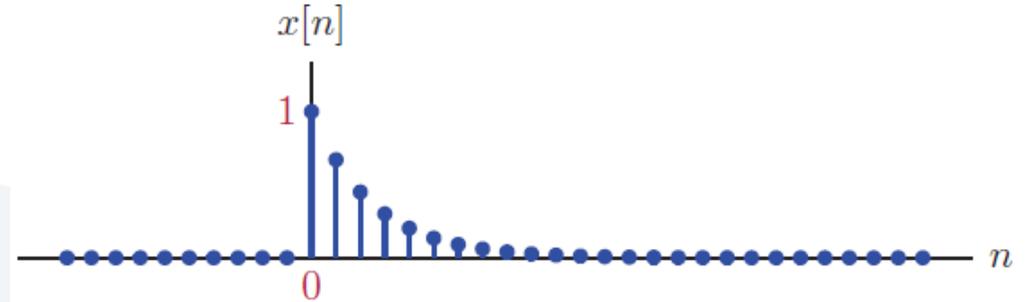
$$\mathcal{F}\{\delta[n]\} = \sum_{n=-\infty}^{\infty} \delta[n] e^{-j\Omega n} = 1$$

- Example 2:** DTFT of right-sided exponential signal

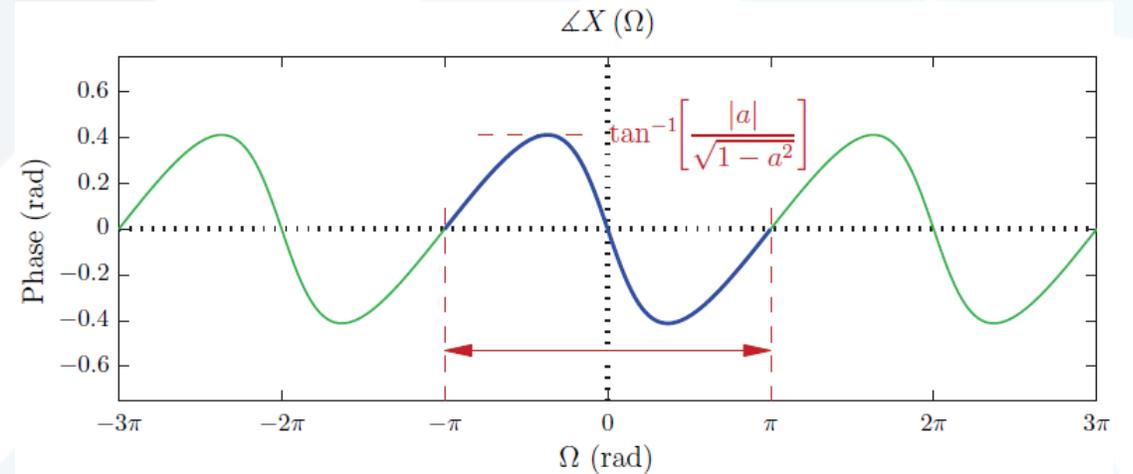
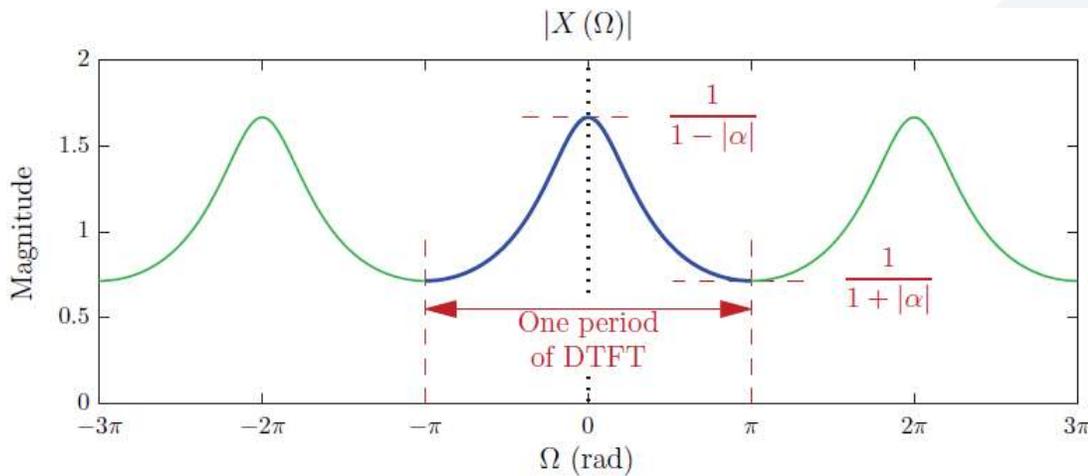
$$x[n] = \alpha^n u[n], \quad |\alpha| < 1$$



$$X(\Omega) = \sum_{n=0}^{\infty} \alpha^n e^{-j\Omega n} = \frac{1}{1 - \alpha e^{-j\Omega}}$$



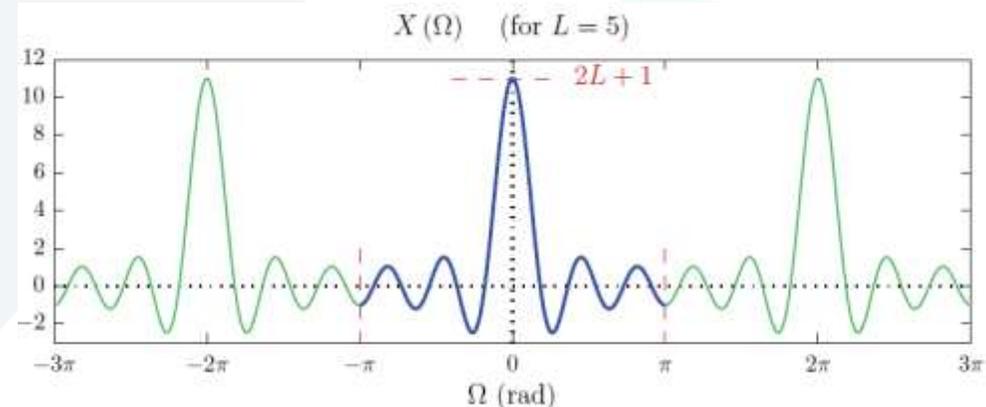
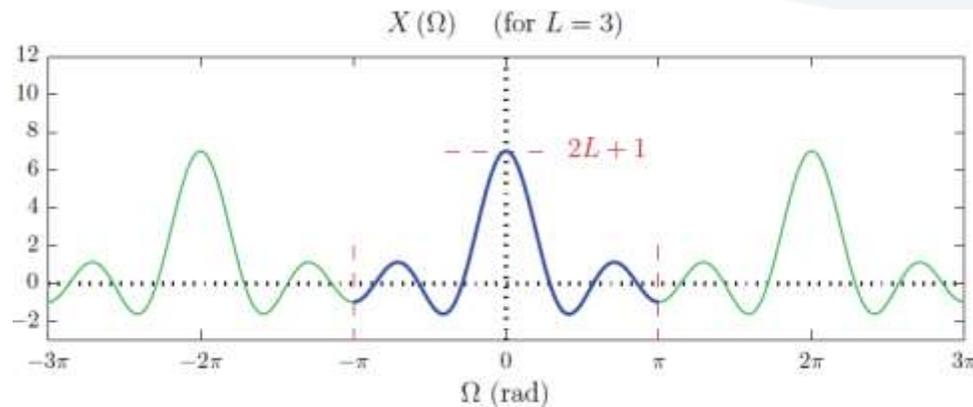
$$|X(\Omega)| = \frac{1}{\sqrt{1 + \alpha^2 - 2\alpha \cos(\Omega)}}, \quad \angle X(\Omega) = -\tan^{-1} \frac{\alpha \sin(\Omega)}{1 - \alpha \cos(\Omega)}$$





■ **Example 3:** DTFT for discrete-time pulse

$$x[n] = \begin{cases} 1, & -L \leq n \leq L \\ 0, & \text{otherwise} \end{cases} \quad X(\Omega) = \sum_{n=-L}^L (1) e^{-j\Omega n} = \frac{e^{j\Omega L} - e^{-j\Omega(L+1)}}{1 - e^{-j\Omega}} = \frac{\sin\left(\frac{\Omega}{2}(2L+1)\right)}{\sin\left(\frac{\Omega}{2}\right)}$$



## Properties of the DTFT

### Periodicity

$$X(\Omega + 2\pi r) = X(\Omega) \text{ for all integer } r$$

### Linearity

$$x_1[n] \xleftrightarrow{F} X_1(\Omega) \quad \text{and} \quad x_2[n] \xleftrightarrow{F} X_2(\Omega)$$



$$\Rightarrow \alpha_1 x_1[n] + \alpha_2 x_2[n] \xleftrightarrow{F} \alpha_1 X_1(\Omega) + \alpha_2 X_2(\Omega)$$

### Time shifting

$$x[n] \xleftrightarrow{F} X(\Omega) \Rightarrow x[n - m] \xleftrightarrow{F} X(\Omega) e^{-j\Omega m}$$

### Time reversal and Conjugation

$$x[n] \xleftrightarrow{F} X(\Omega) \Rightarrow x[-n] \xleftrightarrow{F} X(-\Omega)$$

$$x[n] \xleftrightarrow{F} X(\Omega) \Rightarrow x^*[n] \xleftrightarrow{F} X^*(-\Omega)$$

### Frequency shifting

$$x[n] \xleftrightarrow{F} X(\Omega) \Rightarrow x[n] e^{j\Omega_0 n} \xleftrightarrow{F} X(\Omega - \Omega_0)$$

### Modulation property

$$x[n] \xleftrightarrow{F} X(\Omega) \Rightarrow$$

$$x[n] \cos(\Omega_0 n) \xleftrightarrow{F} \frac{1}{2} [X(\Omega - \Omega_0) + X(\Omega + \Omega_0)]$$

$$x[n] \sin(\Omega_0 n) \xleftrightarrow{F} \frac{1}{2} [X(\Omega - \Omega_0) e^{-j\pi/2} + X(\Omega + \Omega_0) e^{j\pi/2}]$$

### Differentiation in the frequency domain

$$x[n] \xleftrightarrow{F} X(\Omega) \Rightarrow n^m x[n] \xleftrightarrow{F} j^m \frac{d^m X(\Omega)}{d\Omega^m}$$



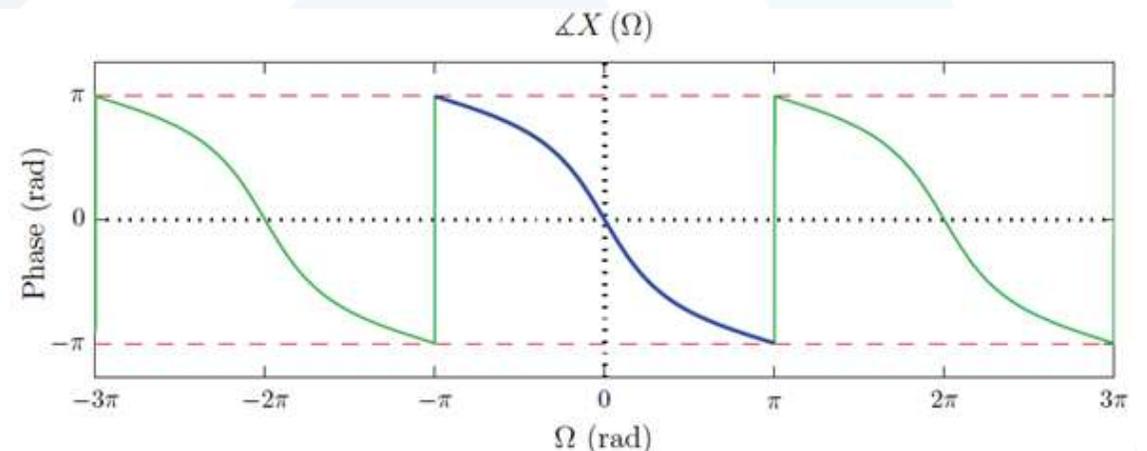
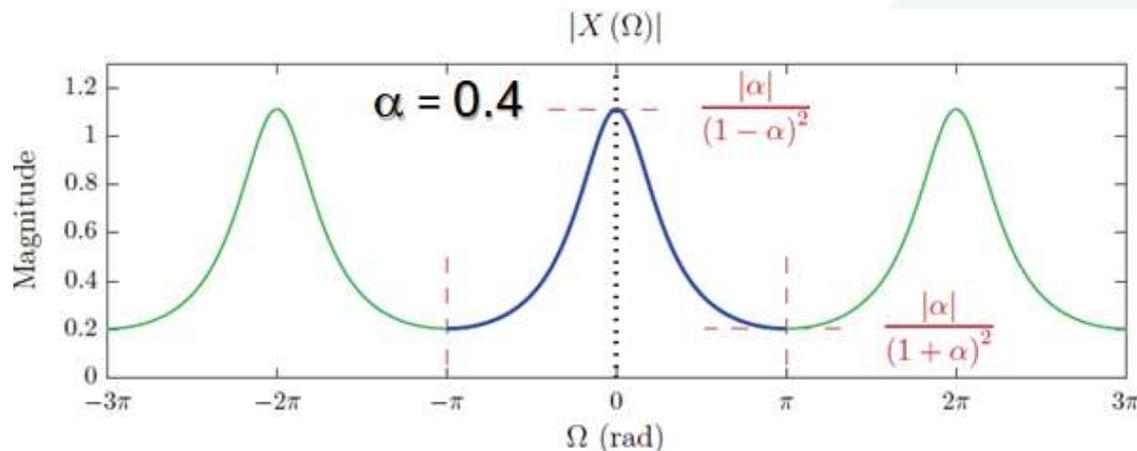
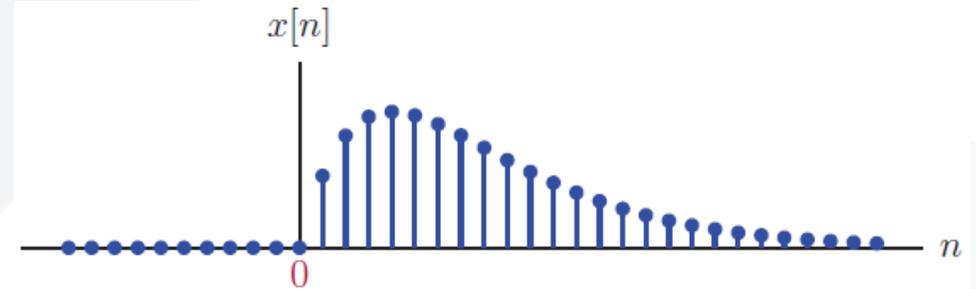
■ **Example 4:** Use of differentiation in frequency property

$$\alpha^n u[n] \xleftrightarrow{F} \frac{1}{1 - \alpha e^{-j\Omega}}$$

$$x[n] = n\alpha^n u[n]. \quad |\alpha| < 1$$

$$n\alpha^n u[n] \xleftrightarrow{F} j \frac{d}{d\Omega} \left[ \frac{1}{1 - \alpha e^{-j\Omega}} \right] = \frac{\alpha e^{-j\Omega}}{(1 - \alpha e^{-j\Omega})^2}$$

$$X(\Omega) = \frac{\alpha e^{-j\Omega}}{(1 - \alpha e^{-j\Omega})^2}$$





**Convolution property**  $x_1[n] \xleftrightarrow{F} X_1(\Omega)$  and  $x_2[n] \xleftrightarrow{F} X_2(\Omega)$   
 $\Rightarrow x_1[n] * x_2[n] \xleftrightarrow{F} X_1(\Omega)X_2(\Omega)$

■ **Example 5:** Convolution using the DTFT

$h[n] = (2/3)^n u[n]$ ,  $x[n] = (3/4)^n u[n]$ . Determine  $y[n] = h[n] * x[n]$  using the DTFT

$$H(\Omega) = \frac{1}{1 - \frac{2}{3} e^{-j\Omega}}, \quad X(\Omega) = \frac{1}{1 - \frac{3}{4} e^{-j\Omega}}$$

$$Y(\Omega) = H(\Omega)X(\Omega) = \frac{1}{\left(1 - \frac{2}{3} e^{-j\Omega}\right)\left(1 - \frac{3}{4} e^{-j\Omega}\right)} = \frac{-8}{1 - \frac{2}{3} e^{-j\Omega}} + \frac{9}{1 - \frac{3}{4} e^{-j\Omega}}$$

$$y[n] = -8(2/3)^n u[n] + 9(3/4)^n u[n]$$

**Multiplication of two signals**  $x_1[n] \xleftrightarrow{F} X_1(\Omega)$  and  $x_2[n] \xleftrightarrow{F} X_2(\Omega)$   
 $\Rightarrow x_1[n]x_2[n] \xleftrightarrow{F} \frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(\tau)X_2(\Omega - \tau) d\tau$



## 2. Energy and Power in the Frequency Domain

### Parseval's theorem

- For a non-periodic energy signal  $x[n]$  with DTFT,  $X(\Omega)$ :

$$E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(\Omega)|^2 d\Omega$$

Let the function  $G_x(\Omega)$  be defined as:

$$G_x = |X(\Omega)|^2 \text{ energy spectral density (ESD) of the signal } x[n]$$

$$E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} G_x(\Omega) d\Omega$$

$$E_x \text{ in } (-\Omega_0, \Omega_0) = \frac{1}{2\pi} \int_{-\Omega_0}^{\Omega_0} G_x(\Omega) d\Omega$$



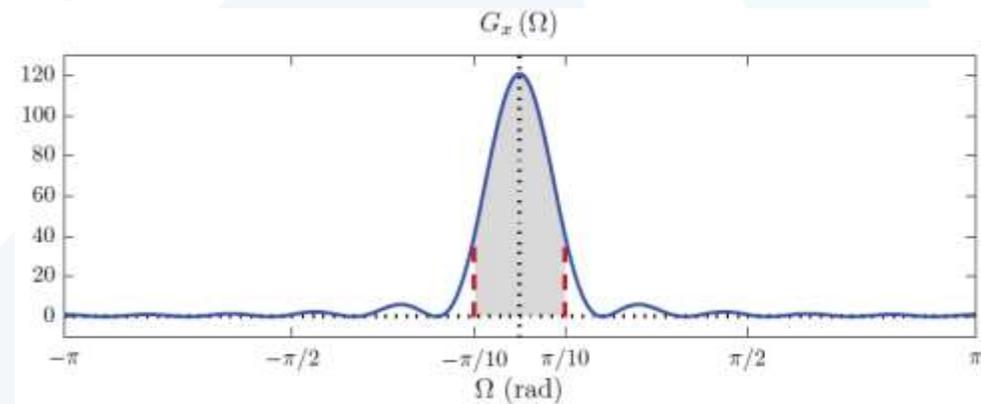
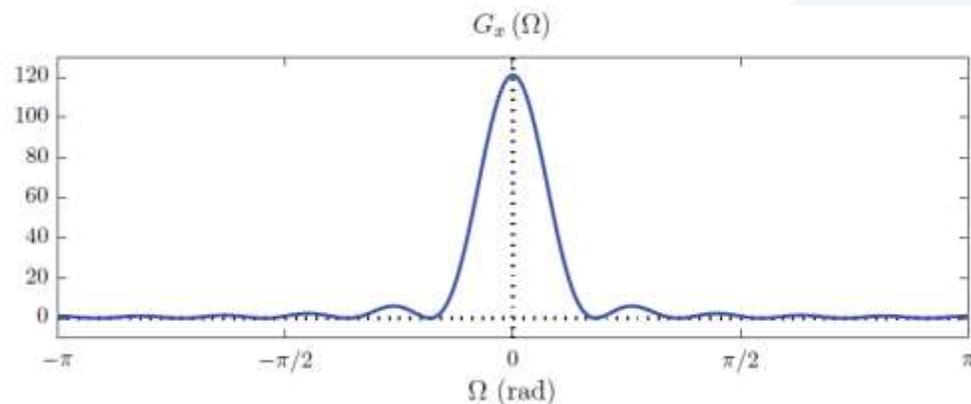
- **Example 6:** Energy spectral density of a discrete-time pulse

Determine the ESD of the rectangular pulse  $x[n] = \begin{cases} 1, & n = -5, \dots, 5 \\ 0, & \text{otherwise} \end{cases}$

The energy of the signal in the frequency interval  $-\pi/10 < \Omega < \pi/10$ :

$$X(\Omega) = \frac{\sin(11\Omega/2)}{\sin(\Omega/2)} \Rightarrow G_x(\Omega) = |X_x(\Omega)|^2 = \frac{\sin^2(11\Omega/2)}{\sin^2(\Omega/2)}$$

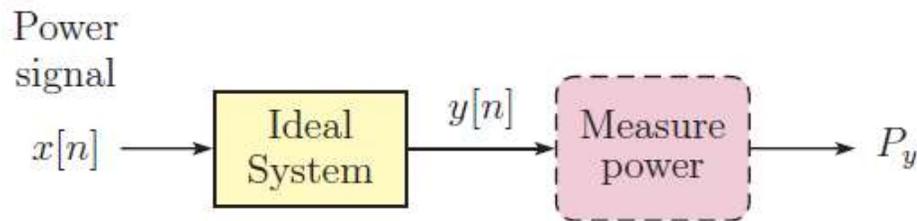
$$E_x \text{ in } (-\pi/10, \pi/10) = \frac{1}{2\pi} \int_{-\pi/10}^{\pi/10} G_x(\Omega) d\Omega \approx 8.9309$$



## Energy or power in a frequency range

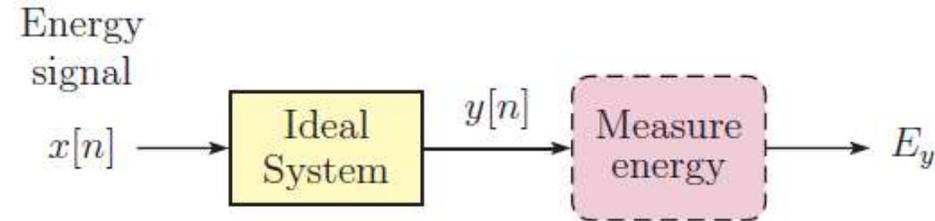
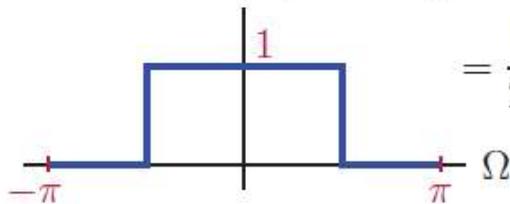
- The power/energy of  $x[n]$  in the frequency range  $-\Omega_0 < \Omega < \Omega_0$  is the same as the power/energy of the output signal of a system with transfer function:

$$H(\Omega) = \begin{cases} 1, & |\Omega| < \Omega_0 \\ 0, & \Omega_0 < |\Omega| < \pi \end{cases}$$


 $H(\Omega)$ 

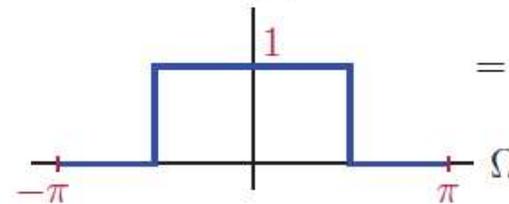
$P_y = P_x \text{ in } (-\Omega_0, \Omega_0)$

$$= \frac{1}{2\pi} \int_{-\Omega_0}^{\Omega_0} S_x(\Omega) d\Omega$$


 $H(\Omega)$ 

$E_y = E_x \text{ in } (-\Omega_0, \Omega_0)$

$$= \frac{1}{2\pi} \int_{-\Omega_0}^{\Omega_0} G_x(\Omega) d\Omega$$





## Autocorrelation

- For an energy signal  $x[n]$  the **autocorrelation function** is defined as:

$$r_{xx}[m] = \sum_{n=-\infty}^{\infty} x[n]x[n+m]$$

- For an energy signal, the energy spectral density is the DTFT of the autocorrelation function, that is,  $\mathcal{F}\{r_{xx}[m]\} = G_x(\Omega)$

## Properties of the autocorrelation function

- $r_{xx}[0] = E_x \geq |r_{xx}[m]|$  for all  $m$ .
- $r_{xx}[-m] = r_{xx}[m]$  for all  $m$ , the autocorrelation function has **even symmetry**.
- If the signal  $\tilde{x}[n]$  is **periodic** with period  $N$ , then its autocorrelation function  $\tilde{r}_{xx}[m]$  is also periodic with the **same** period.



### 3. Transfer Function Concept

- In **time-domain** analysis of systems two distinct descriptions for DTLTI systems:
  1. A **linear constant-coefficient difference** equation that describes the relationship between the input and the output signals.
  2. An **impulse response** which can be used with the **convolution operation** for determining the response of the system to an arbitrary input signal.
- The concept of **Transfer function** will be introduced as the third method for describing the characteristics of a system.

$$H(\Omega) = \mathcal{F}\{h[n]\} = \sum_{n=-\infty}^{\infty} h[n]e^{-j\Omega n}$$

- The transfer function concept is valid for LTI systems **only**.
- In general,  $H(\Omega)$  is a **complex function** of  $\Omega$ ,  $H(\Omega) = |H(\Omega)|e^{j\Theta(\omega)}$ .



## Obtaining the transfer function from the difference equation

$$y[n] = h[n] * x[n] \xleftrightarrow{F} Y(\Omega) = H(\Omega)X(\Omega) \Rightarrow H(\Omega) = \frac{Y(\Omega)}{X(\Omega)}$$

$$y[n - m] \xleftrightarrow{F} e^{-j\Omega m} Y(\Omega), \quad m = 0, 1, \dots$$

$$x[n - m] \xleftrightarrow{F} e^{-j\Omega m} X(\Omega), \quad m = 0, 1, \dots$$

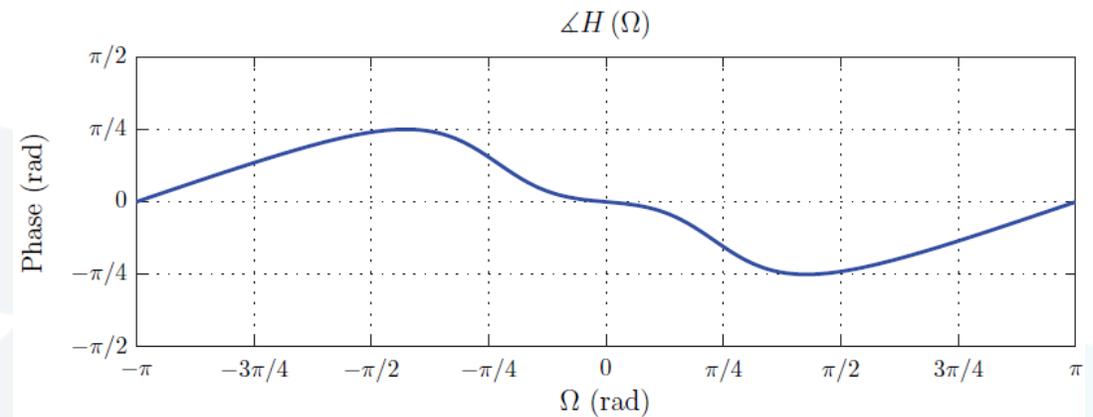
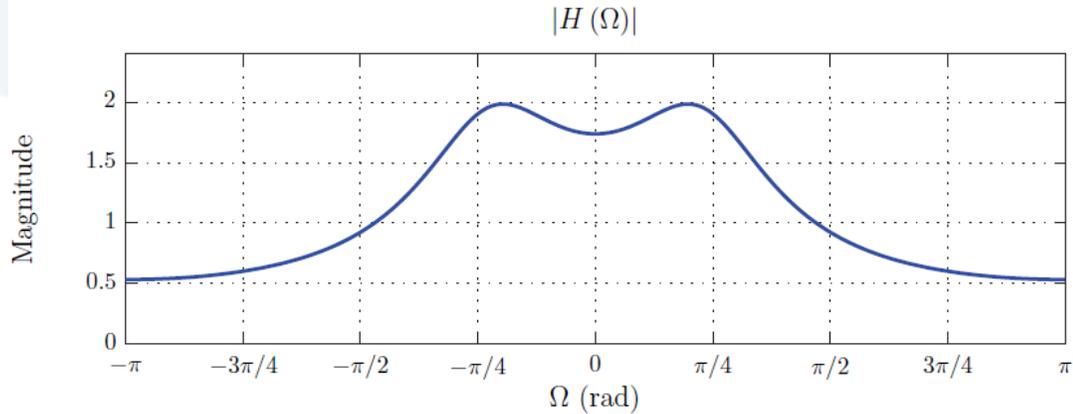
- **Example 7:** Finding the transfer function from the difference equation

$$y[n] - 0.9y[n - 1] + 0.36y[n - 2] = x[n] - 0.2x[n - 1]$$

$$Y(\Omega) - 0.9Y(\Omega)e^{-j\Omega} + 0.36Y(\Omega)e^{-j2\Omega} = X(\Omega) - 0.2X(\Omega)e^{-j\Omega}$$

$$Y(\Omega)[1 - 0.9e^{-j\Omega} + 0.36e^{-j2\Omega}] = X(\Omega)[1 - 0.2e^{-j\Omega}]$$

$$H(\Omega) = \frac{Y(\Omega)}{X(\Omega)} = \frac{1 - 0.2e^{-j\Omega}}{1 - 0.9e^{-j\Omega} + 0.36e^{-j2\Omega}}$$

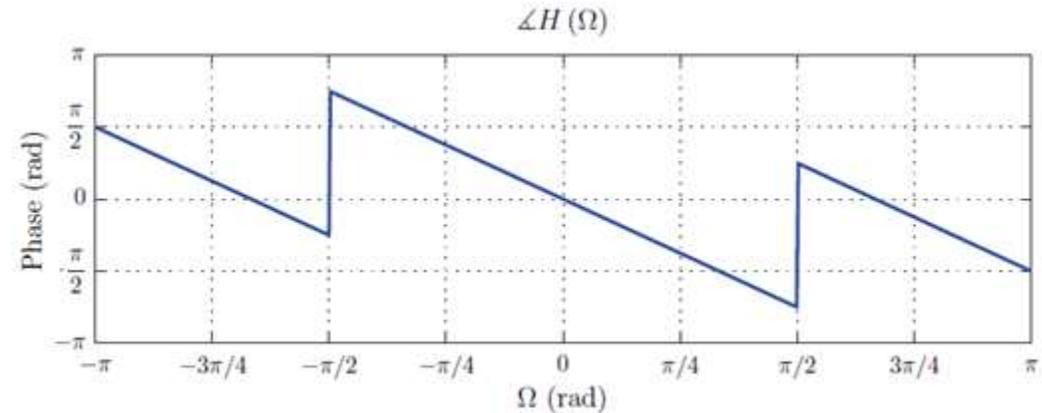
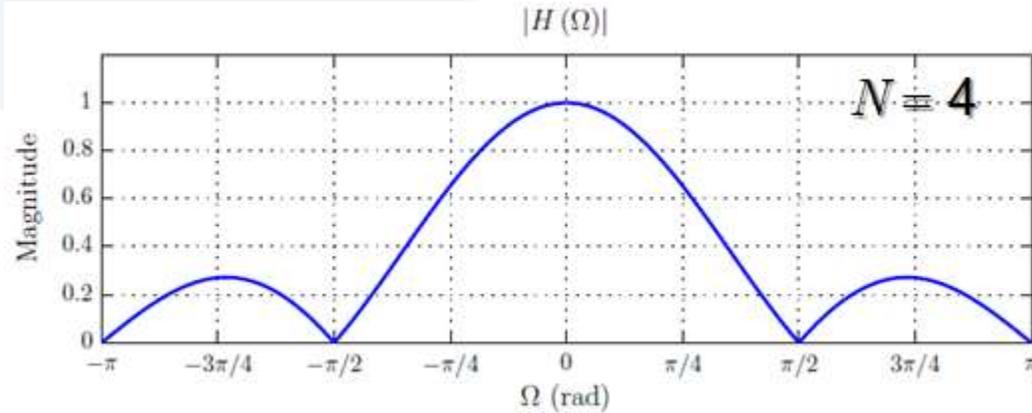


- **Example 8:** Transfer function for length- $N$  moving average filter

$$y[n] = \frac{1}{N} \sum_{k=0}^{N-1} x[n-k]$$

$$Y(\Omega) = \frac{1}{N} \sum_{k=0}^{N-1} e^{-j\Omega k} X(\Omega) \Rightarrow H(\Omega) = \frac{1}{N} \sum_{k=0}^{N-1} e^{-j\Omega k} = \frac{1}{N} \frac{1 - e^{-j\Omega N}}{1 - e^{-j\Omega}}$$

$$H(\Omega) = \frac{\sin(\Omega N/2)}{N \sin(\Omega/2)} e^{j\Omega(N-1)/2}$$



#### 4. DTLTI Systems with Non Periodic Input Signals

For a non-periodic signal  $x[n]$  as input to a DTLTI system. The output of the system  $y[n]$  is given by:  $y[n] = h[n] * x[n]$

- Let us assume that The system is stable ensuring that  $H(\Omega)$  converges, and the DTFT of the input signal also converges.

$$Y(\Omega) = H(\Omega)X(\Omega) \quad |Y(\Omega)| = |H(\Omega)||X(\Omega)|, \quad \angle Y(\Omega) = \angle X(\Omega) + \Theta(\Omega)$$



## 5. Discrete Fourier Transform

- The DTFT of a DT signal  $x[n]$  is a transform  $X(\Omega)$  which, if it exists, is a  **$2\pi$ -periodic** function of the **continuous variable**  $\Omega \Rightarrow$  **Storing** the DTFT of a signal on a digital computer is impractical because of the continuous nature of  $\Omega$ .
- Sometimes it is desirable to have a transform that is also discrete. This can be accomplished through the use of the **discrete Fourier transform** (DFT) provided that the signal under consideration is **finite-length**.

Discrete Fourier Transform (DFT):

1. Analysis equation (Forward transform): 
$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi nk/N}, \quad k = 0, \dots, N-1$$

2. Synthesis equation (Inverse transform): 
$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j2\pi nk/N}, \quad n = 0, \dots, N-1$$



- **Example 9:** DFT of simple signal

$$\tilde{x}[n] = \left\{ \underset{\substack{\uparrow \\ n=0}}{1}, -1, 2 \right\}$$

$$X[k] = e^{-j(2\pi/3)k(0)} - e^{-j(2\pi/3)k(1)} + e^{-j(2\pi/3)k(2)} = 1 - e^{-j2\pi k/3} + e^{-j4\pi k/3}$$

$$X[0] = 2, \quad X[1] = 0.5 + j2.5981, \quad X[2] = 0.5 - j2.5981$$

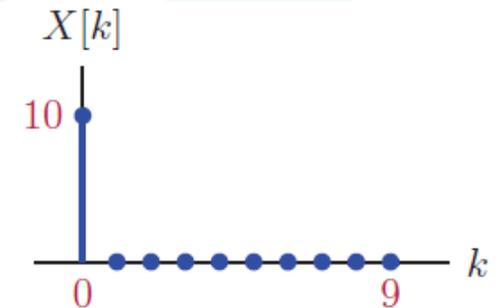
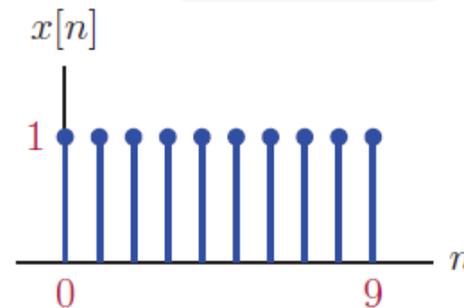
- **Example 10:** DFT of discrete-time pulse

$$x[n] = u[n] - u[n - 10]$$

$$\tilde{x}[n] = \left\{ \underset{\substack{\uparrow \\ n=0}}{1}, 1, 1, 1, 1, 1, 1, 1, 1, 1 \right\}$$

$$X[k] = \sum_{n=0}^9 e^{-j(2\pi/10)kn}, \quad k = 0, \dots, 9$$

$$X[k] = \frac{1 - e^{-j2\pi k}}{1 - e^{-j2\pi k/10}} = \begin{cases} 10, & k = 0 \\ 0, & k = 1, \dots, 9 \end{cases}$$





## Relationship of the DFT to the DTFT

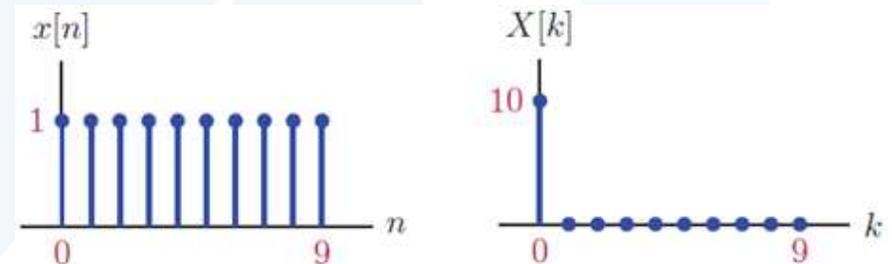
- The DFT of a length- $N$  signal is equal to its DTFT evaluated at a set of  $N$  angular frequencies equally spaced in the interval  $[0, 2\pi)$ . Let an indexed set of angular frequencies be defined as:  $\Omega_k = 2\pi k/N$ ,  $k = 0, 1, \dots, N - 1$ .

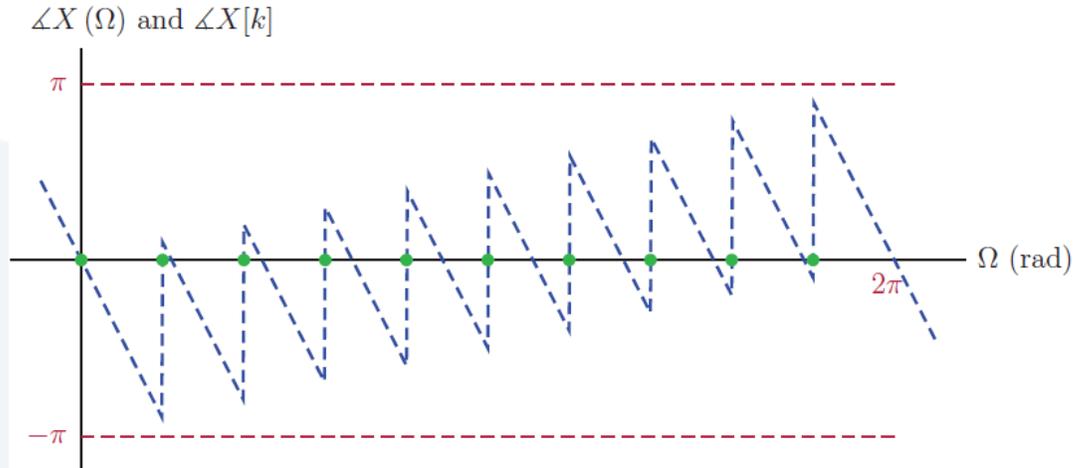
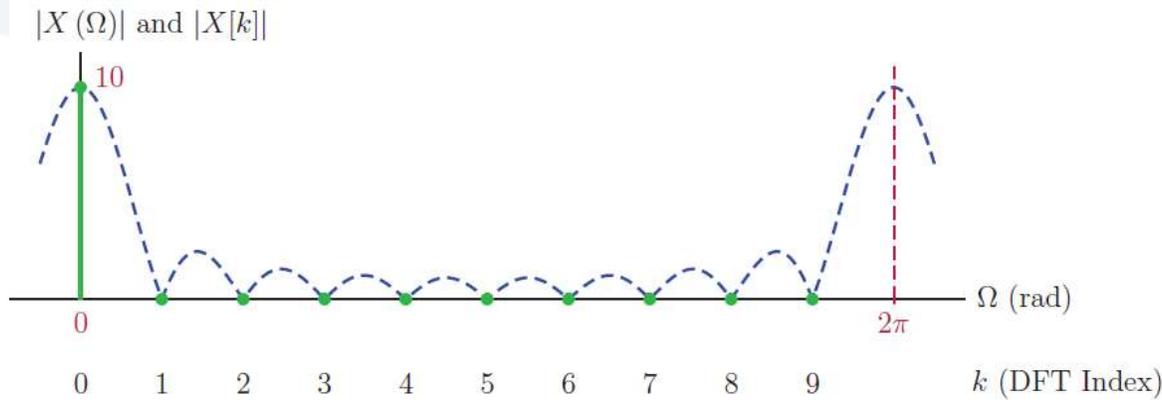
The DFT of the signal is written as  $X[k] = X(\Omega_k) = \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N}$

- Example 11:** DFT of discrete-time pulse revisited

$$x[n] = u[n] - u[n - 10]$$

$$X(\Omega) = \sum_{n=0}^9 e^{-j\Omega n} = \frac{\sin(5\Omega)}{\sin(0.5\Omega)} e^{-j4.5\Omega}$$





## Properties of the DFT

- The properties of the DFT are similar to those of DTFT with one significant difference: Any shifts in the time domain or the transform domain are **circular shifts** rather than **linear shifts**.
- Also, any time reversals used in conjunction with the DFT are **circular time reversals** rather than linear ones.



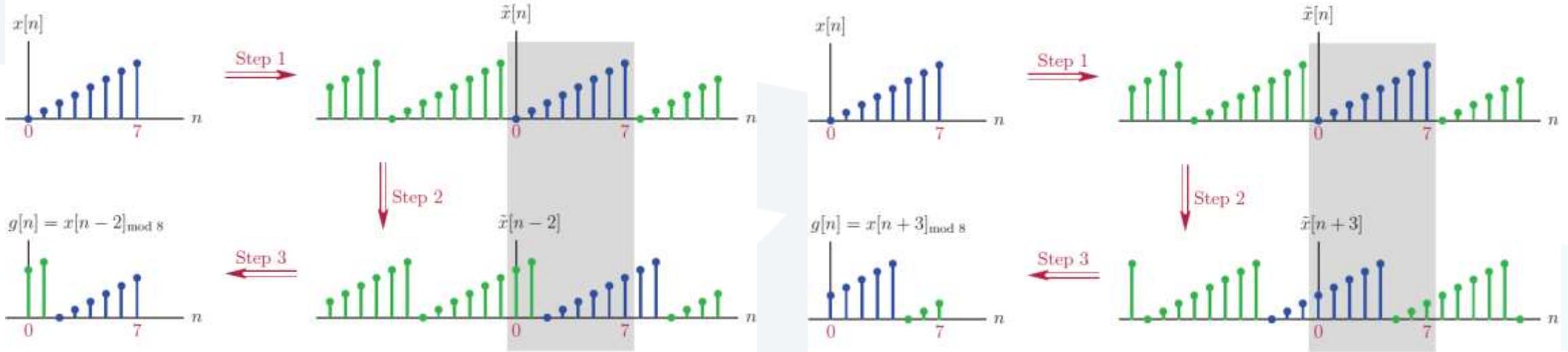
1. Obtain periodic extension  $\tilde{x}[n]$  from  $x[n]$ : 
$$\tilde{x}[n] = \sum_{m=-\infty}^{\infty} x[n - mN]$$
2. Apply a time shift to  $\tilde{x}[n]$  to obtain  $\tilde{x}[n - m]$ . The amount of the time shift may be positive or negative.
3. Obtain an length- $N$  signal  $g[n]$  by extracting the main period of  $\tilde{x}[n - m]$ .

$$g[n] = \begin{cases} \tilde{x}[n - m], & n = 0, 1, \dots, N - 1 \\ 0, & \text{otherwise} \end{cases}$$

The resulting signal  $g[n]$  is a **circularly shifted** version of  $x[n]$ ,  $g[n] = x[n - m]_{\text{mod } N}$

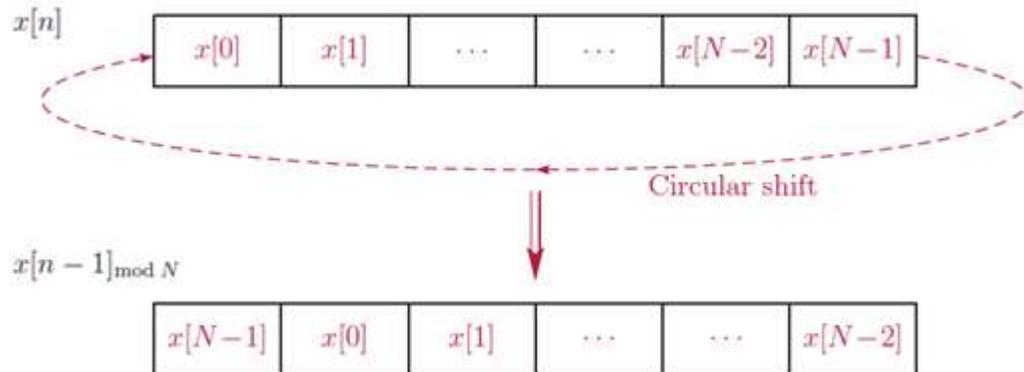
- For the time reversal operation:

$$\tilde{x}[n - m] \rightarrow \tilde{x}[-n] \text{ and } g[n] = x[n - m]_{\text{mod } N} \rightarrow g[n] = x[-n]_{\text{mod } N}$$

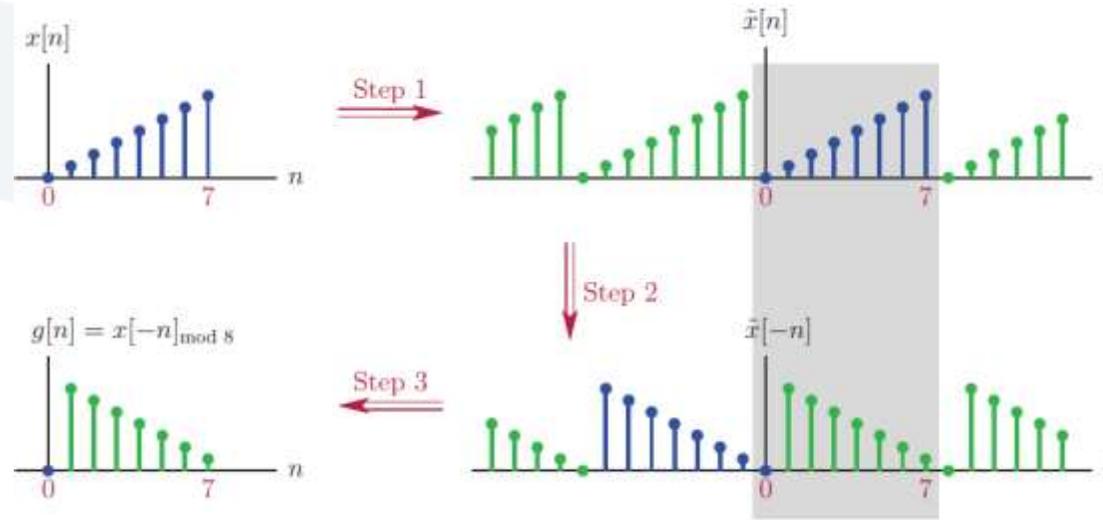


*Circular shift to the right by two samples*

*Circular shift to the left by three samples*



*Circular shifting a length-N signal*



*Circular time reversal of a length-8 signal*

- A length- $N$  signal  $x[n]$  is **circularly conjugate symmetric** if it satisfies  $x^*[n] = x[-n]_{\text{mod } N}$  or **circularly conjugate antisymmetric** if it satisfies  $x^*[n] = -x[-n]_{\text{mod } N}$
- Every signal can be decomposed into two components such that one is circularly conjugate symmetric  $x_E[n]$  and the other is circularly conjugate antisymmetric  $x_O[n]$ :  $x[n] = x_E[n] + x_O[n]$ .



$$x_E[n] = \frac{1}{2} \{x[n] + x^*[-n]_{\text{mod } N}\}, \quad x_O[n] = \frac{1}{2} \{x[n] - x^*[-n]_{\text{mod } N}\}$$

## Linearity

$$x_1[n] \xleftrightarrow{DFT} X_1[k] \quad \text{and} \quad x_2[n] \xleftrightarrow{DFT} X_2[k]$$

$$\Rightarrow \alpha_1 x_1[n] + \alpha_2 x_2[n] \xleftrightarrow{DFT} \alpha_1 X_1[k] + \alpha_2 X_2[k]$$

## Time shifting

$$x[n] \xleftrightarrow{DFT} X[k] \Rightarrow x[n - m]_{\text{mod } N} \xleftrightarrow{DFT} X[k] e^{-j(2\pi/N)km}$$

## Time reversal

$$x[n] \xleftrightarrow{DFT} X[k] \Rightarrow x[-n]_{\text{mod } N} \xleftrightarrow{DFT} X[-k]_{\text{mod } N}$$

## Conjugation property

$$x[n] \xleftrightarrow{DFT} X[k] \Rightarrow x^*[n] \xleftrightarrow{DFT} X^*[-k]_{\text{mod } N}$$

## Symmetry of the DFT

$$x[n]: \text{Real}, \text{Im}\{x[n]\} = 0 \Rightarrow X^*[k] = X[-k]_{\text{mod } N}$$

$$x[n]: \text{Imag}, \text{Re}\{x[n]\} = 0 \Rightarrow X^*[k] = -X[-k]_{\text{mod } N}$$

$$x^*[n] = x[-n]_{\text{mod } N} \Rightarrow X[k]: \text{Real} \quad x^*[n] = -x[-n]_{\text{mod } N} \Rightarrow X[k]: \text{Imag}$$



**Frequency shifting**  $x[n] \xleftrightarrow{DFT} X[k] \Rightarrow x[n]e^{j(2\pi/N)mn} \xleftrightarrow{DFT} X[k - m]_{\text{mod } N}$

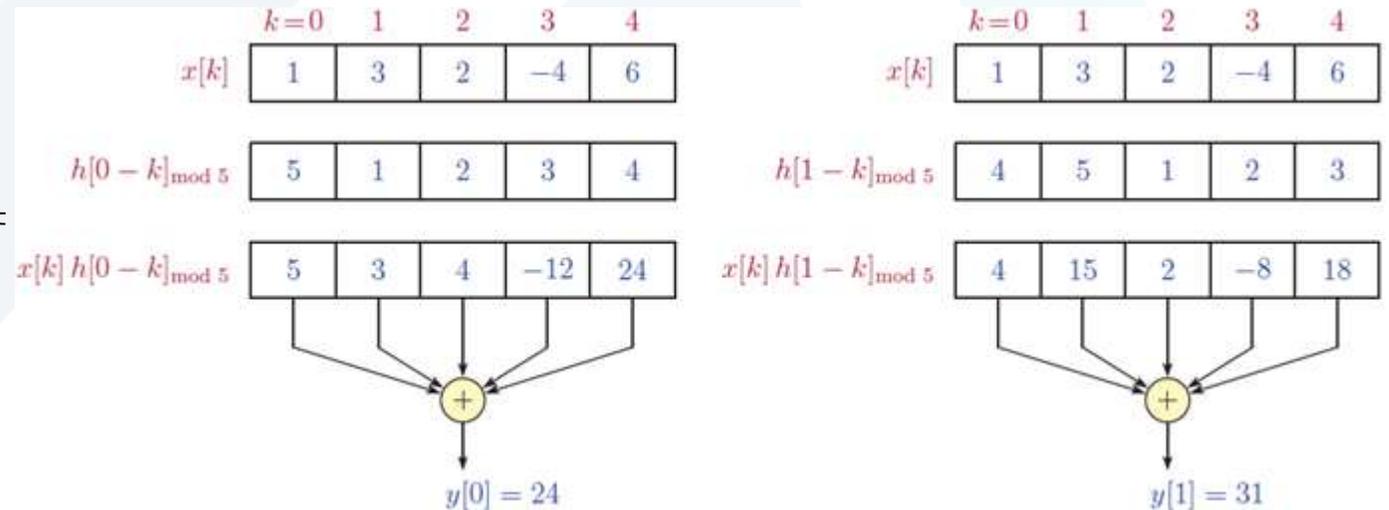
**Circular convolution**  $y[n] = x[n] \otimes h[n] = \sum_{k=0}^{N-1} x[k]h[n - k]_{\text{mod } N}, \quad n = 0, 1, \dots, N - 1$

▪ **Example 12:** Circular convolution of two signals

$x[n] = \{1, 3, 2, -4, 6\}$        $h[n] = \{5, 4, 3, 2, 1\}$

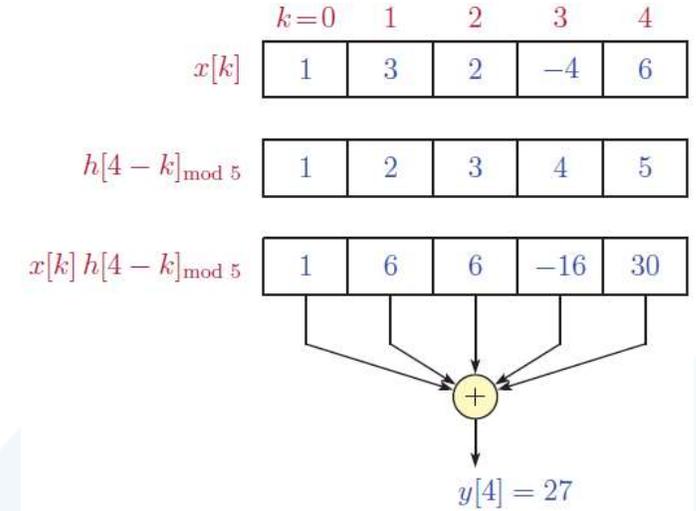
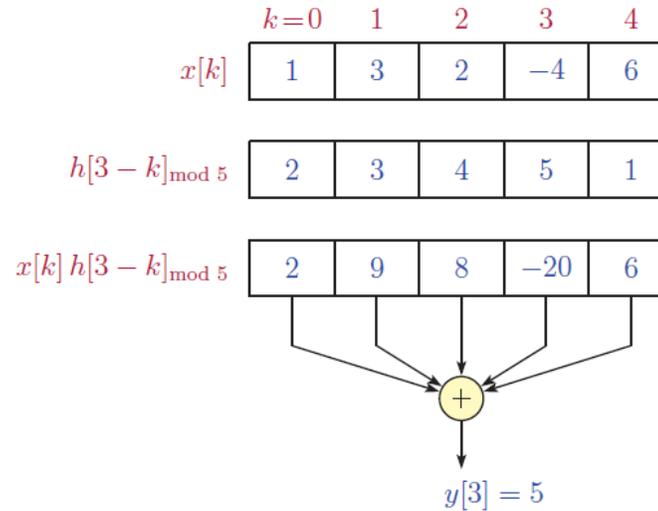
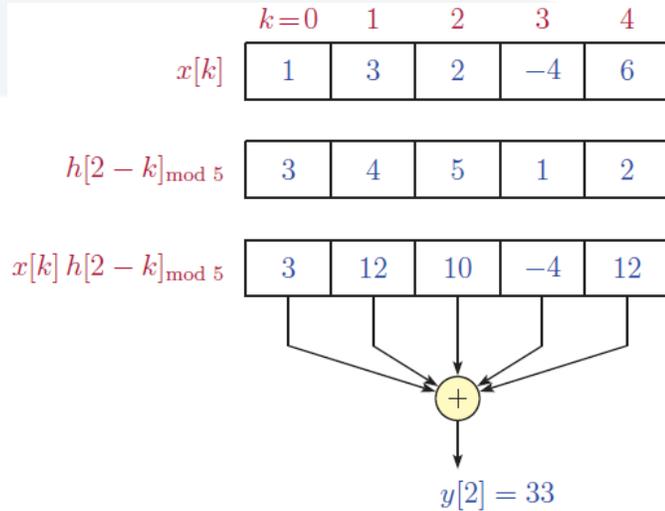
$y[n] = x[n] \otimes h[n]$   
 $= \sum_{k=0}^4 x[k]h[n - k]_{\text{mod } 5}, \quad n = 0, \dots, 4$

$y[n] = \{24, 31, 33, 5, 27\}$





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- The **circular** convolution property of the discrete Fourier transform:

$$x[n] \xleftrightarrow{DFT} X[k] \quad \text{and} \quad h[n] \xleftrightarrow{DFT} H[k]$$

$$\Rightarrow x[n] \otimes h[n] \xleftrightarrow{DFT} X[k] H[k]$$

- Example 13:** Circular convolution through DFT  
Verify the circular convolution property of example 12



$k$	$X[k]$	$H[k]$	$Y[k]$
0	$8.0000 + j 0.0000$	$15.0000 + j 0.0000$	$120.0000 + j 0.0000$
1	$5.3992 + j 0.6735$	$2.5000 + j 3.4410$	$11.1803 + j 20.2622$
2	$-6.8992 - j 7.4697$	$2.5000 + j 0.8123$	$-11.1803 - j 24.2784$
3	$-6.8992 + j 7.4697$	$2.5000 - j 0.8123$	$-11.1803 + j 24.2784$
4	$5.3992 - j 0.6735$	$2.5000 - j 3.4410$	$11.1803 - j 20.2622$

$$Y[k] = X[k] H[k]$$

Obtaining circular convolution  $y[n] = x[n] \otimes h[n]$

1. Compute the DFTs:  $X[k] = \text{DFT}\{x[n]\}$ , and  $H[k] = \text{DFT}\{h[n]\}$ .
  2. Multiply the two DFTs to obtain  $Y[k]$ :  $Y[k] = X[k] H[k]$ .
  3. Compute  $y[n]$  through inverse DFT:  $y[n] = \text{DFT}^{-1}\{Y[k]\}$ .
- The output signal of a DTLTI system is equal to the **linear convolution** of its impulse response with the input signal.



▪ **Example 14:** Linear vs. circular convolution

$$x[n] = \{ \underset{\uparrow}{1}, 3, 2, -4, 6 \} \quad h[n] = \{ \underset{\uparrow}{5}, 4, 3, 2, 1 \}$$

$$y[n] = x[n] \otimes h[n] = \{ \underset{\uparrow}{24}, 31, 33, 5, 27 \}$$

$$y_l[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \{ 5, 19, 25, -1, 27, 19, 12, 8, 6 \} \text{ linear convolution}$$

How does  $y_l[n]$  relate to  $y[n]$ ?

- The most obvious difference between the two results  $y_l[n]$  and  $y[n]$  is the length of each (9 vs. 5).

$n=0$	1	2	3	4	
$y_l[0]$	$y_l[1]$	$y_l[2]$	$y_l[3]$	$y_l[4]$	
$y_l[5]$	$y_l[6]$	$y_l[7]$	$y_l[8]$		
Totals:	$y[0]$	$y[1]$	$y[2]$	$y[3]$	$y[4]$

$n=0$	1	2	3	4	
5	19	25	-1	27	
19	12	8	6		
Totals:	24	31	33	5	27

*Relationship between linear and circular convolution*



Computing linear convolution using the DFT:

Given two finite length signals with  $N_x$  and  $N_h$  samples respectively:

$$x[n], n = 0, \dots, N_x - 1 \text{ and } h[n], n = 0, \dots, N_h - 1$$

1. Anticipating the length of the linear convolution result to be  $N_y = N_x + N_h - 1$ , extend the length of each signal to  $N_y$  through zero padding:

$$x_p[n] = \begin{cases} x[n], & n = 0, \dots, N_x - 1 \\ 0, & n = N_x, \dots, N_y - 1 \end{cases} \quad h_p[n] = \begin{cases} h[n], & n = 0, \dots, N_h - 1 \\ 0, & n = N_h, \dots, N_y - 1 \end{cases}$$

2. Compute the DFTs of the zero-padded signals  $x_p[n]$  and  $h_p[n]$ :

$$X_p[k] = \text{DFT}\{x_p[n]\}, \text{ and } H_p[k] = \text{DFT}\{h_p[n]\}$$

3. Multiply the two DFTs to obtain  $Y_p[k]$ :  $Y_p[k] = X_p[k] H_p[k]$ .

4. Compute  $y_p[n]$  through inverse DFT:  $y_p[n] = \text{DFT}^{-1}\{Y_p[k]\}$ :

$$y_p[n] = y_l[n] \text{ for } n = 0, \dots, N_y - 1$$