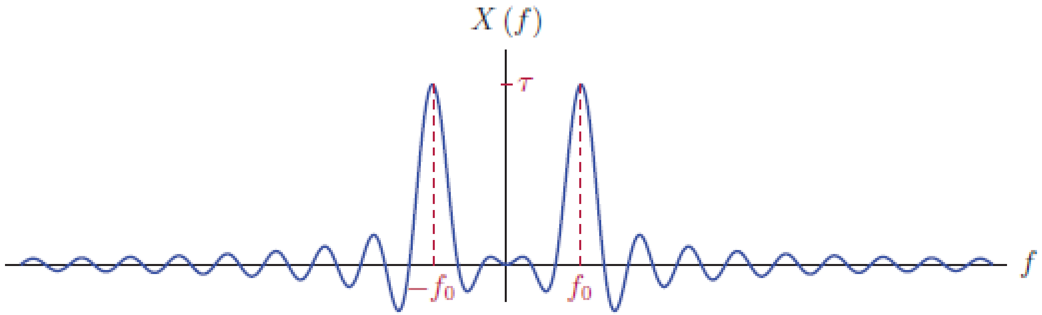
CECC507: Signals and Systems

Lecture Notes 5: Fourier Analysis for Continuous Time Signals and Systems



Ramez Koudsieh, Ph.D.
Faculty of Engineering
Department of Mechatronics
Manara University

Chapter 4

Fourier Analysis for Continuous Time Signals and Systems

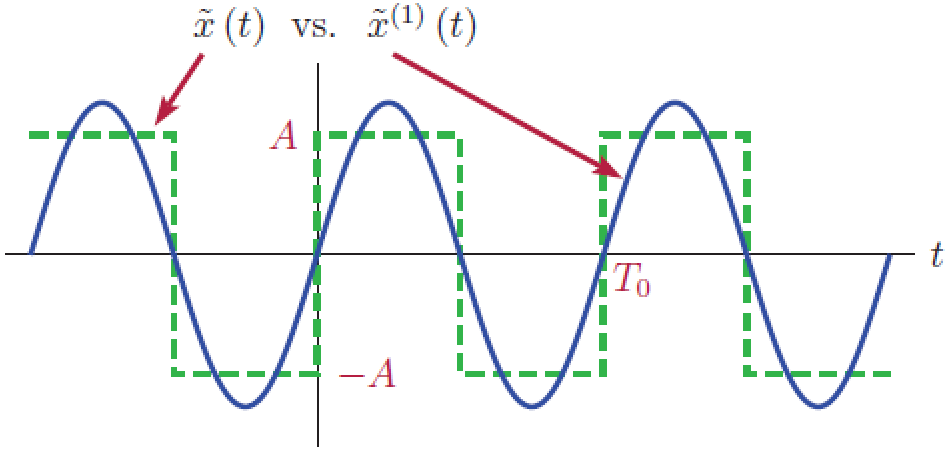
1. Introduction

- Fourier analysis leads to the frequency spectrum of a continuous-time signal.
- The frequency spectrum displays the various sinusoidal components that make up the signal.
- In the frequency domain linear systems are described by linear algebraic equations that can be easily solved, in contrast to the time-domain representation, where they are described by linear differential equations.
- A weighted summation of Sines and Cosines of different frequencies can be used to represent periodic (Fourier Series), or non-periodic (Fourier Transform) functions.

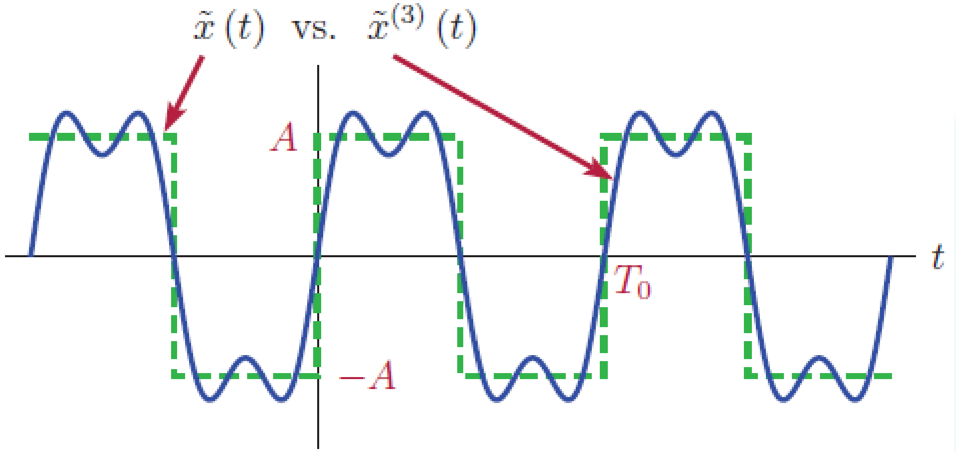
2. Analysis of Periodic Continuous-Time Signals

- We will study methods of expressing periodic continuous-time signals in two different but equivalent formats, namely the trigonometric Fourier series (TFS) and the exponential Fourier series (EFS).

Approximating a periodic signal with trigonometric functions



$\tilde{x}^{(1)}(t) = b_1 \sin(\omega_0 t)$



$\tilde{x}^{(3)}(t) = b_1 \sin(\omega_0 t) + b_2 \sin(2\omega_0 t) + b_3 \sin(3\omega_0 t)$

Trigonometric Fourier series (TFS)

$$\tilde{x}(t) = a_0 + a_1\cos(\omega_0 t) + a_2\cos(2\omega_0 t) + \ldots + a_k\cos(k\omega_0 t) + \ldots$$
$$+ b_1\sin(\omega_0 t) + b_2\sin(2\omega_0 t) + \ldots + b_k\sin(k\omega_0 t) + \ldots$$

- In a compact notation (trigonometric Fourier Series TFS of the periodic signal $\tilde{x}(t)$):

$$\tilde{x}(t) = a_0 + \sum_{k=1}^{\infty} a_k\cos(k\omega_0 t) + \sum_{k=1}^{\infty} b_k\sin(k\omega_0 t)$$

  where $\omega_0 = 2\pi f_0$ is the fundamental frequency in rad/s.

- We call the frequencies that are integer multiples of the fundamental frequency the harmonics.

- The frequencies $2\omega_0$, $3\omega_0$, ..., $k\omega_0$ are the second, the third, and the k-th harmonics of the fundamental frequency respectively.

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Trigonometric Fourier series (TFS)

1. Synthesis equation:

$$\tilde{x}(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(k\omega_0 t) + \sum_{k=1}^{\infty} b_k \sin(k\omega_0 t)$$

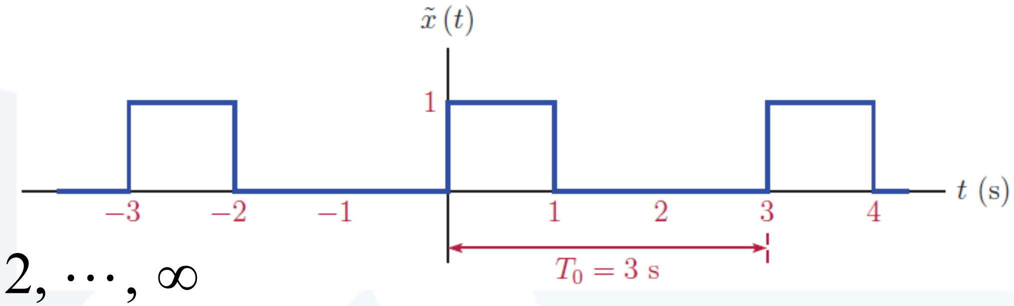
2. Analysis equation:

$$a_0 = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} \tilde{x}(t)\,dt \quad \text{(dc component)}$$

$$a_k = \frac{2}{T_0}\int_{t_0}^{t_0+T_0} \tilde{x}(t)\cos(k\omega_0 t)\,dt, \quad \text{for } k = 1, 2, \cdots, \infty$$

$$b_k = \frac{2}{T_0}\int_{t_0}^{t_0+T_0} \tilde{x}(t)\sin(k\omega_0 t)\,dt, \quad \text{for } k = 1, 2, \cdots, \infty$$

- **Example 1:** Trigonometric Fourier series of a periodic pulse train



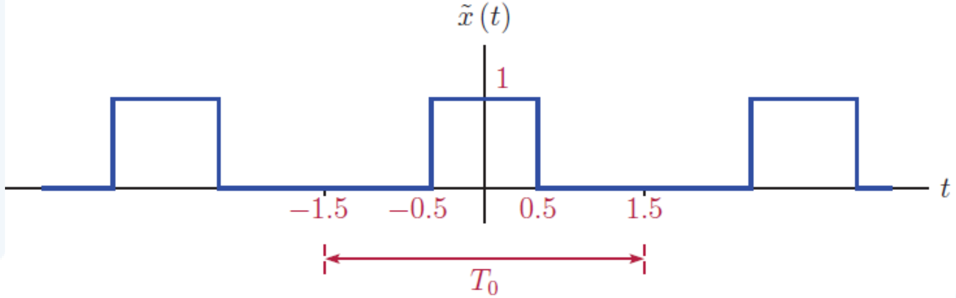
$$a_0 = \frac{1}{3}\int_0^1 (1)\,dt = \frac{1}{3}$$

$$a_k = \frac{2}{3}\int_0^1 (1)\cos\left(\frac{2\pi kt}{3}\right)dt = \frac{\sin(2\pi k/3)}{\pi k}, \text{ for } k = 1, 2, \cdots, \infty$$

$$b_k = \frac{2}{3}\int_0^1 (1)\sin\left(\frac{2\pi kt}{3}\right)dt = \frac{1-\cos(2\pi k/3)}{\pi k}, \text{ for } k = 1, 2, \cdots, \infty$$

$$\tilde{x}(t) = \frac{1}{3} + \sum_{k=1}^{\infty}\frac{\sin(2\pi k/3)}{\pi k}\cos\left(\frac{2\pi kt}{3}\right) + \sum_{k=1}^{\infty}\frac{1-\cos(2\pi k/3)}{\pi k}\sin\left(\frac{2\pi kt}{3}\right)$$

- **Example 2:** Periodic pulse train



$$\tilde{x}(t) = \frac{1}{3} + \sum_{k=1}^{\infty}\frac{2\sin(\pi k/3)}{\pi k}\cos\left(\frac{2\pi kt}{3}\right)$$

Exponential Fourier series (EFS)

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

Single-tone signals:

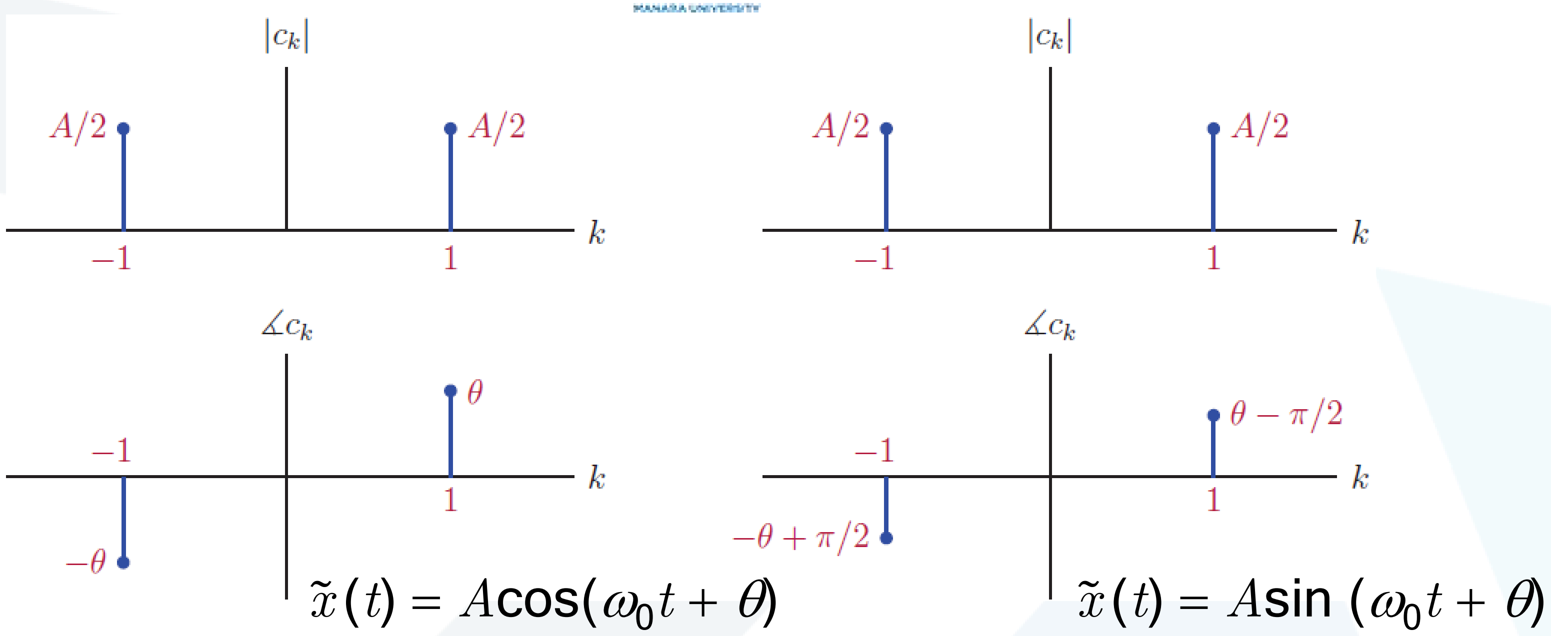
$$\tilde{x}(t) = A\cos(\omega_0 t + \theta) = \frac{A}{2} e^{j(\omega_0 t+\theta)} + \frac{A}{2} e^{-j(\omega_0 t+\theta)} = \frac{A}{2} e^{j\theta} e^{j\omega_0 t} + \frac{A}{2} e^{-j\theta} e^{-j\omega_0 t}$$

$$c_1 = \frac{A}{2} e^{j\theta}, \quad c_{-1} = \frac{A}{2} e^{-j\theta}, \quad \text{and} \quad c_k = 0 \text{ for all other } k$$

$$\tilde{x}(t) = A\sin(\omega_0 t + \theta) = \frac{A}{2} e^{j(\theta-\pi/2)} e^{j\omega_0 t} + \frac{A}{2} e^{-j(\theta-\pi/2)} e^{-j\omega_0 t}$$

$$c_1 = \frac{A}{2} e^{j(\theta-\pi/2)}, \quad c_{-1} = \frac{A}{2} e^{-j(\theta-\pi/2)}, \quad \text{and} \quad c_k = 0 \text{ for all other } k$$

- The EFS representations of the two signals are shown graphically, in the form of a line spectrum.



The general case: $\tilde{x}(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} = a_0 + \sum_{k=1}^{\infty} a_k \cos(k\omega_0 t) + \sum_{k=1}^{\infty} b_k \sin(k\omega_0 t)$

$$c_0 = a_0$$

$$c_k + c_{-k} = a_k \quad \text{and} \quad j(c_k - c_{-k}) = b_k, \quad \text{for } k = 1, \cdots, \infty$$

$$c_k = \frac{1}{2}(a_k - jb_k) \quad \text{and} \quad c_{-k} = \frac{1}{2}(a_k + jb_k), \quad \text{for } k = 1, \cdots, \infty$$

Exponential Fourier series (EFS):

1. Synthesis equation: $\tilde{x}(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$

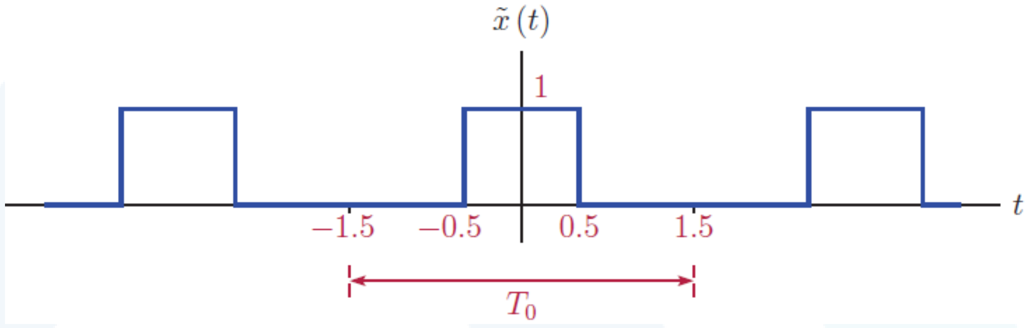
2. Analysis equation: $c_k = \dfrac{1}{T_0}\int_{t_0}^{t_0+T_0} \tilde{x}(t)\, e^{-jk\omega_0 t} dt$

- In general, the coefficients of the EFS representation of a periodic signal $\tilde{x}(t)$ are complex valued. They can be graphed in the form of a line spectrum if each coefficient is expressed in polar complex form: $c_k = |c_k| e^{j\theta_k}$

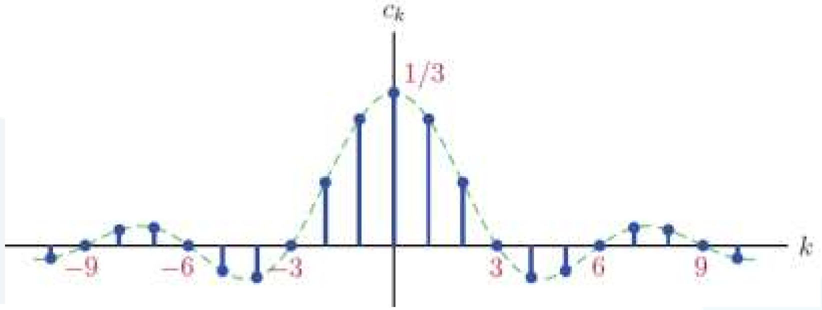
- Example 3: EFS for periodic pulse train

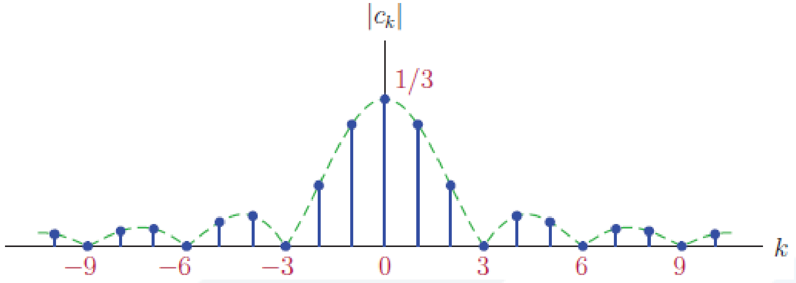
$$c_k = \frac{1}{3}\int_{-0.5}^{0.5} (1)\, e^{-j2\pi kt/3} dt = \frac{\sin(\pi k/3)}{\pi k}$$

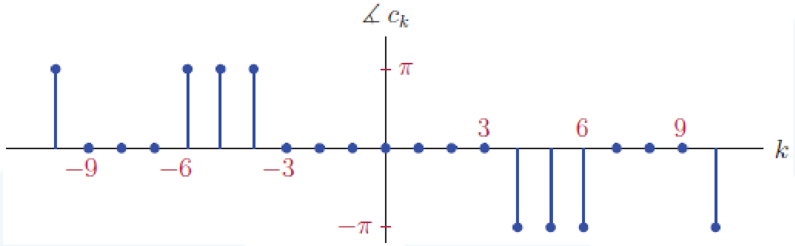
$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} \frac{\sin(\pi k/3)}{\pi k} e^{j2\pi kt/3}$$



- A line graph of the set of coefficients c_k is useful for illustrating the make-up of the signal $\tilde{x}(t)$ in terms of its harmonics.







Properties of Fourier series

Linearity

$$\alpha_1 \tilde{x}(t) + \alpha_2 \tilde{y}(t) = \sum_{k=-\infty}^{\infty} [\alpha_1 c_k + \alpha_2 d_k] e^{jk\omega_0 t}$$

Symmetry of Fourier series

$$\tilde{x}(t)\text{: real, } \mathrm{Im}\{\tilde{x}(t)\} = 0 \Rightarrow c_{-k} = c_k^*, \qquad \tilde{x}(t)\text{: imag, } \mathrm{Re}\{\tilde{x}(t)\} = 0 \Rightarrow c_{-k} = -c_k^*$$

3. Analysis of Non-Periodic Continuous-Time Signals

- Consider the non-periodic signal $x(t)$.

Fourier transform for continuous-time signals:

1. Synthesis equation: (Inverse transform)

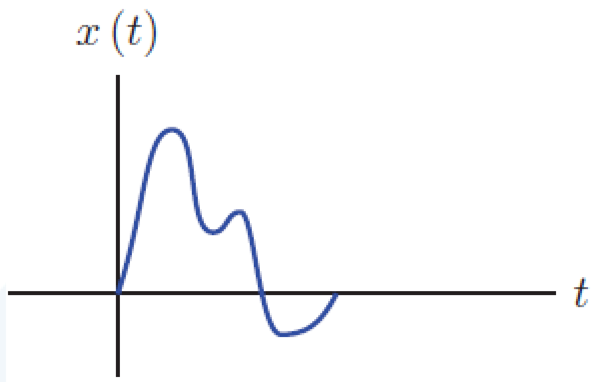
$$x(t) = \mathcal{F}^{-1}\{X(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t} d\omega$$

$$x(t) = \mathcal{F}^{-1}\{X(f)\} = \int_{-\infty}^{\infty} X(f)\, e^{j2\pi f t} df \quad \text{(using } f\text{)}$$

2. Analysis equation: (Forward transform)

$$X(\omega) = \mathcal{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t} dt$$

$$X(f) = \mathcal{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t} dt \quad \text{(using } f\text{)}$$



What frequencies are contained in this signal?

Fourier transforms of some signals

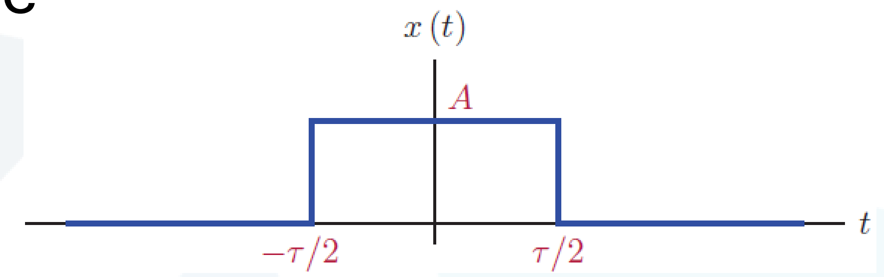
Example 4: Fourier transform of a rectangular pulse

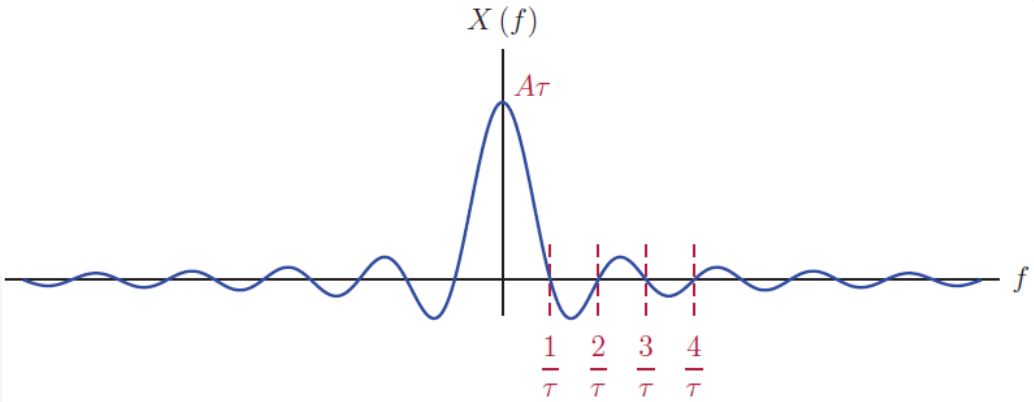
$$x(t) = A\Pi\left(\frac{t}{\tau}\right)$$

$$X(\omega) = \int_{-\tau/2}^{\tau/2} (A)\, e^{-j\omega t} dt = A \left.\frac{e^{-j\omega t}}{-j\omega}\right|_{-\tau/2}^{\tau/2} = \frac{2A}{\omega} \sin\left(\frac{\omega\tau}{2}\right)$$

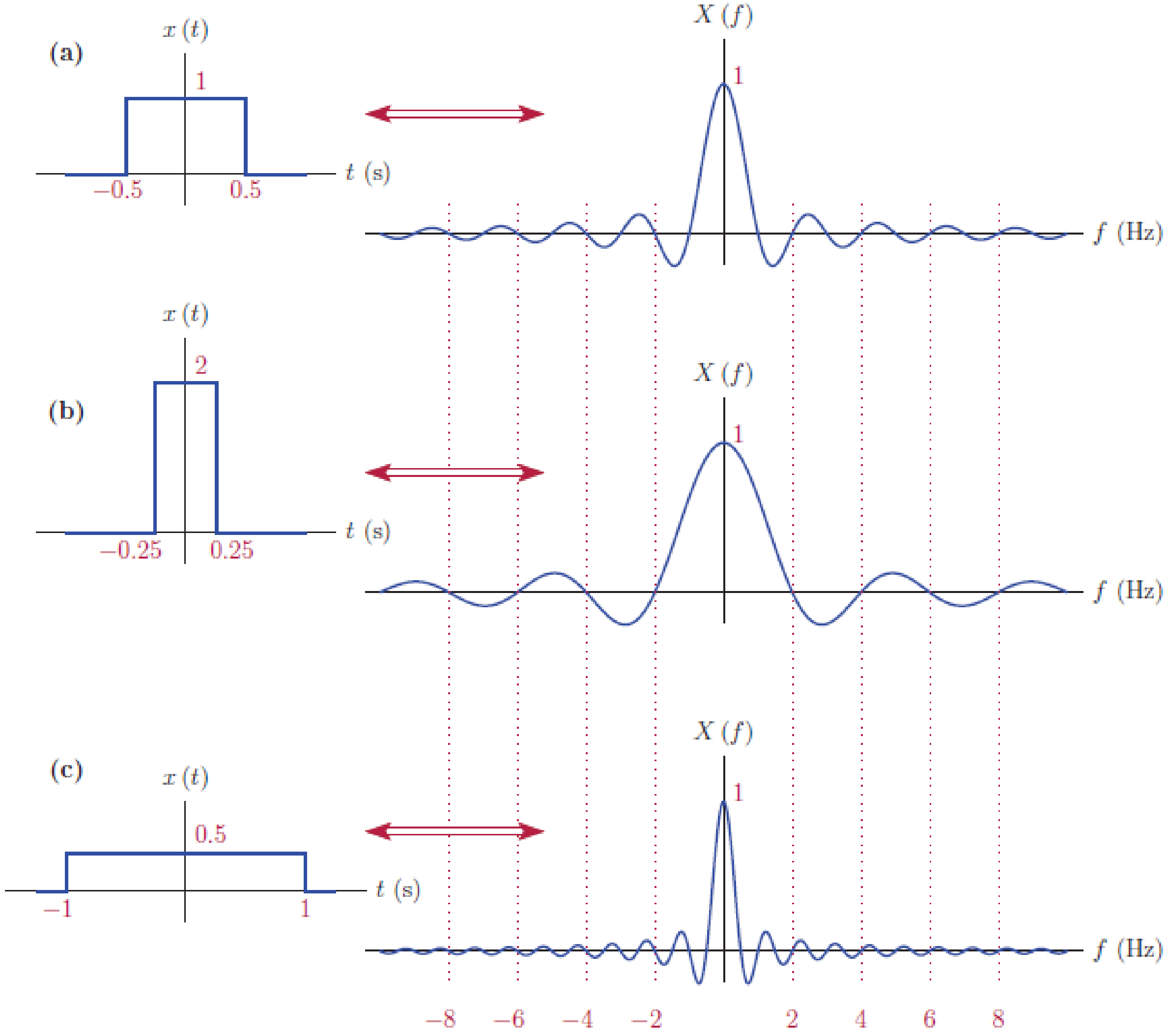
$$X(\omega) = A\tau \frac{\sin(\omega\tau/2)}{(\omega\tau/2)} = A\tau\, \text{sinc}\left(\frac{\omega\tau}{2\pi}\right)$$

$$X(f) = A\tau\, \text{sinc}(f\tau)$$





- Effects of changing the pulse width on the frequency spectrum:

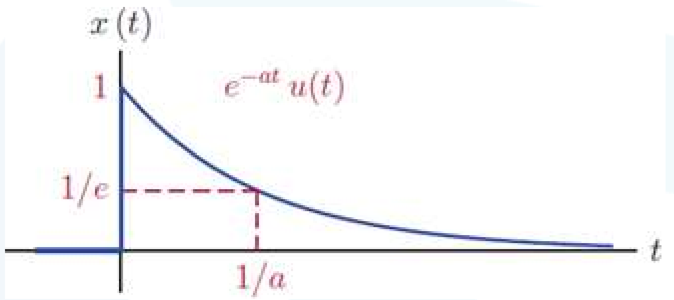

(a)
x (t)
1
−0.5
0.5
t (s)
X (f)
1
f (Hz)
(b)
x (t)
2
−0.25
0.25
t (s)
X (f)
1
f (Hz)
(c)
x (t)
0.5
−1
1
t (s)
X (f)
1
f (Hz)
−8 −6 −4 −2 2 4 6 8

Example 5: Transform of the unit-impulse function

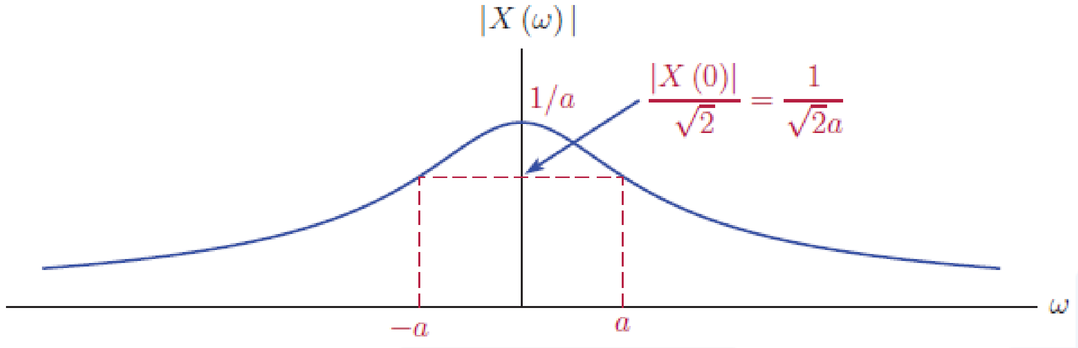
$$\mathcal{F}\{\delta(t)\} = \int_{-\infty}^{\infty} \delta(t)\, e^{-j\omega t} dt = \int_{-\infty}^{\infty} \delta(t) dt = 1$$

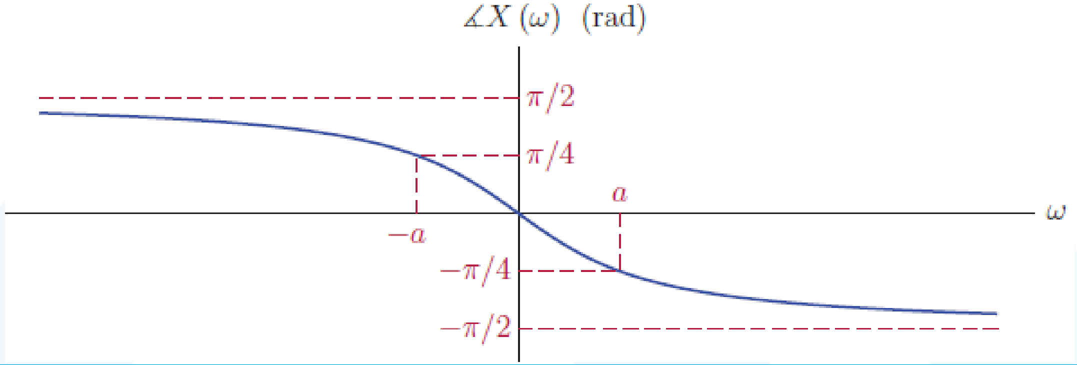
Example 6: FT of a right-sided exponential signal $x(t) = e^{-at}u(t),\ a > 0$

$$X(\omega) = \int_{-\infty}^{\infty} e^{-at}\, u(t) e^{-j\omega t} dt = \int_{0}^{\infty} e^{-at}\, e^{-j\omega t} dt = \frac{1}{a + j\omega}$$



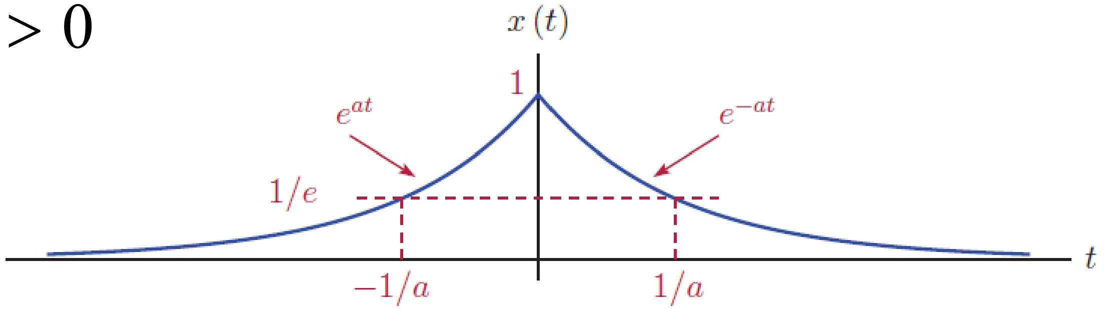
$$|X(\omega)| = \left|\frac{1}{a + j\omega}\right| = \frac{1}{\sqrt{a^2 + \omega^2}}, \qquad \theta(\omega) = -\tan^{-1}(\omega/a)$$

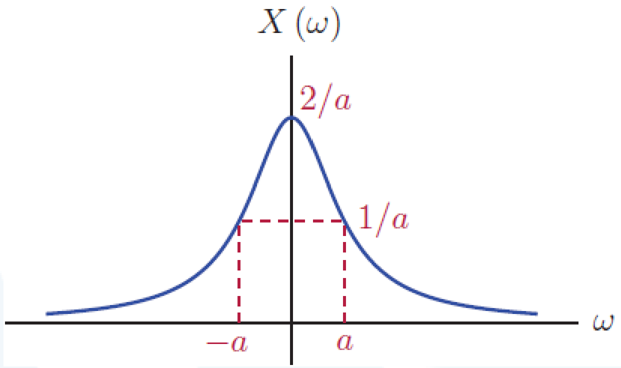




- **Example 7:** Fourier transform of a two-sided exponential signal

$$x(t) = e^{-a|t|},\ a > 0$$

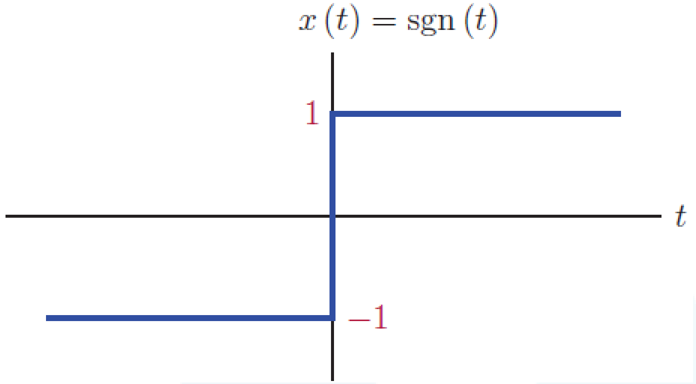




$$X(\omega) = \int_{-\infty}^{\infty} e^{-a|t|} e^{-j\omega t} dt = \int_{-\infty}^{0} e^{at} e^{-j\omega t} dt + \int_{0}^{\infty} e^{-at} e^{-j\omega t} dt = \frac{2a}{a^2 + \omega^2}$$

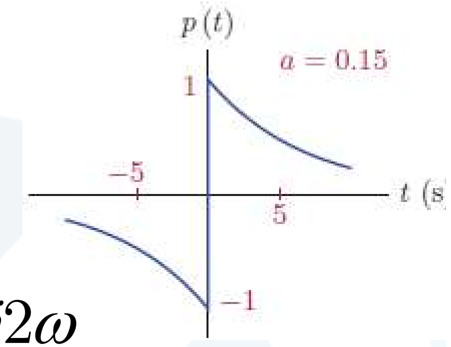
- **Example 8:** Fourier transform of the signum function

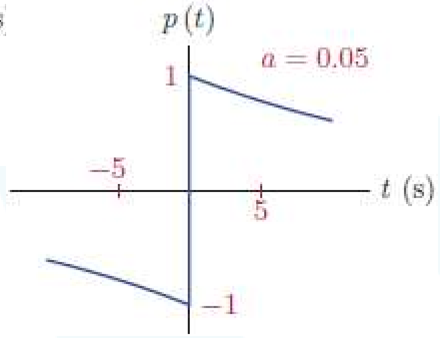
$$x(t) = \mathrm{sgn}(t) = \begin{cases} -1, & t < 0 \\ 1, & t > 0 \end{cases}$$



$$X(\omega) = \int_{-\infty}^{0} (-1)\, e^{-j\omega t} dt + \int_{0}^{\infty} (1)\, e^{-j\omega t} dt$$

The two integrals cannot be evaluated. Instead, we will define an intermediate signal $p(t)$ as:





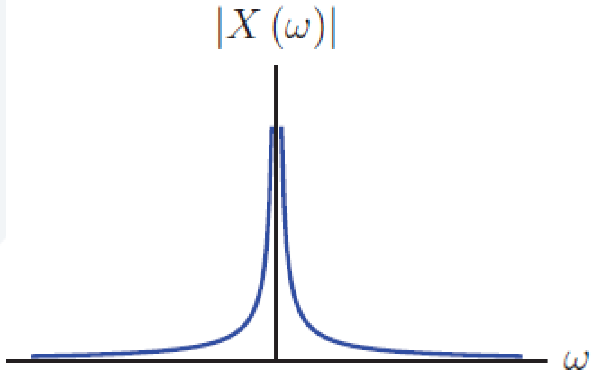
$$p(t) = \begin{cases} -e^{at}, & t < 0 \\ e^{-at}, & t > 0 \end{cases}, \text{ where } a \geq 0$$

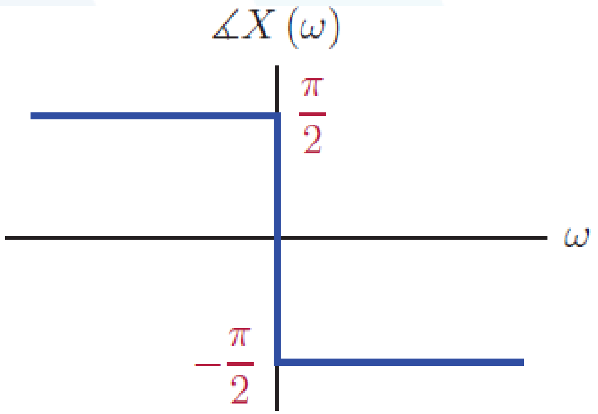
$$P(\omega) = \int_{-\infty}^{0} (-e^{at})\, e^{-j\omega t} dt + \int_{0}^{\infty} (e^{-at})\, e^{-j\omega t} dt = \frac{-j2\omega}{a^2 + \omega^2}$$

$$X(\omega) = \mathcal{F}\{\mathrm{sgn}(t)\} = \lim_{a \to 0} \frac{-j2\omega}{a^2 + \omega^2} = \frac{2}{j\omega}$$

$$|X(\omega)| = \frac{2}{|\omega|}$$

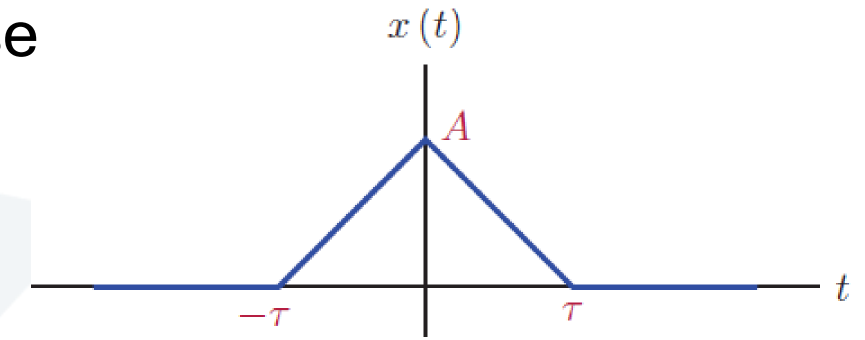
$$\theta(\omega) = \begin{cases} \frac{\pi}{2}, & \omega < 0 \\ -\frac{\pi}{2}, & \omega > 0 \end{cases}$$





- Example 9: Fourier transform of a triangular pulse

$$x(t) = A\Lambda\left(\frac{t}{\tau}\right) = \begin{cases} A + At/\tau, & -\tau < t < 0 \\ A - At/\tau, & 0 < t < \tau \\ 0, & |t| \geq \tau \end{cases}$$

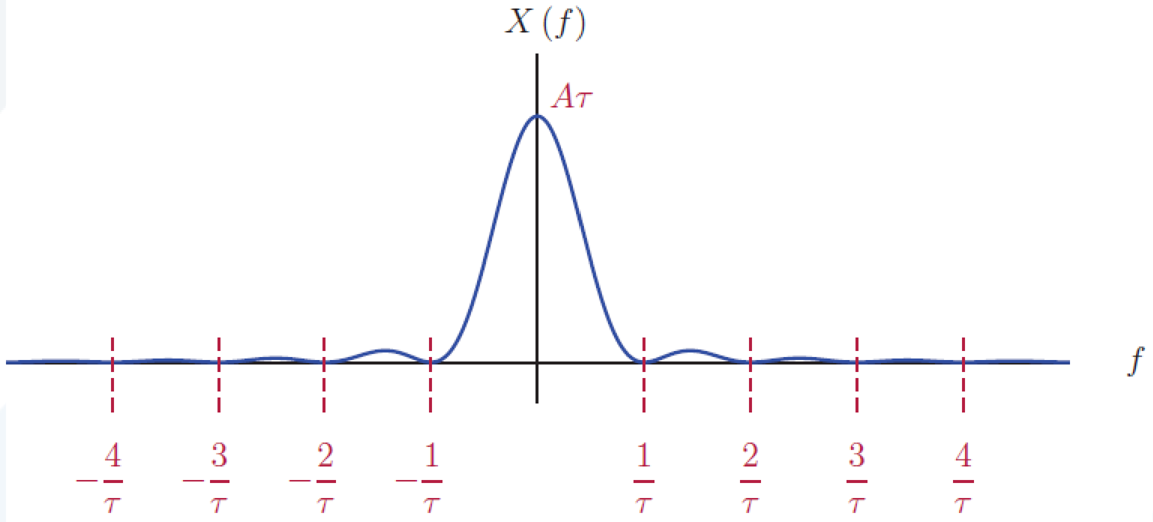


$$X(\omega) = \int_{-\tau}^{0} (A + At/\tau)\, e^{-j\omega t} dt + \int_{0}^{\tau} (A - At/\tau)\, e^{-j\omega t} dt = \frac{2A}{\omega^2 \tau}[1 - \cos(\omega\tau)]$$

$$\operatorname{sinc}\left(\frac{\omega\tau}{2\pi}\right) = \frac{\sin(\omega\tau/2)}{\omega\tau/2} = \frac{2}{\omega\tau}\sin\left(\frac{\omega\tau}{2}\right)$$

$$X(\omega) = A\tau \operatorname{sinc}^2\left(\frac{\omega\tau}{2\pi}\right)$$

$$X(f) = A\tau \operatorname{sinc}^2(f\tau)$$



Properties of Fourier transform

Linearity of the Fourier transform: $\mathcal{F}\{\alpha_1 x(t) + \alpha_2 y(t)\} = \alpha_1 \mathcal{F}\{x(t)\} + \alpha_2 \mathcal{F}\{y(t)\}$

Duality property: $x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega) \quad \Rightarrow \quad X(t) \overset{\mathcal{F}}{\longleftrightarrow} 2\pi x(-\omega)$

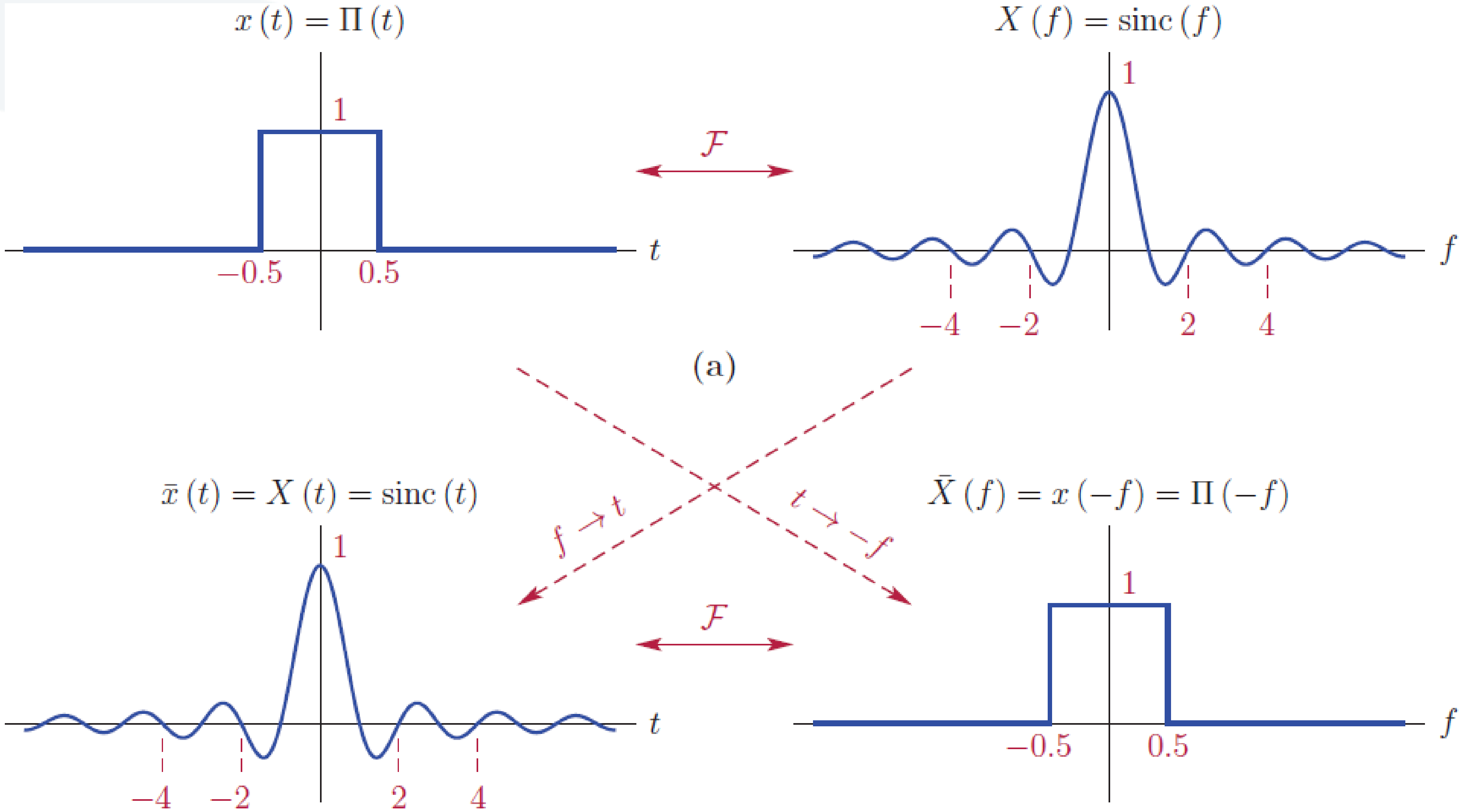
Duality property (using f): $x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(f) \quad \Rightarrow \quad X(t) \overset{\mathcal{F}}{\longleftrightarrow} x(-f)$

- Example 10: Fourier transform of the sinc function

$$\mathcal{F}\left\{\frac{1}{2\pi}\Pi\left(\frac{t}{2\pi}\right)\right\} = \text{sinc}(\omega) \Rightarrow$$

$$\mathcal{F}\{\text{sinc}(t)\} = \Pi\left(\frac{-\omega}{2\pi}\right) = \Pi\left(\frac{\omega}{2\pi}\right)$$

$$\mathcal{F}\{\text{sinc}(t)\} = \Pi(f)$$



$x(t) = \Pi(t)$
$X(f) = \text{sinc}(f)$
$\mathcal{F}$
(a)
$\bar{x}(t) = X(t) = \text{sinc}(t)$
$\bar{X}(f) = x(-f) = \Pi(-f)$
$f \to t$
$t \to -f$
$\mathcal{F}$
1
−0.5
0.5
t
f
−4
−2
2
4

- Example 11: Transform of a constant-amplitude signal

  $F\{\delta(t)\} = 1$, all ω $\Rightarrow$ $F\{1\} = 2\pi\delta(-\omega) = 2\pi\delta(\omega)$, $F\{1\} = \delta(f)$ (duality)

- Example 12: Fourier transform of the unit-step function

  $X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t} dt = \int_{0}^{\infty} e^{-j\omega t} dt$ could not be evaluated

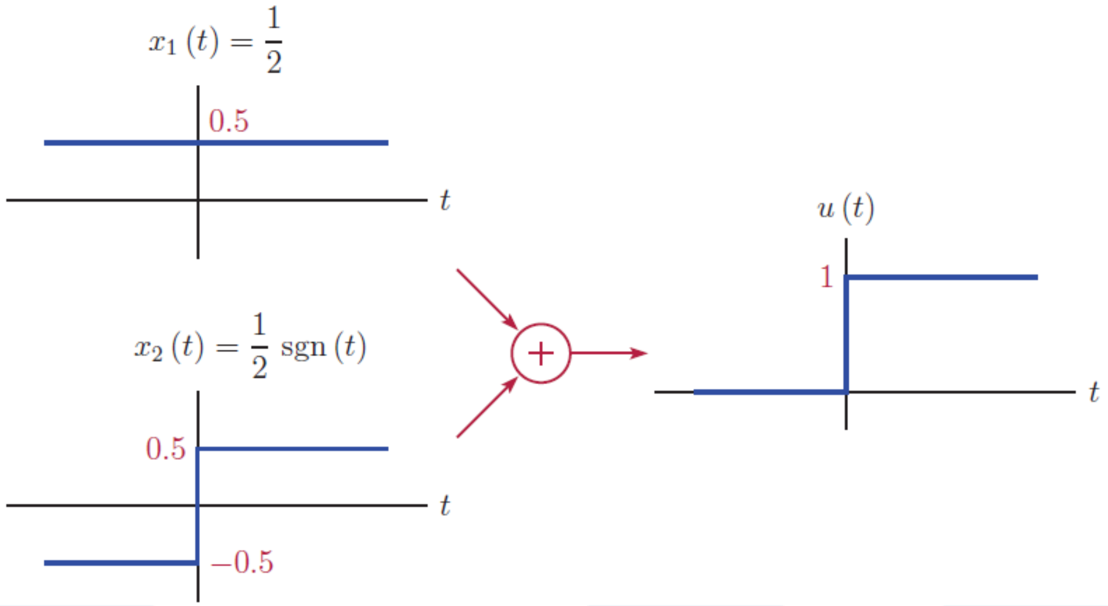
  $u(t) = ½ + ½\ \text{sgn}(t)$

  $\mathcal{F}\{u(t)\} = \mathcal{F}\{½ + ½\ \text{sgn}(t)\}$

  $= ½F\{1\} + ½\mathcal{F}\{\text{sgn}(t)\}$

  $\mathcal{F}\{u(t)\} = \pi\,\delta(\omega) + \dfrac{1}{j\omega}$

  $\mathcal{F}\{u(t)\} = \dfrac{1}{2}\delta(f) + \dfrac{1}{j2\pi f}$



Symmetry of the Fourier transform

$$x(t)\text{: real, } \mathrm{Im}\{x(t)\} = 0 \Rightarrow X^*(\omega) = X(-\omega)$$

$$\tilde{x}(t)\text{: imag, } \mathrm{Re}\{\tilde{x}(t)\} = 0 \Rightarrow X^*(\omega) = -X(-\omega)$$

Transforms of even and odd signals

- If the real-valued signal $x(t)$ is an even function of time, the resulting Fourier transform $X(\omega)$ is real-valued for all ω.

$$x(-t) = x(t), \text{ for all } t \Rightarrow \mathrm{Im}\{X(\omega)\} = 0, \text{ for all } \omega$$

- If the real-valued signal $x(t)$ has odd-symmetry, the resulting Fourier transform $X(\omega)$ is purely imaginary.

$$x(-t) = -x(t), \text{ for all } t \Rightarrow \mathrm{Re}\{X(\omega)\} = 0, \text{ for all } \omega$$

Time shifting $x(t) \xleftrightarrow{\mathcal{F}} X(\omega) \quad \Rightarrow \quad x(t-\tau) \xleftrightarrow{\mathcal{F}} X(\omega)e^{-j\omega\tau}$

Frequency shifting $x(t) \xleftrightarrow{\mathcal{F}} X(\omega) \quad \Rightarrow \quad x(t)e^{j\omega_0 t} \xleftrightarrow{\mathcal{F}} X(\omega-\omega_0)$

Modulation property $x(t) \xleftrightarrow{\mathcal{F}} X(\omega) \quad \Rightarrow$

$$x(t)\cos(\omega_0 t) \xleftrightarrow{\mathcal{F}} \tfrac{1}{2}\left[X(\omega-\omega_0)+X(\omega+\omega_0)\right]$$

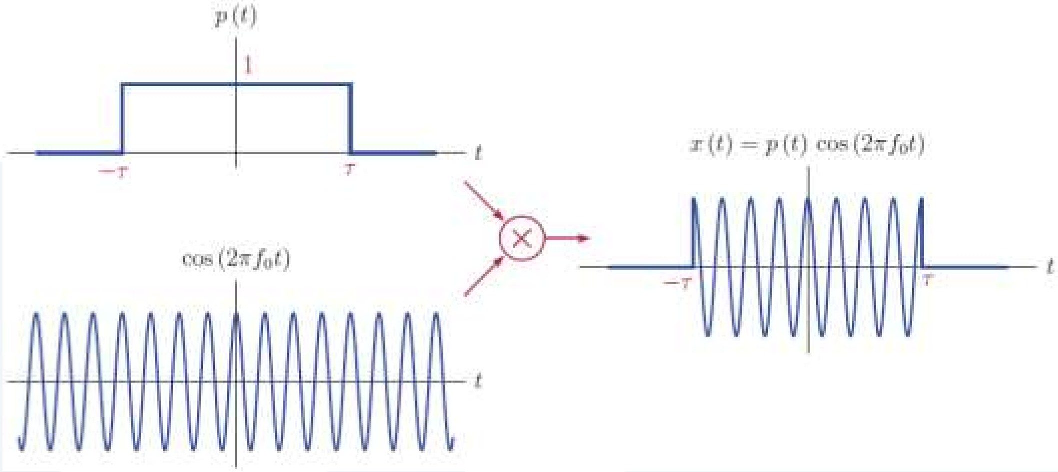
$$x(t)\sin(\omega_0 t) \xleftrightarrow{\mathcal{F}} \tfrac{1}{2}\left[X(\omega-\omega_0)e^{-j\pi/2}+X(\omega+\omega_0)e^{j\pi/2}\right]$$

- Example 13: Modulated pulse

$$x(t)=\begin{cases}\cos(2\pi f_0 t), & |t|<\tau \\ 0, & |t|>\tau\end{cases}$$

Using $p(t)$, the signal $x(t)$ can be expressed as $x(t)=p(t)\cos(2\pi f_0 t)$

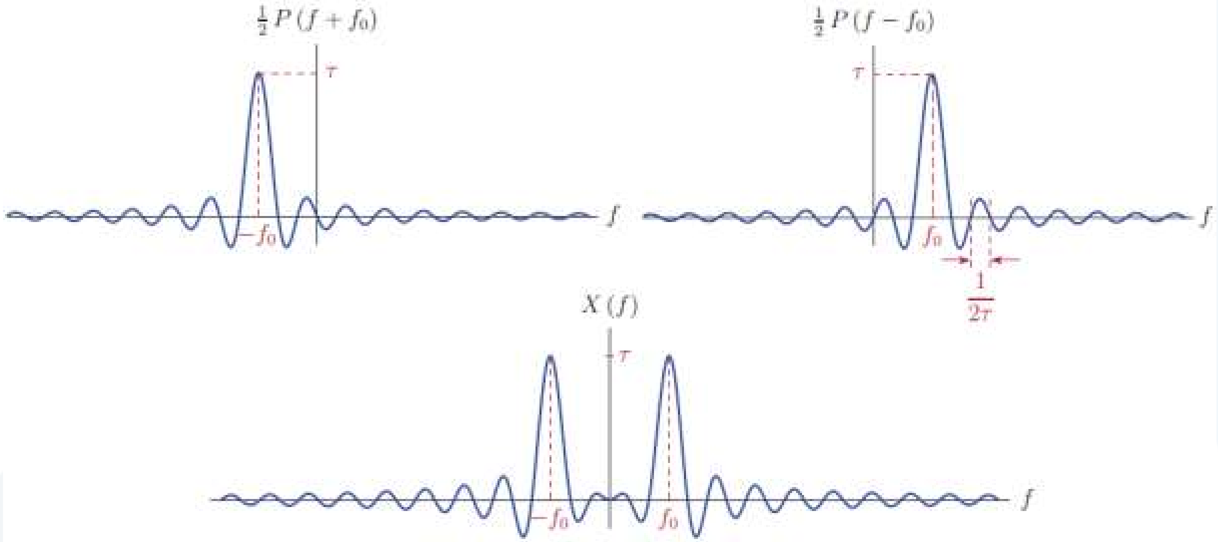
where $p(t) = \Pi\left(\frac{t}{2\tau}\right)$



$$P(f) = 2\tau\,\mathrm{sinc}\,(2\tau f)$$

$$X(f) = \tfrac{1}{2}\left[P(f - f_0) + P(f + f_0)\right]$$

$$= \tau\,\mathrm{sinc}\,(2\tau(f + f_0)) + \tau\,\mathrm{sinc}\,(2\tau(f - f_0))$$



Time and frequency scaling

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega) \quad \Rightarrow \quad x(at) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$$

The parameter a is any non-zero and real-valued constant.

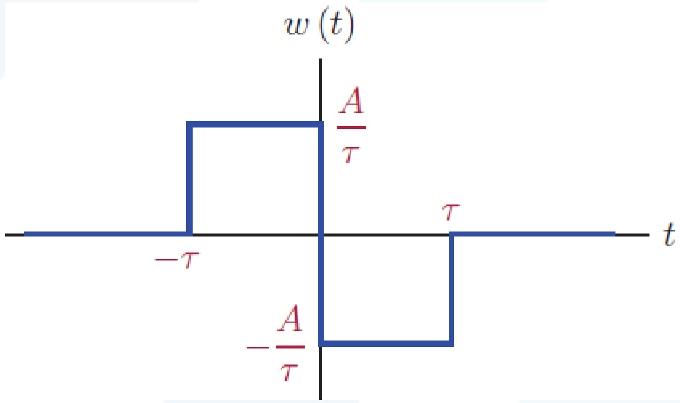
Differentiation in the time domain

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega) \Rightarrow \frac{d^n}{dt^n}[x(t)] \overset{\mathcal{F}}{\longleftrightarrow} (j\omega)^n X(\omega), \quad \frac{d^n}{dt^n}[x(t)] \overset{\mathcal{F}}{\longleftrightarrow} (j2\pi f)^n X(f)$$

- Example 14: Triangular pulse revisited

$$x(t) = A\Lambda(t/\tau)$$

$$w(t) = \frac{dx(t)}{dt} = \frac{A}{\tau}\left[\Pi\left(\frac{t+\tau/2}{\tau}\right) - \Pi\left(\frac{t-\tau/2}{\tau}\right)\right]$$



$$W(f) = A\,\mathrm{sinc}(f\tau)e^{j2\pi f(\tau/2)} - A\,\mathrm{sinc}(f\tau)e^{-j2\pi f(\tau/2)} = 2jA\,\mathrm{sinc}(f\tau)\sin(\pi f\tau)$$

$$W(f) = (j2\pi f)X(f) \Rightarrow X(f) = \frac{W(f)}{j2\pi f} = \frac{2jA\,\mathrm{sinc}(f\tau)\sin(\pi f\tau)}{j2\pi f} = A\tau\,\mathrm{sinc}^2(f\tau)$$

Differentiation in the frequency domain

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega) \quad \Rightarrow \quad (-jt)^n x(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{d^n}{d\omega^n}[X(\omega)]$$

Convolution property

$$x_1(t) \overset{\mathcal{F}}{\longleftrightarrow} X_1(\omega) \quad \text{and} \quad x_2(t) \overset{\mathcal{F}}{\longleftrightarrow} X_2(\omega)$$

$$\Rightarrow \quad x_1(t) * x_2(t) \overset{\mathcal{F}}{\longleftrightarrow} X_1(\omega)X_2(\omega)$$

Multiplication of two signals

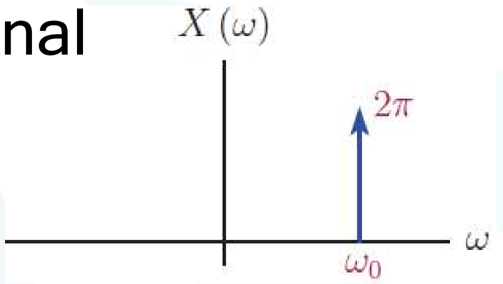
$$x_1(t) \overset{\mathcal{F}}{\longleftrightarrow} X_1(\omega) \quad \text{and} \quad x_2(t) \overset{\mathcal{F}}{\longleftrightarrow} X_2(\omega)$$

$$\Rightarrow x_1(t)x_2(t) \overset{\mathcal{F}}{\longleftrightarrow} \tfrac{1}{2\pi} X_1(\omega) * X_2(\omega), \quad x_1(t)x_2(t) \overset{\mathcal{F}}{\longleftrightarrow} X_1(f) * X_2(f)$$

Integration $x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega) \quad \Rightarrow \quad \int_{-\infty}^{t} x(\tau)d\tau \overset{\mathcal{F}}{\longleftrightarrow} \frac{X(\omega)}{j\omega} + \pi X(0)\delta(\omega)$

Applying Fourier transform to periodic signals

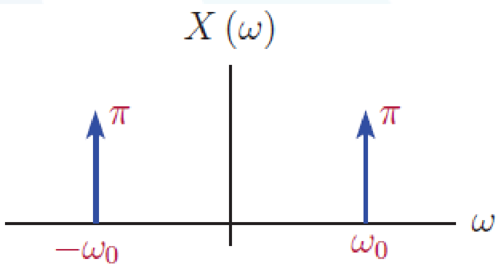
- **Example 15:** Fourier transform of complex exponential signal



$$x(t) = e^{j\omega_0 t}$$

$$\mathcal{F}\{1\} = 2\pi\delta(\omega) \quad \Rightarrow \quad \mathcal{F}(e^{j\omega_0 t}) = 2\pi\delta(\omega - \omega_0)$$

- **Example 16:** Fourier transform of sinusoidal signal



$$x(t) = \cos(\omega_0 t)$$

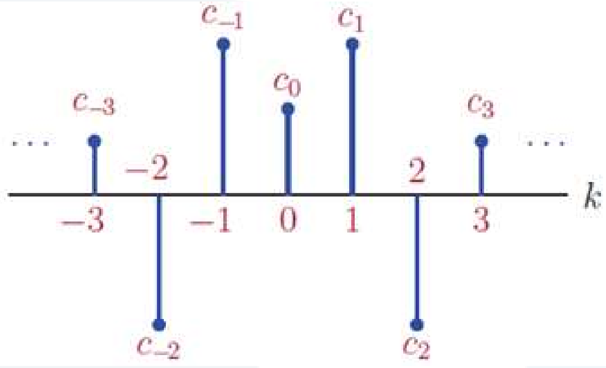
$$\mathcal{F}\{1\} = 2\pi\delta(\omega) \Rightarrow \mathcal{F}\{\cos(\omega_0 t)\} = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$

- The idea can be generalized to apply to any periodic continuous-time signal that has an EFS representation:

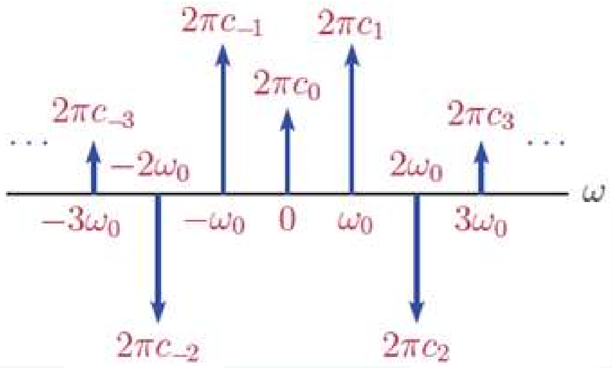
$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \Rightarrow X(\omega) = \int_{-\infty}^{\infty} \tilde{x}(t)\, e^{-j\omega t} dt = \int_{-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \right] e^{-j\omega t} dt$$

$$X(\omega) = \sum_{k=-\infty}^{\infty} c_k \left[\int_{-\infty}^{\infty} e^{jk\omega_0 t} e^{-j\omega t} dt \right]$$

$$= \sum_{k=-\infty}^{\infty} c_k \left[2\pi\delta(\omega - k\omega_0) \right]$$



EFS coefficients for a signal



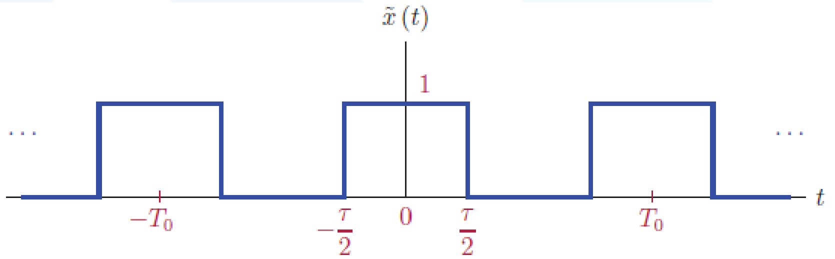
Fourier transform obtained

- Example 17: Fourier transform of periodic pulse train

Determine the FT of the periodic pulse train with duty cycle $d = \tau / T_0$

$c_k = d\,\text{sinc}\,(kd)$ $\quad X(\omega) = \sum_{k=-\infty}^{\infty} 2\pi d\, \text{sinc}(kd) \delta(\omega - k\omega_0)$

$\omega_0 = 1/T_0$ is the fundamental radian frequency.



4. Energy and Power in the Frequency Domain

Parseval's theorem

- For a periodic power signal $\tilde{x}(t)$ with period T_0 and EFS coefficients $\{c_k\}$:

$$\frac{1}{T_0}\int_{t_0}^{t_0+T_0}\left|\tilde{x}(t)\right|^2 dt = \sum_{k=-\infty}^{\infty}\left|c_k\right|^2$$

- For a non-periodic energy signal $x(t)$ with a Fourier transform $X(f)$:

$$\int_{-\infty}^{\infty}\left|x(t)\right|^2 dt = \int_{-\infty}^{\infty}\left|X(f)\right|^2 df$$

Energy and power spectral density

$$S_x(f) = \sum_{k=-\infty}^{\infty}\left|c_k\right|^2\delta(f-kf_0)$$

power spectral density of the signal $x(t)$

$$\int_{-\infty}^{\infty} S_x(f)\,df = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_x(\omega)\,d\omega = \sum_{k=-\infty}^{\infty} |c_k|^2$$

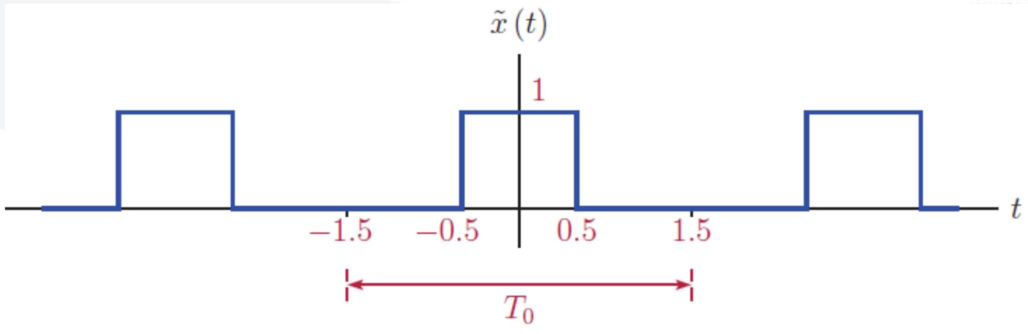
$$P_x \text{ in } (-f_0, f_0) = \int_{-f_0}^{f_0} S_x(f)\,df$$

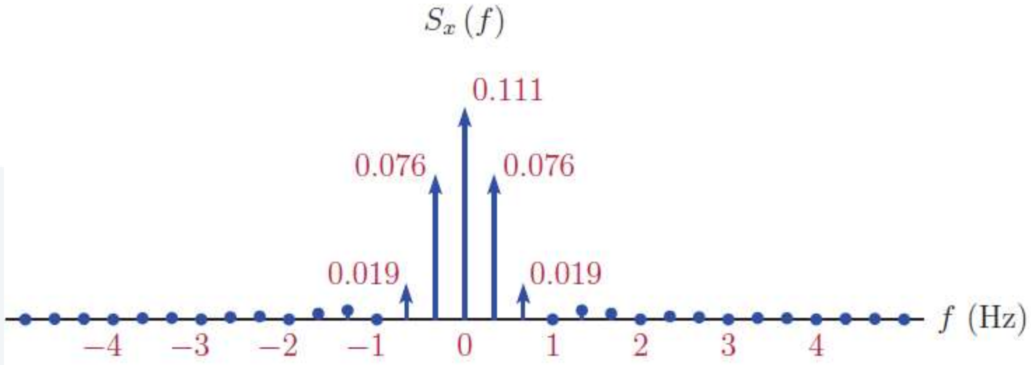
$G_x(f) = |X(f)|^2$ energy spectral density of the signal $x(t)$

$$\int_{-\infty}^{\infty} |x(t)|^2\,dt = \int_{-\infty}^{\infty} G_x(f)\,df = \frac{1}{2\pi}\int_{-\infty}^{\infty} G_x(\omega)\,d\omega$$

- Example 18: Power spectral density of a periodic pulse train
  Determine the power spectral density for $x(t)$. Also find the total power, the dc power, the power in the first three harmonics, and the power above 1 Hz.

$$c_k = \tfrac{1}{3}\operatorname{sinc}(k/3) \qquad S_x(f) = \sum_{k=-\infty}^{\infty} \left|\tfrac{1}{3}\operatorname{sinc}(k/3)\right|^2 \delta(f - k/3)$$





The total power in the signal $x(t)$: $\frac{1}{T_0}\int_{t_0}^{t_0+T_0} |\tilde{x}(t)|^2\, dt = \frac{1}{3}\int_{-0.5}^{0.5} (1)^2\, dt = \frac{1}{3}$

$$P_1 = |c_{-1}|^2 + |c_1|^2 = \frac{3}{2\pi^2} \approx 0.1520, \quad P_2 = |c_{-2}|^2 + |c_2|^2 = \frac{3}{8\pi^2} \approx 0.0380, \quad P_3 = 0$$

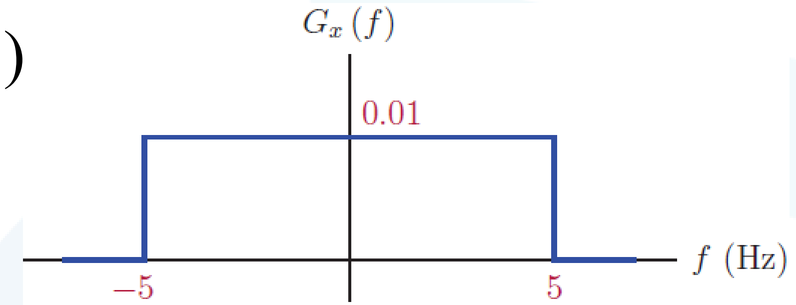
The third harmonic is at frequency $f = 1$ Hz. Thus, the power above 1 Hz:

$$P_{hf} = P_x - P_{dc} - P_1 - P_2 - P_3 = 0.3333 - 0.1111 - 0.1520 - 0.0380 - 0 = 0.0322$$

- Example 19: Energy spectral density of the sinc function

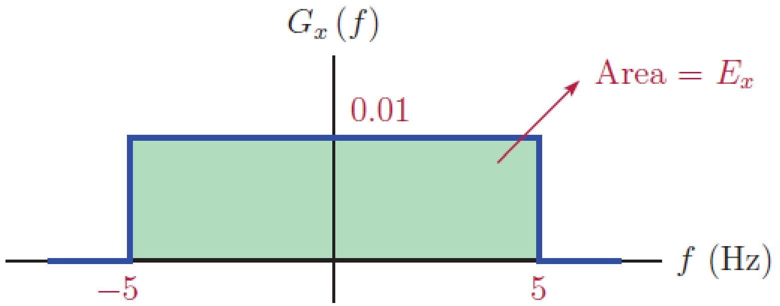
Determine the energy spectral density of $x(t) = \text{sinc}(10t)$. Afterwards, compute the total energy, and the energy in the sinc pulse at frequencies up to 3 Hz.

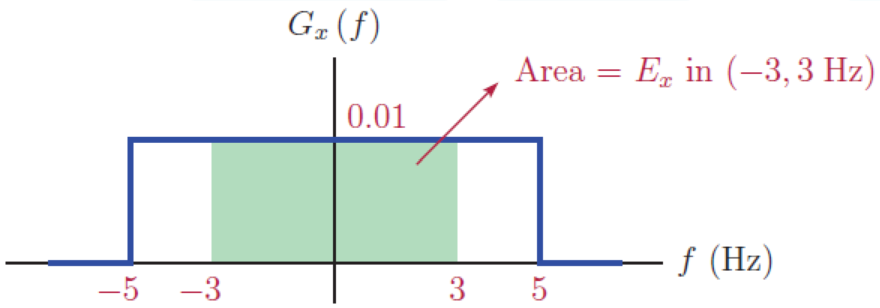
$$X(f) = \frac{1}{10}\Pi\left(\frac{f}{10}\right), \quad G_x(f) = |X(f)|^2 = \frac{1}{100}\text{sinc}^2\left(\frac{f}{10}\right)$$



$$E_x = \int_{-\infty}^{\infty} G_x(f)df = \int_{-5}^{5}\frac{1}{100}df = 0.1$$

$$E_x \text{ in } (-3, 3 \text{ Hz}) = \int_{-3}^{3} G_x(f)df = \int_{-3}^{3}\frac{1}{100}df = 0.06$$





Autocorrelation

- For an energy signal $x(t)$ the autocorrelation function is defined as

$$r_{xx}(\tau) = \int_{-\infty}^{\infty} x(t)x(t+\tau)dt$$

- For a periodic power signal $\tilde{x}(t)$ with period T_0, the corresponding definition of the autocorrelation function is:

$$\tilde{r}_{xx}(\tau) = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} \tilde{x}(t)\tilde{x}(t+\tau)dt$$

- The energy spectral density is the FT of the autocorrelation function:

$$F\{r_{xx}(\tau)\} = G_x(f)$$

- The power spectral density is the FT of the autocorrelation function:

$$F\{\tilde{r}_{xx}(\tau)\} = S_x(f)$$

- Example 20: Power spectral density of a sinusoidal signal revisited

$$\tilde{x}(t) = 5\cos(200\pi t)$$

$$\tilde{r}_{xx}(\tau) = \frac{1}{0.01}\int_{-0.005}^{0.005} 25\cos(200\pi t)\cos(200[t+\tau])dt = \frac{25}{2}\cos(200\pi\tau)$$

$$S_x(f) = F\{\tilde{r}_{xx}(\tau)\} = \frac{25}{4}\delta(f+100) + \frac{25}{4}\delta(f-100)$$

Properties of the autocorrelation function

- $r_{xx}(0) \geq |r_{xx}(\tau)|$ for all τ
- $r_{xx}(-\tau) = r_{xx}(\tau)$ for all τ, that is, the autocorrelation function has even symmetry.
- If the signal $x(t)$ is periodic with period T, then its autocorrelation function $\tilde{r}_{xx}(\tau)$ is also periodic with the same period.

5. Transfer Function Concept

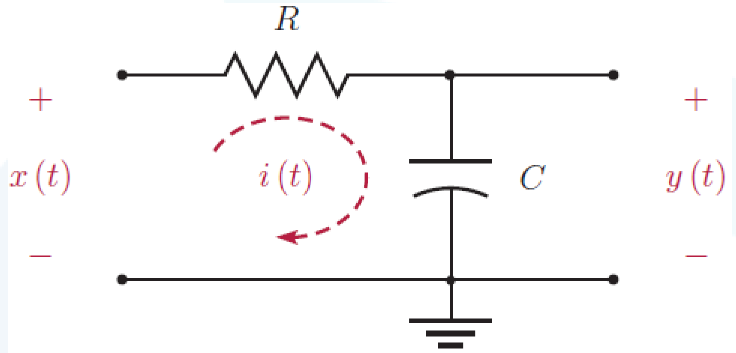
- In time-domain analysis of systems we have relied on two distinct description forms for CTLTI systems:

  1. A linear constant-coefficient differential equation that describes the relationship between the input and the output signals.
  2. An impulse response which can be used with the convolution operation for determining the response of the system to an arbitrary input signal.

- The concept of Transfer function will be introduced as the third method for describing the characteristics of a system.

$$H(\omega) = F\{h(t)\} = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt$$

- The transfer function concept is valid for LTI systems only.
- In general, $H(\omega)$ is a complex function of ω, $H(\omega) = |H(\omega)| e^{j\Theta(\omega)}$.
- Example 21: Transfer function for the simple RC circuit



$$h(t) = \frac{1}{RC} e^{-t/RC} u(t)$$

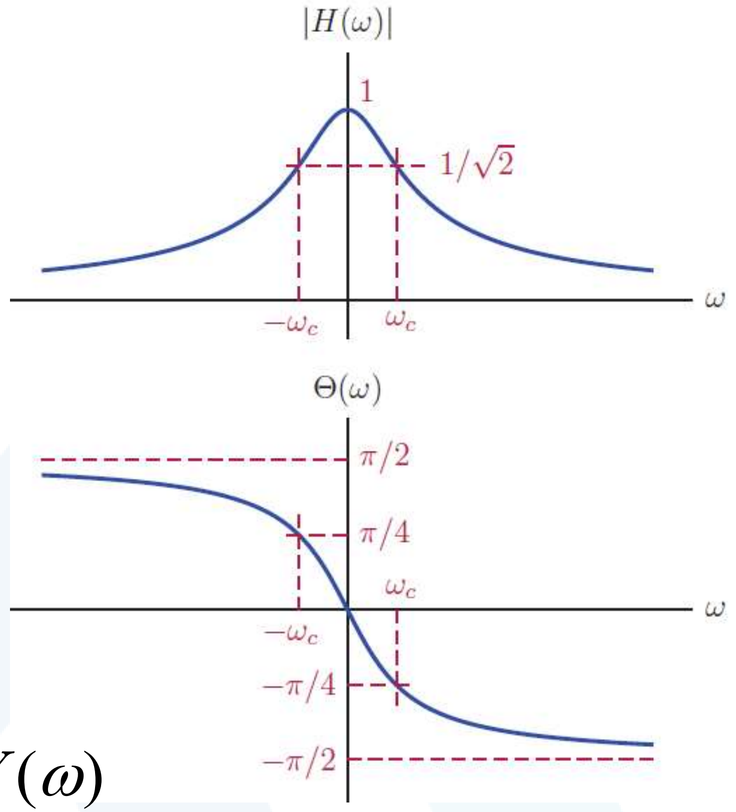
$$H(\omega) = \int_0^\infty \frac{1}{RC} e^{-t/RC} e^{-j\omega t} dt = \frac{1}{1 + j\omega RC} = \frac{1}{1 + j(\omega/\omega_c)},$$

$$\omega_c = \frac{1}{RC}$$

$$|H(\omega)| = \frac{1}{\sqrt{1 + (\omega/\omega_c)^2}}, \quad \Theta(\omega) = -\tan^{-1}(\omega/\omega_c)$$

$$H(\omega_c) = \frac{1}{1 + j}, \quad |H(\omega_c)| = \frac{1}{\sqrt{2}}$$

- ω_c represents the frequency at which the magnitude of the transfer function is 3 decibels below its peak value at $\omega = 0$, $$20\log_{10}\frac{|H(\omega_c)|}{|H(0)|} = 20\log_{10}\frac{1}{\sqrt{2}} \approx -3\text{dB}$$
- The frequency ω_c is often referred to as the 3-dB cutoff frequency of the system.



Obtaining the TF from the differential equation

$$y(t) = h(t) * x(t) \overset{\mathcal{F}}{\longleftrightarrow} Y(\omega) = H(\omega)X(\omega) \Rightarrow H(\omega) = \frac{Y(\omega)}{X(\omega)}$$

$$\frac{d^k y(t)}{dt^k} \overset{\mathcal{F}}{\longleftrightarrow} (j\omega)^k\, Y(\omega), \quad \frac{d^k x(t)}{dt^k} \overset{\mathcal{F}}{\longleftrightarrow} (j\omega)^k\, X(\omega), \qquad k = 0, 1, \cdots$$

- Example 22: Transfer function from the differential equation

$$\frac{d^2y(t)}{dt^2} + 2\frac{dy(t)}{dt} + 26y(t) = x(t)$$

$$(j\omega)^2 Y(\omega) + 2(j\omega)Y(\omega) + 26Y(\omega) = X(\omega)$$

$$[(26-\omega^2) + j2\omega]Y(\omega) = X(\omega) \Rightarrow H(\omega) = \frac{1}{(26-\omega^2)+j2\omega}$$

6. CTLTI Systems with Periodic Input Signals

$$\tilde{x}(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(k\omega_0 t) + \sum_{k=1}^{\infty} b_k \sin(k\omega_0 t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

Response of a CTLTI system to complex exponential signal

$$\tilde{x}(t) = e^{j\omega_0 t}$$

$$y(t) = h(t) * \tilde{x}(t) = \int_{-\infty}^{\infty} h(\tau)\,\tilde{x}(t-\tau)\,d\tau = \int_{-\infty}^{\infty} h(\tau)\,e^{j\omega_0(t-\tau)}d\tau$$

$$= e^{j\omega_0 t}\int_{-\infty}^{\infty} h(\tau)\,e^{-j\omega_0\tau}d\tau = e^{j\omega_0 t}H(\omega_0) = |H(\omega_0)|\,e^{j[\omega_0 t+\Theta(\omega_0)]}$$

- That is, $e^{j\omega t}$ is an eigenfunction of a LTI system and $H(\omega)$ is the corresponding eigenvalue. We refer to H as the frequency response of the system.

Response of a CTLTI system to sinusoidal signal

$$\tilde{x}(t) = \cos(\omega_0 t)$$

$$\tilde{x}(t) = \cos(\omega_0 t) = \tfrac{1}{2}e^{j\omega_0 t} + \tfrac{1}{2}e^{-j\omega_0 t}$$

$$y(t) = \tfrac{1}{2}e^{j\omega_0 t}H(\omega_0) + \tfrac{1}{2}e^{-j\omega_0 t}H(-\omega_0)$$

$$= \tfrac{1}{2}e^{j\omega_0 t}\,|H(\omega_0)|\,e^{j\Theta(\omega_0)} + \tfrac{1}{2}e^{-j\omega_0 t}\,|H(-\omega_0)|\,e^{-j\Theta(\omega_0)}$$

If the impulse response $h(t)$ is real-valued:

$$|H(-\omega_0)| = |H(\omega_0)|, \quad \Theta(-\omega_0) = -\Theta(\omega_0)$$

$$y(t) = \frac{1}{2}|H(\omega_0)|e^{j[\omega_0 t + \Theta(\omega_0)]} + \frac{1}{2}|H(\omega_0)|e^{-j[\omega_0 t + \Theta(\omega_0)]} = |H(\omega_0)|\cos(\omega_0 t + \Theta(\omega_0))$$

Response of a CTLTI system to periodic input signal

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

$$T\{\tilde{x}(t)\} = T\left\{\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}\right\} = \sum_{k=-\infty}^{\infty} T\left\{c_k e^{jk\omega_0 t}\right\} = \sum_{k=-\infty}^{\infty} c_k T\left\{e^{jk\omega_0 t}\right\} = \sum_{k=-\infty}^{\infty} c_k H(k\omega_0) e^{jk\omega_0 t}$$

7. CTLTI Systems with Non-Periodic Input Signals

$$y(t) = h(t) * x(t) \Rightarrow Y(\omega) = H(\omega)X(\omega)$$

$$|Y(\omega)| = |H(\omega)||X(\omega)|, \quad \measuredangle Y(\omega) = \measuredangle X(\omega) + \Theta(\omega)$$

- Example 23: Pulse response of RC circuit

Consider again the RC circuit. Let $f_c = 1/RC = 80$ Hz. Determine the FT of the response of the system to the unit-pulse input signal $x(t) = \Pi(t)$.

$$H(f) = \frac{1}{1 + j(f/f_c)}, \qquad X(f) = \mathrm{sinc}(f),$$

$$Y(f) = \frac{1}{1 + j(f/80)}\mathrm{sinc}(f),$$

$$|Y(f)| = \frac{1}{\sqrt{1 + (f/80)^2}}|\mathrm{sinc}(f)|,$$

$$\measuredangle Y(f) = -\tan^{-1}(f/80) + \measuredangle[\mathrm{sinc}(f)]$$