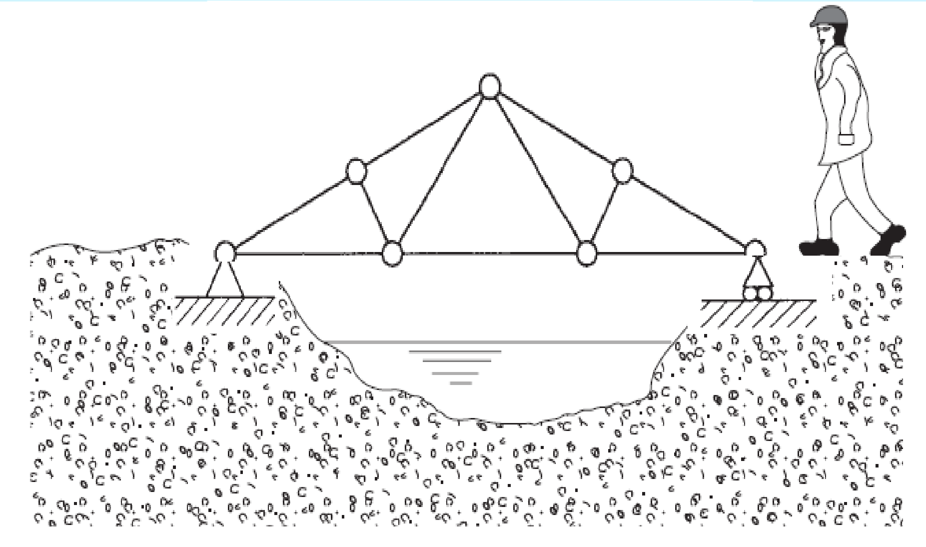
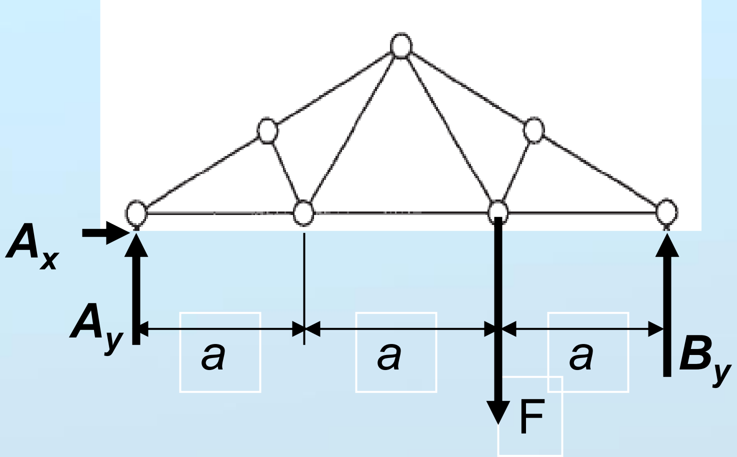
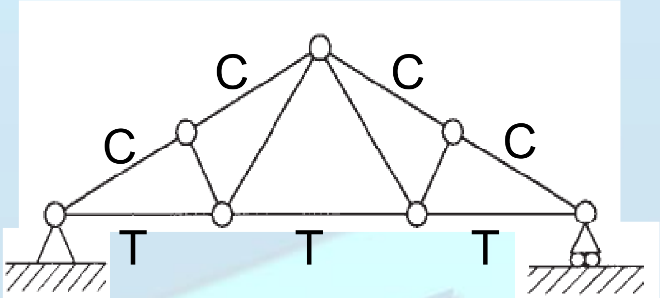
Beams, Frames,… Arches

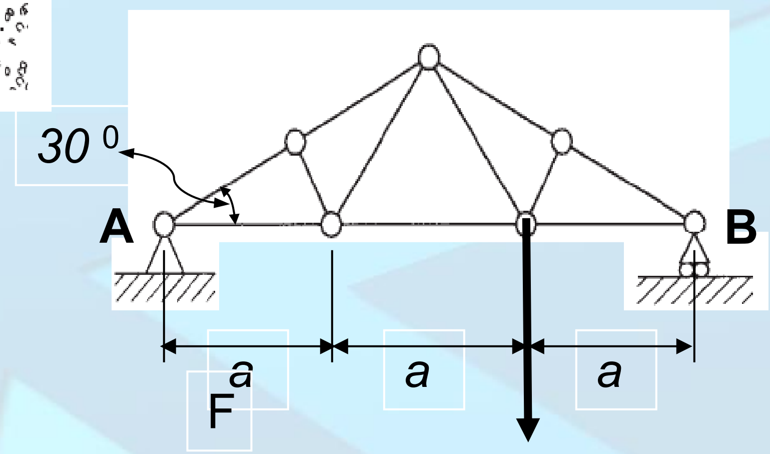
Beams are slender structural members that offer resistance to bending. They are among the most important elements in structural engineering.

الجيزان (ومفردها جائز) عناصر إنشائية هامة نحيلة تتلقى حمولات (أي قوى وعزوم) عرضية وتقاومها بالانعطاف



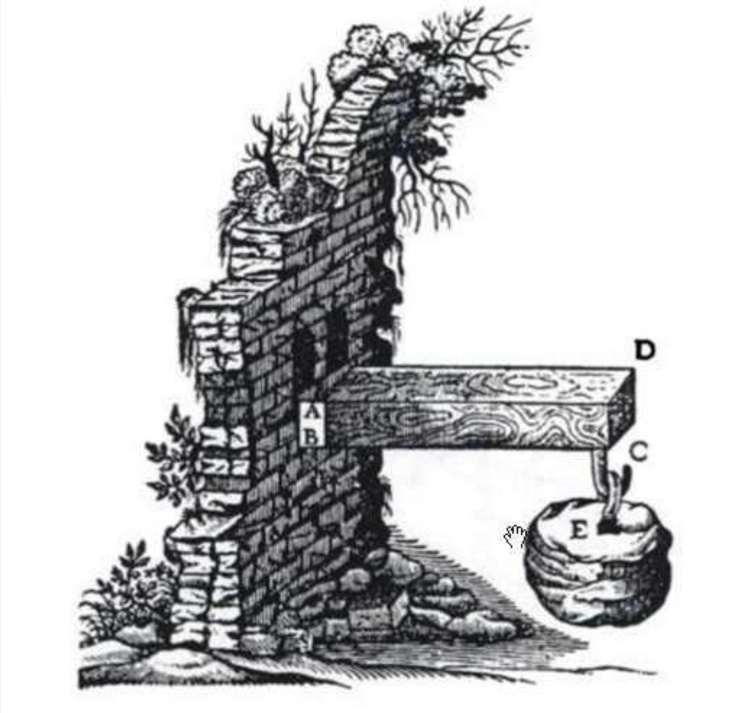






الجيزان (ومفردها جائز) عناصر إنشائية هامة نحيلة تتلقى حمولات (أي قوى وعزوم) عرضية وتقاومها بالانعطاف

The basic function of a structure is to carry loads for which it has been designed & transmit forces from their points of applications to the supports. For example



Observing… the behavior of structures under the action of external loads, led to 2Qs (two questions) then to 2Ss+2Ss:

How strong is a str.? Strength notion & Stress concept.

How stiff is a str.? Stiffness notion & Strain concept.(*deformation concept*)

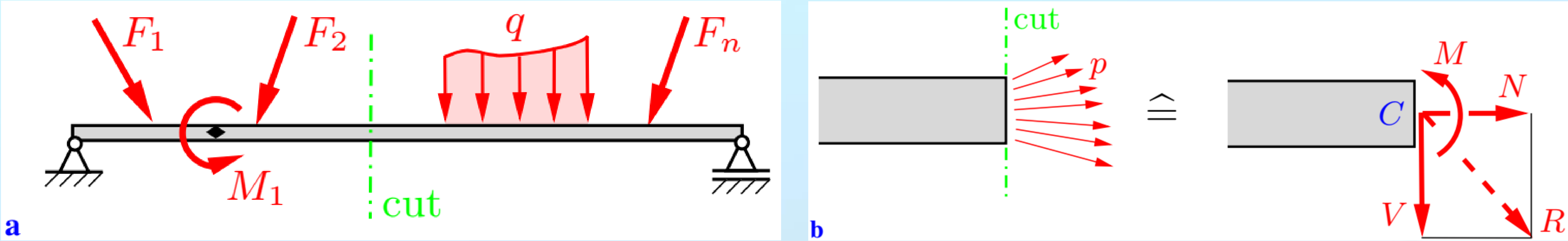
يهتم ميكانيك المواد بدراسة سلوك العناصر الإنشائية من خلال : المقاومة ومعيار قياسها الإجهادات، والصلابة ومعيار قياسها التشوهات.

Knowledge of the internal forces is important in order :

1. to determine the load-bearing capacity of a beam,
2. to compute the properties (area,...) of the cross-section required to sustain a given load,
3. or to compute the deformation as we will see later.

1. قدرة التحمل
2. خصائص المقطع القادرعلى تلقي الحمولة ونقلها بأمان
3. التشوهات

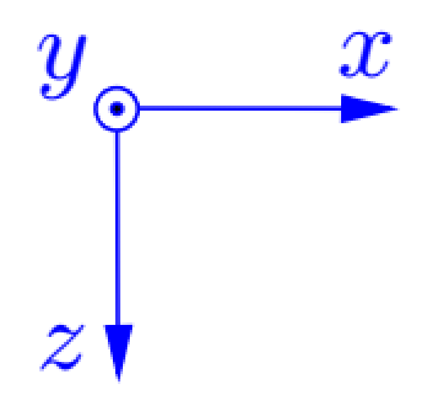
For simplicity, the discussion is limited to statically determinate plane problems, المسائل المقررة والمستوية

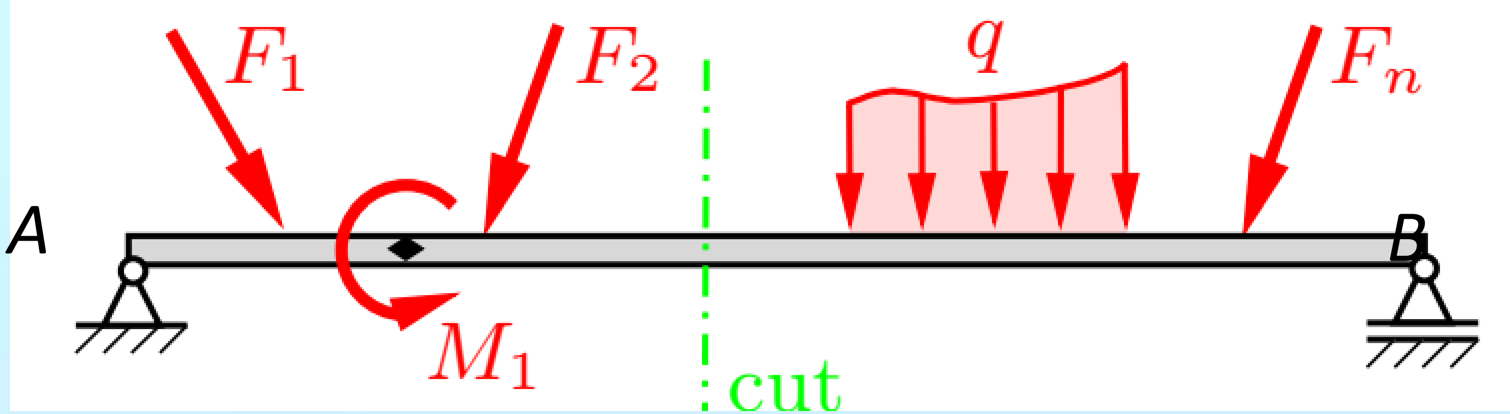


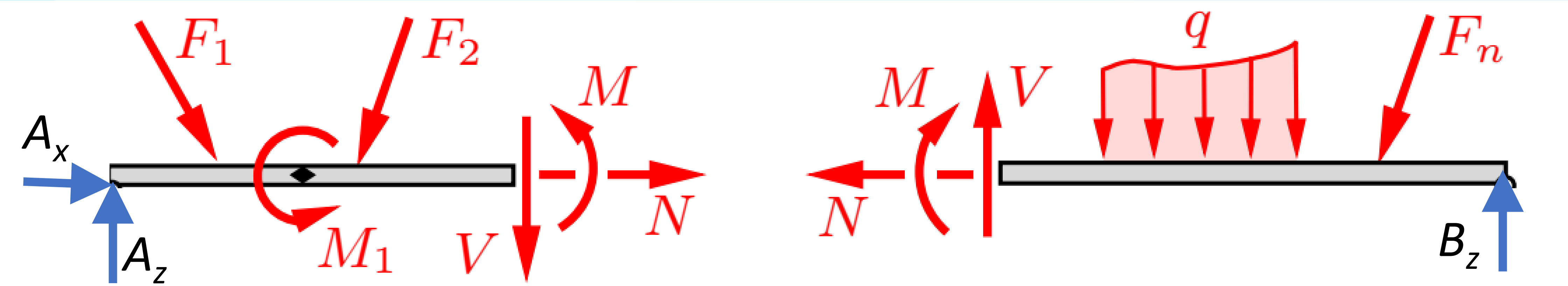
The quantities *N*, *V* & *M* are called the stress resultants (قوى المقطع أو القوى الداخلية أو محصلات الاجهادات).

In particular *N* is called the normal force (القوة الناظمية), *V* is called the shear force (قوة القص) and *M* is called the bending moment (عزم الانعطاف).

In order to determine the stress resultants (القوى الداخلية), the beam is divided by a cut into two segments (method of sections) (طريقة المقطع)



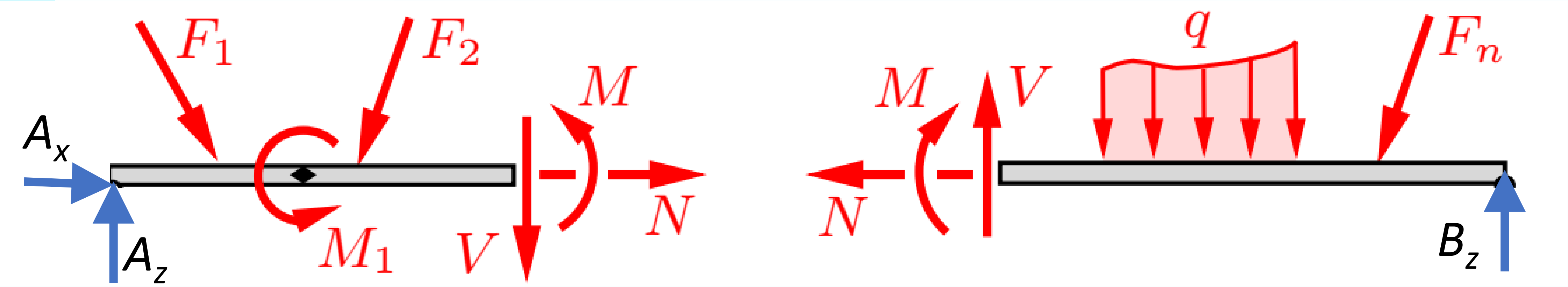




A F. B. D. of each segment includes all of the forces acting on it, i.e., the applied loads (forces and couples), the support reactions & the stress resultants acting at the cut sections. (مخطط جسم حر لكل جزء مع حمولاته وردود أفعال مسانده وقوى المقطع)

According to Newton's third law (action = reaction) stress resultant act in opposite directions at the two faces of the segments of the beam.

وفق قانون نيوتن الثالث (رد الفعل يساوي ويعاكس الفعل) تكون القوى الداخلية على وجهي القطع متساوية ومتعاكسة.

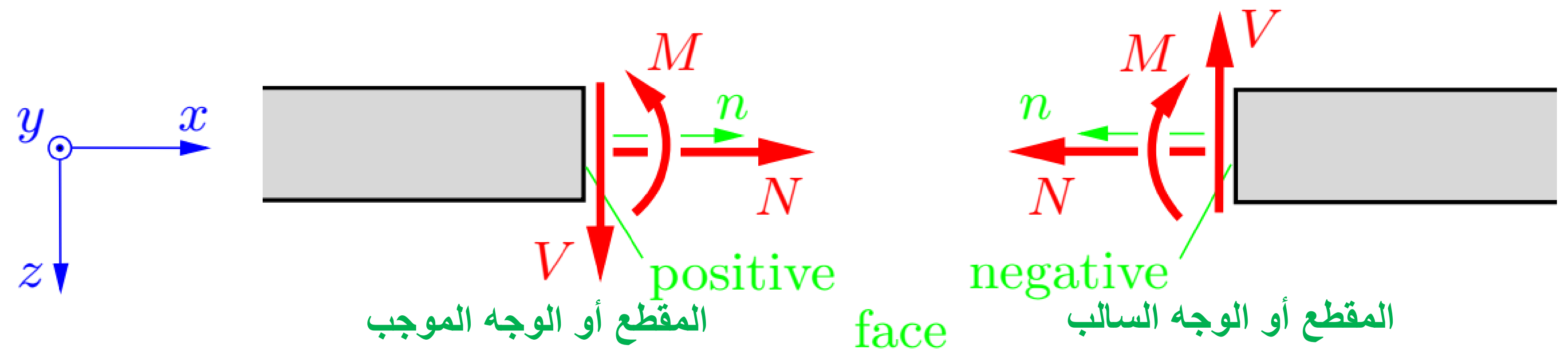


Since each part of the beam is in equilibrium, the 3 conditions of equilibrium for either part can be used to compute the three unknown stress resultants.

إن جزئي الجائزعلى طرفي القطع في حالة توازن، لذا فمن الممكن استخدام شروط التوازن الثلاثة لكل منهما من أجل حساب القوى الداخلية (قوى المقطع) المؤثرة على كل منهما.

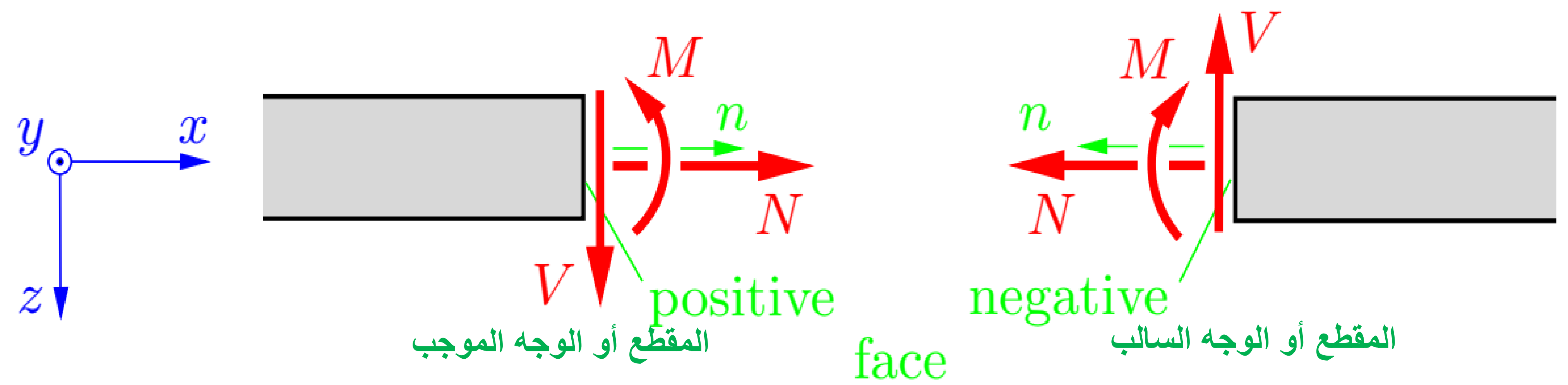
Before we can provide examples for the determination of the stress resultants, a sign convention must be introduced. Consider the two adjoining portions of the same beam shown in the next figure. The coordinate x coincides with the direction of the axis of the beam and points to the right; the coordinate z points downward.

جملة الاحداثيات والوجهين (المقطعين) الموجب والسالب



By cutting the beam, a left-hand face and a right-hand face are obtained (figure above). They are characterized by a normal vector n that points outward from the interior of the beam. If the vector n points in the positive (negative) direction of the *x-axis*, the corresponding face is called positive (negative).

The following sign convention is adopted: اصطلاح إشارة القوى الداخلية:



Positive stress resultants at a positive (negative) face point in the positive (negative) directions of the coordinates.

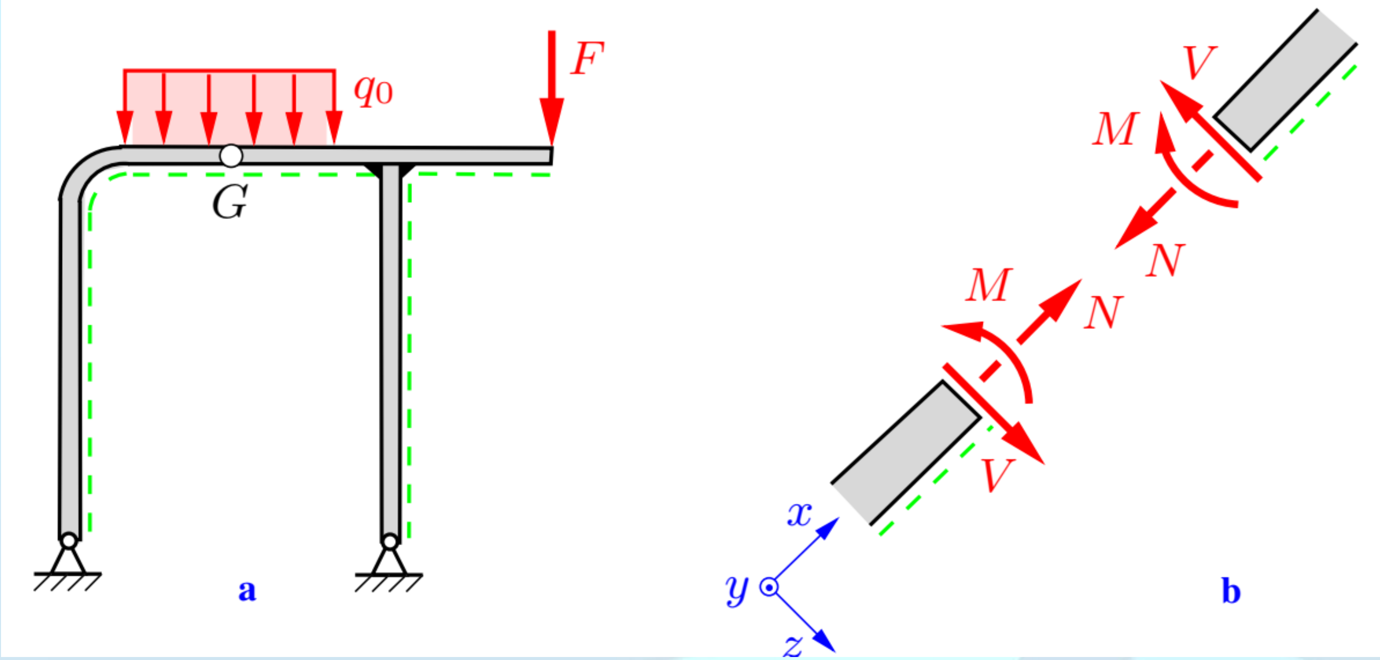
Here, the bending moment *M* has to be interpreted as a vector pointing in the direction of the *y-axis* (*positive direction according to the right-hand rule*)

The above figure shows the stress resultants with their positive directions. In the following examples, we shall strictly adhere to this sign convention.

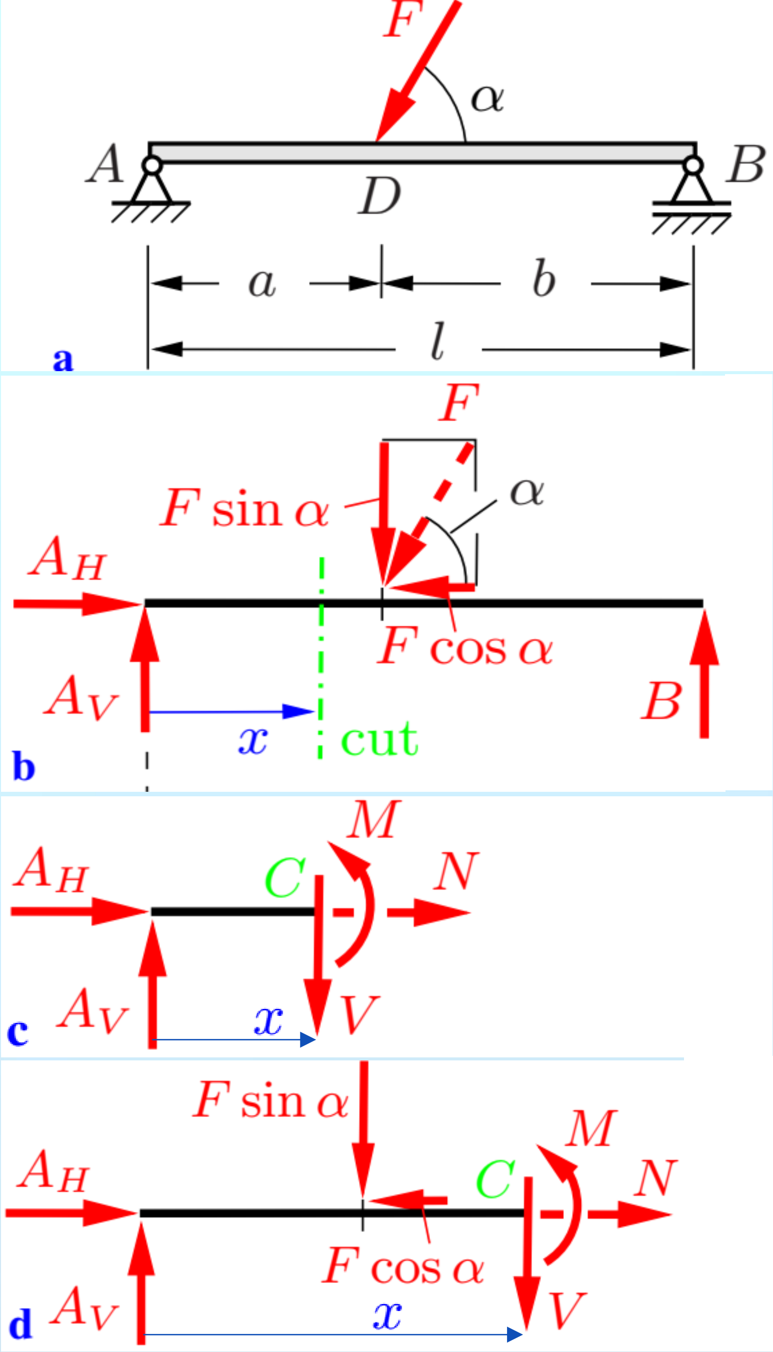
It should be noted, however, that different sign conventions exist.

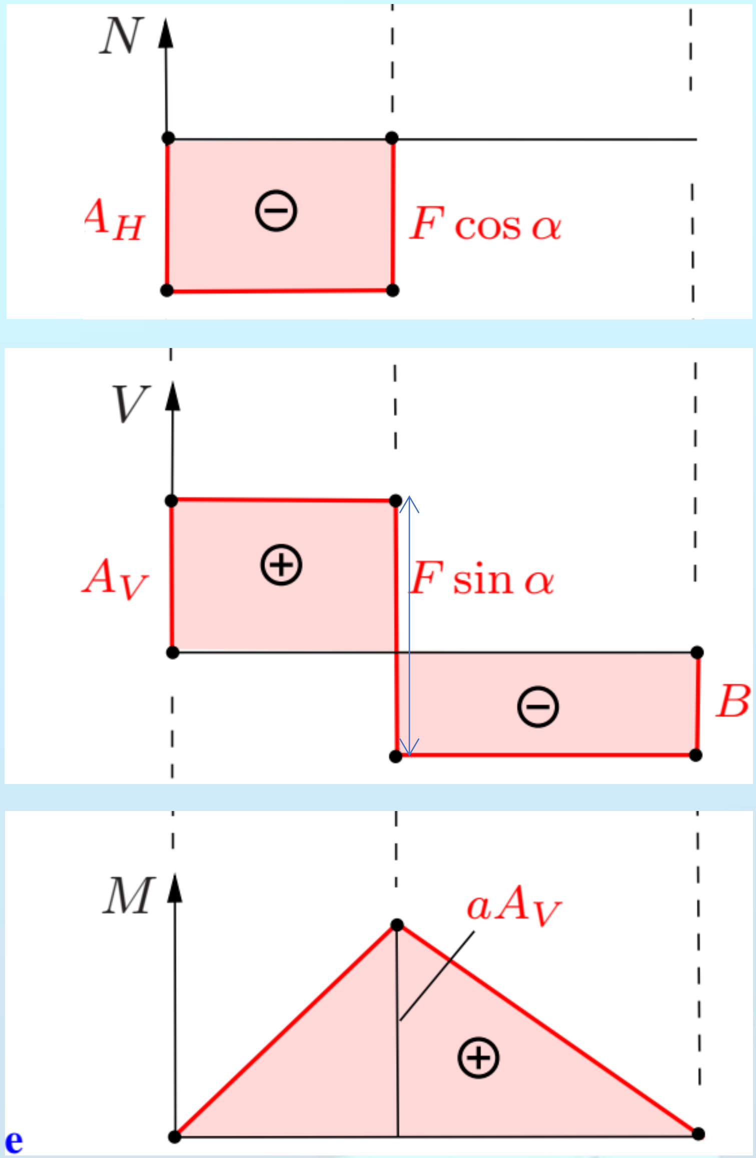
In the case of a horizontal beam, very often only the *x* coordinate is given. Then it is understood that the *z-axis* points downward.

The sign convention for frames and arches may be introduced by drawing a dashed line at one side of each part of the system. The side with the dashed line can then be interpreted as the "underneath side" of the respective part and the coordinate system can be chosen as the one for a beam: *x*-axis in the direction of the dashed line, *z-axis* toward the dashed line ("downward").



Example 1. We will now determine the stress resultants for the simply supported beam shown in Fig.a.





Solution:

1. Reactions, Fig.b:

$A_H = F\cos\alpha$,

$A_V = (b/l)F\sin\alpha$, $B = (a/l)F\sin\alpha$

2. Cut, Fig.c, $0 < x < a$:

$N = -A_H = -F\cos\alpha$

$V = A_V = (b/l)F\sin\alpha$,

$M = xA_V = [(b/l)F\sin\alpha]x$.

$x = 0 \Rightarrow M = 0$, $x = a \Rightarrow M = aA_V$

3. Cut, Fig.d, $a < x < l$:

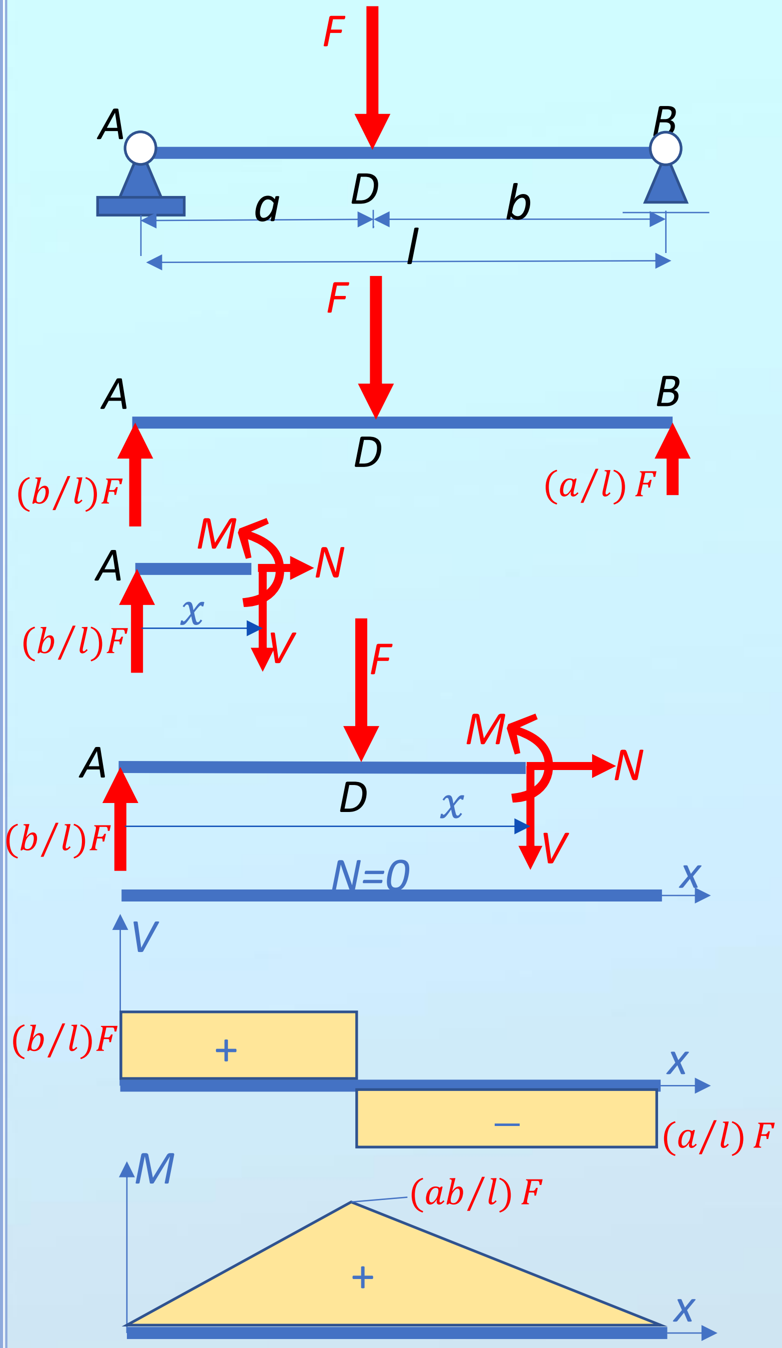
$N = A_H - F\cos\alpha = 0$

$V = A_V - F\sin\alpha = -B$,

$M = xA_V - (x - a)F\sin\alpha$

$x = a \Rightarrow M = aA_V$

$x = l \Rightarrow M = lA_V - bF\sin\alpha = 0$



Example 2. Draw the diagrams of the stress resultants for the beam shown in figure.

Solution:

1. Reactions:

$\rightarrow: A_x = 0$

$\overset{\curvearrowleft}{B}: A_z = (b/l)F, \overset{\curvearrowleft}{A}: B_z = (a/l)F, \uparrow: A_z + B_z = F$ (yes)

1. Cut: A…D, $0 < x < a$:

$\rightarrow: N = 0;$

$\uparrow: (b/l)F - V = 0 \Rightarrow V = (b/l)F;$

$\overset{\curvearrowleft}{X}: M - x(b/l)F = 0 \Rightarrow M = x(b/l)F$

$x = 0: M = 0; x = a: M = (ab/l)F$

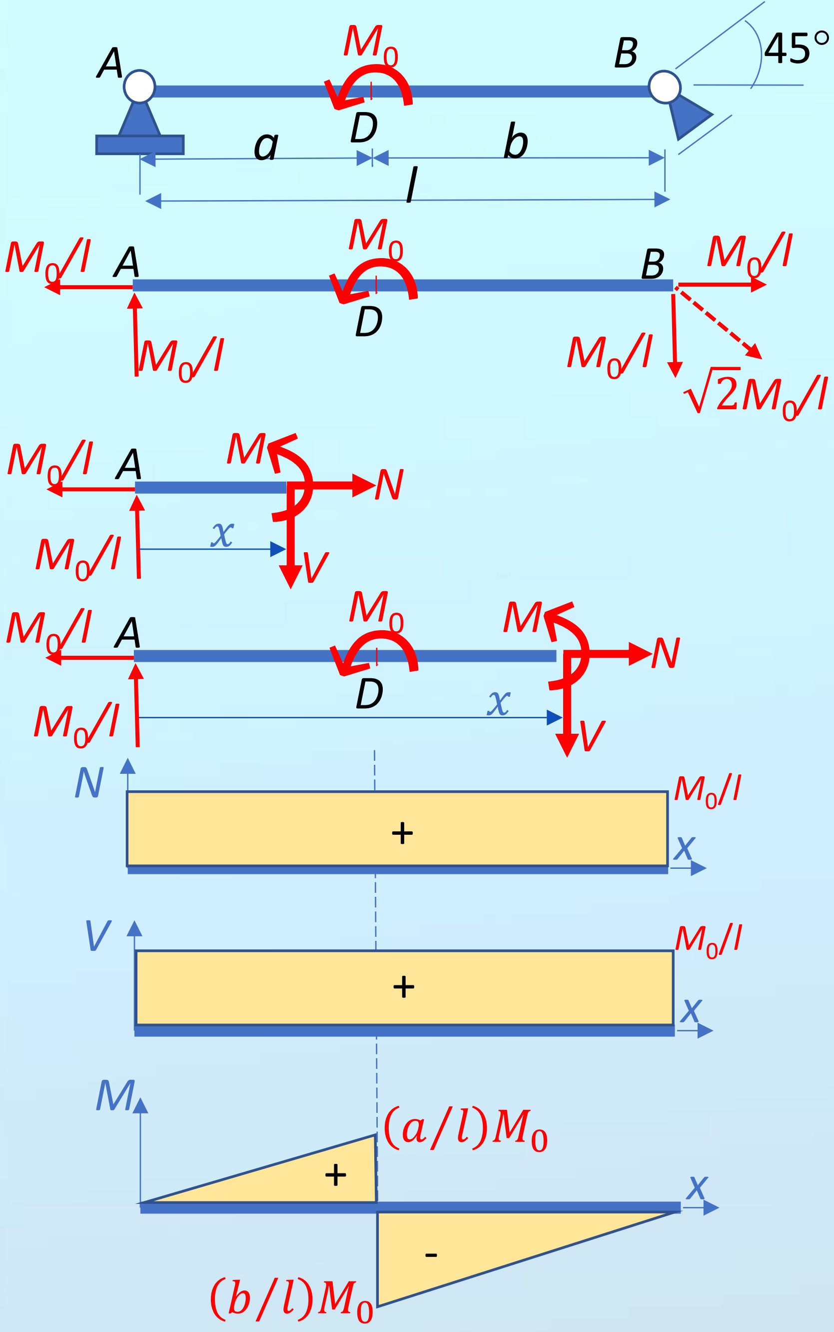
2. Cut: D…B, $a < x < l$:

$\rightarrow: N = 0;$

$\uparrow: (b/l)F - F - V = 0 \Rightarrow V = (b/l)F - F = [(b-l)/l]F = -(a/l)F;$

$\overset{\curvearrowleft}{X}: M + (x-a)F - x(b/l)F = 0 \Rightarrow M = (l-x)(a/l)F;$

$x = a: M = (ab/l)F; x = l: M = 0$



Example 3. Draw the diagrams of the stress resultants for the beam shown in figure.

Solution:

0. Reactions:

$\overset{\curvearrowleft}{B}: -lA_z + M_0 = 0 \Rightarrow A_z = M_0/l;$

$\overset{\curvearrowleft}{A}: + lB_z + M_0 = 0 \Rightarrow B_z = -M_0/l\,(\downarrow) \Rightarrow B_x = -M_0/l\,(\rightarrow)$

$\rightarrow: -A_x + B_x = 0 \Rightarrow A_x = M_0/l(\leftarrow)$

1. Cut: A…D, $0 < x < a$:

$\rightarrow: N = M_0/l;$

$\uparrow: M_0/l - V = 0 \Rightarrow V = M_0/l$

$\overset{\curvearrowleft}{X}: M - x(M_0/l) = 0 \Rightarrow M = (x/l)M_0,$

$x = 0: M = 0; x = a: M = (a/l)M_0.$

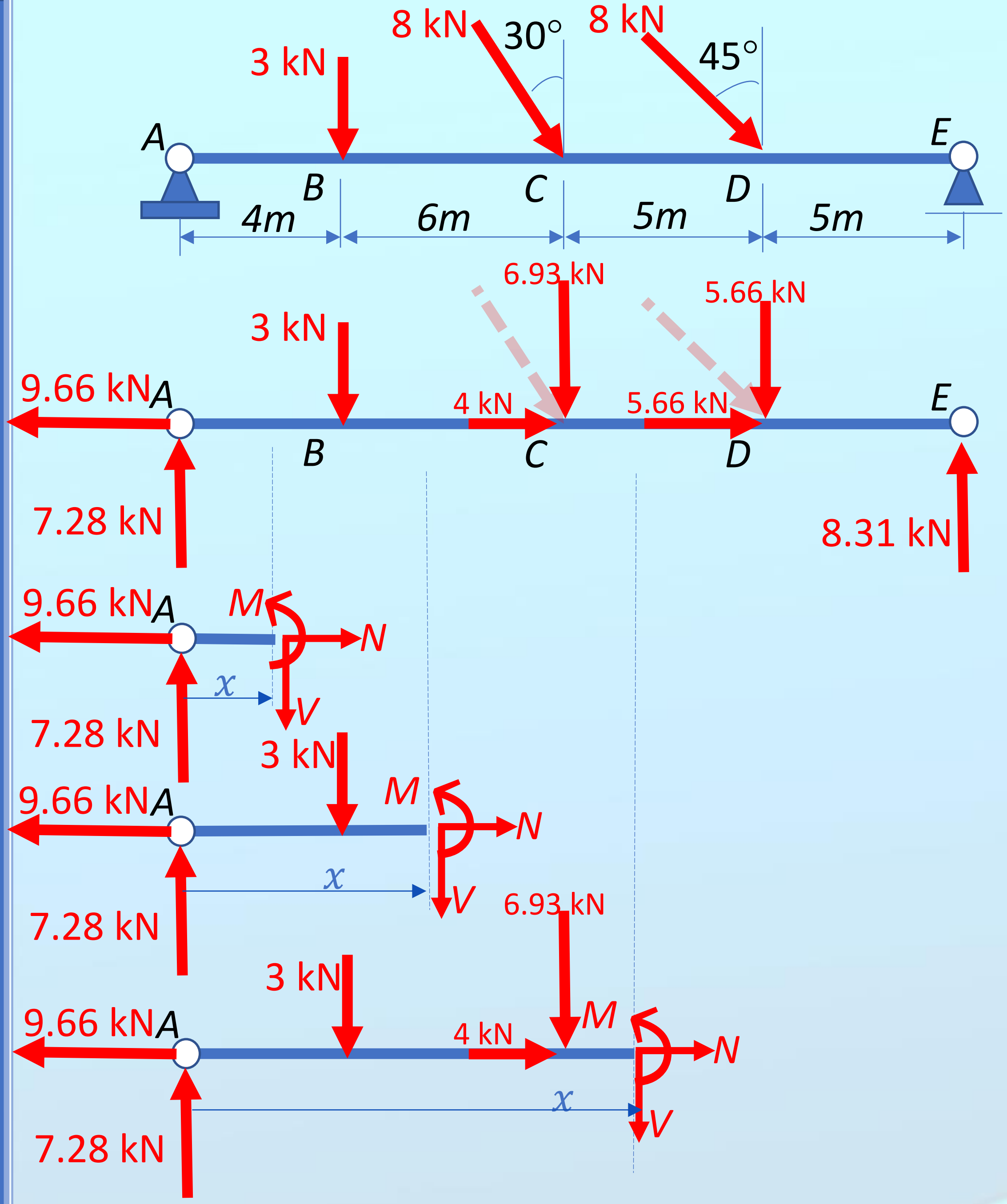
2. Cut: D…B, $a < x < l$:

$\rightarrow: N = M_0/l;$

$\uparrow: M_0/l - V = 0 \Rightarrow V = M_0/l$

$\overset{\curvearrowleft}{X}: M + M_0 - x(M_0/l) = 0 \Rightarrow M = -M_0 + (x/l)M_0 = (\frac{x-l}{l})M_0;$

$x = a: M = -(b/l)M_0; x = l: M = 0.$



Problem 1. Draw the diagrams of the stress resultants for the beam shown in figure.

Solution:

0. Reactions:

$\rightarrow: -A_x + 4 + 5.66 = 0 \Rightarrow A_x = 9.66 \text{ kN } (\leftarrow)$

$\overset{\curvearrowleft}{E}: +5(5.66) + 10(6.93) + 16(3) - 20A_z = 0 \Rightarrow \; A_z = 7.28 \text{ kN}$

$\overset{\curvearrowleft}{A}: -4(3) - 10(6.93) - 15(5.66) + 20E_z = 0 \Rightarrow \; E_z = 8.31 \text{ kN}$

1. Cut: A…B, $0 < x < 4\text{m}$:

$\rightarrow: N = 9.66 \text{ kN}; \quad \uparrow: V = 7.28 \text{ kN}$

$\overset{\curvearrowleft}{X}: \; M - x(7.28) = 0 \Rightarrow M = 7.28x,$

$x = 0: M = 0; x = 4: M = 29.1 \text{ kNm}.$

2. Cut: B…C, $4 < x < 10\text{m}$:

$\rightarrow: N = 9.66 \text{ kN}; \quad \uparrow: V = 7.28 - 3 = 4.28 \text{ kN}$

$\overset{\curvearrowleft}{X}: \; M - x(7.28) + (x - 4)(3) = 0 \Rightarrow M = 4.28x + 12,$

$x = 4: M = 29.1 \text{ kNm}; x = 10: M = 54.8 \text{ kNm}.$

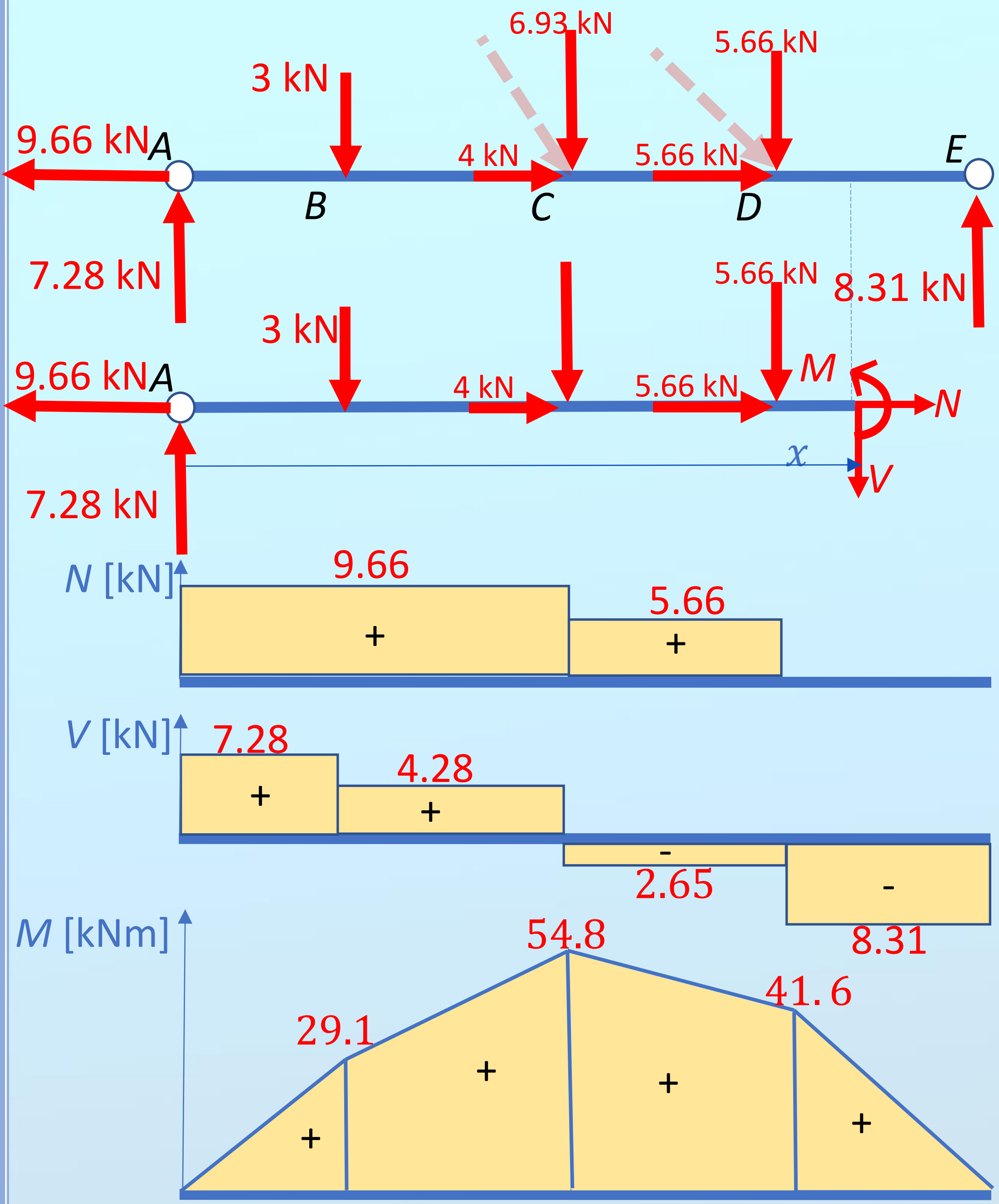
3. Cut: C…D, $10 < x < 15\text{m}$:

$\rightarrow: N = 5.66 \text{ kN}; \quad \uparrow: V = 7.28 - 3 - 6.93 = -2.65\text{kN}$

$\overset{\curvearrowleft}{X}: \; M - x(7.28) + (x - 4)(3) + (x - 10)(6.93) = 0$

$\Rightarrow M = -2.65x + 81.3,$

$x = 10: M = 54.8 \text{ kNm}; x = 15: M = 41.6 \text{ kNm}.$



4. Cut: D…E, $15 < x < 20\text{m}$:

$\rightarrow: N = 0; \uparrow: V = 7.28 - 3 - 6.93 - 5.66 = -8.31\text{kN}$

$\overset{\curvearrowleft}{X}: M - x(7.28) + (x-4)(3) + (x-10)(6.93) + (x-15)(5.66) = 0 \Rightarrow M = -8.31x + 166.2,$

$x = 15: M = 41.6\text{kNm}; x = 20: M = 0.$

Stress Resultants in Straight Beams

Beams are usually subjected to forces perpendicular to their axes. If there is no loading (external or reactions) in the direction of the beam axis, the normal force vanishes $N = 0$.

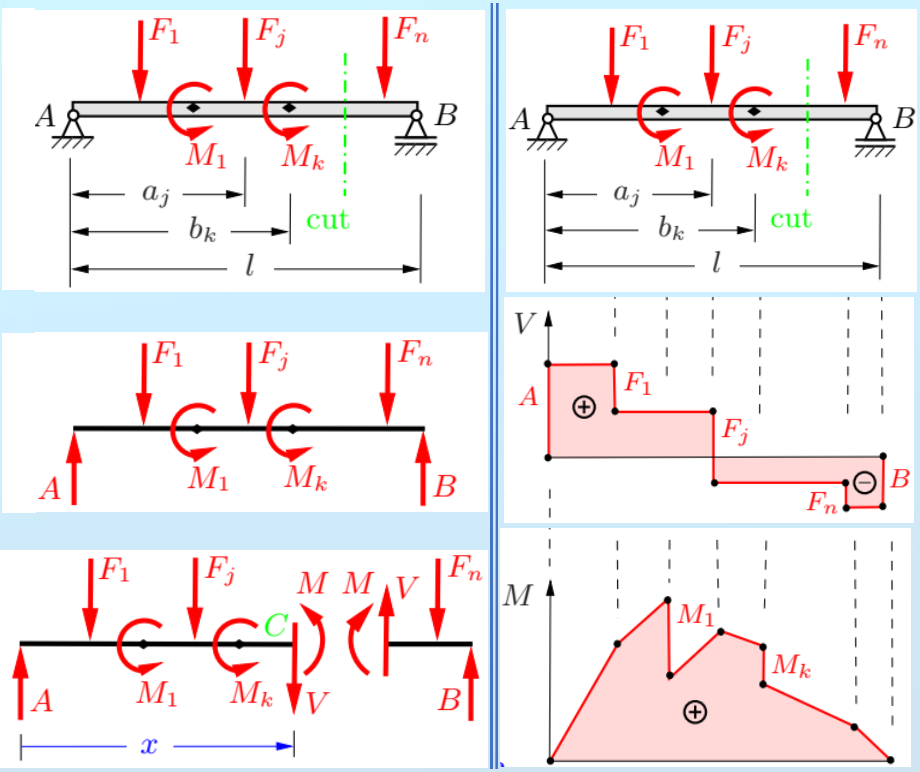
تنعدم القوة الناظمية في الجيزان المستقيمة المحملة عمودياً على محاورها، $N = 0$.

Beams under Concentrated Loads

حالة الجيزان الخاضعة لقوى وعزوم مركزة.

To determine V & M choose a coordinate system and cut at an arbitrary x. Represent V & M with their positive directions in the F. B. Ds.; use Eq. Eqs. for either portion of the beam. Results are a shear-force and a bending-moment diagram.

جملة احداثيات، قطع، معادلات توازن لأي من الجزئين



1. Reactions

$$\overset{\curvearrowleft}{A}: lB - \sum a_i F_i + \sum M_i = 0 \quad \rightarrow B = \frac{1}{l}[\sum a_i F_i - \sum M_i]$$

$$\overset{\curvearrowleft}{B}: -lA + \sum (l - a_i) F_i + \sum M_i = 0 \quad \rightarrow A = \frac{1}{l}[\sum (l - a_i) F_i + \sum M_i]$$

2. Cut at X

$$\uparrow: -V + A - \sum F_i = 0 \quad \rightarrow V = A - \sum F_i$$

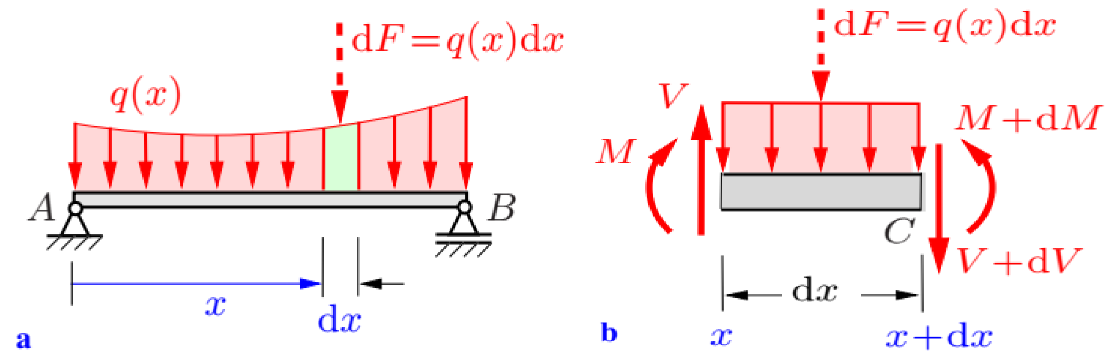
$$\overset{\curvearrowleft}{X}: M - xA + \sum (x - a_i) F_i + \sum M_i = 0$$

$$\rightarrow M = xA - \sum (x - a_i) F_i - \sum M_i$$

$$\frac{dM}{dx} = A - \sum F_i = V$$ مشتق تابع عزم الانعطاف يساوي تابع قوة القص

عند القوى أو العزوم المركزة توجد قفزة مساوية عكسا في المخطط المقابل

Relationship between distributed Loading and Stress Resultants (General case)



تابع الحمولة *q*	تابع قوة القص *V*	تابع عزم الانعطاف *M*
0	Constant ثابت	Linear خطي
Constant ثابت	Linear خطي	Quadratic تربيعي
Linear خطي	Quadratic تربيعي	Cubic تكعيبي

Support المسند	قيمة قوة القص	قيمة عزم الانعطاف
Pin or roller مفصل ثابت أو متدحرج	$V \neq 0$	$M = 0$
Fixed وثاقة	$V \neq 0$	$M \neq 0$
Free End طرف حر	$V = 0$	$M = 0$

Any part of the beam is in Equilibrium

Eq. Eqs. of [**dx**]:

$$\uparrow: V - q(x)\mathrm{d}x - (V + \mathrm{d}V) = 0 \Rightarrow \boxed{\frac{\mathrm{d}V}{\mathrm{d}x} = -q(x)}$$

$$\overset{\frown}{C}: (M + \mathrm{d}M) + \left(\frac{\mathrm{d}x}{2}\right) q(x)\mathrm{d}x - \mathrm{d}xV - M = 0$$

with $\mathrm{d}x \to 0, \Rightarrow \boxed{\frac{\mathrm{d}M}{\mathrm{d}x} = V(x)}$ & $\boxed{\frac{d^2M}{dx^2} = -q(x)}$

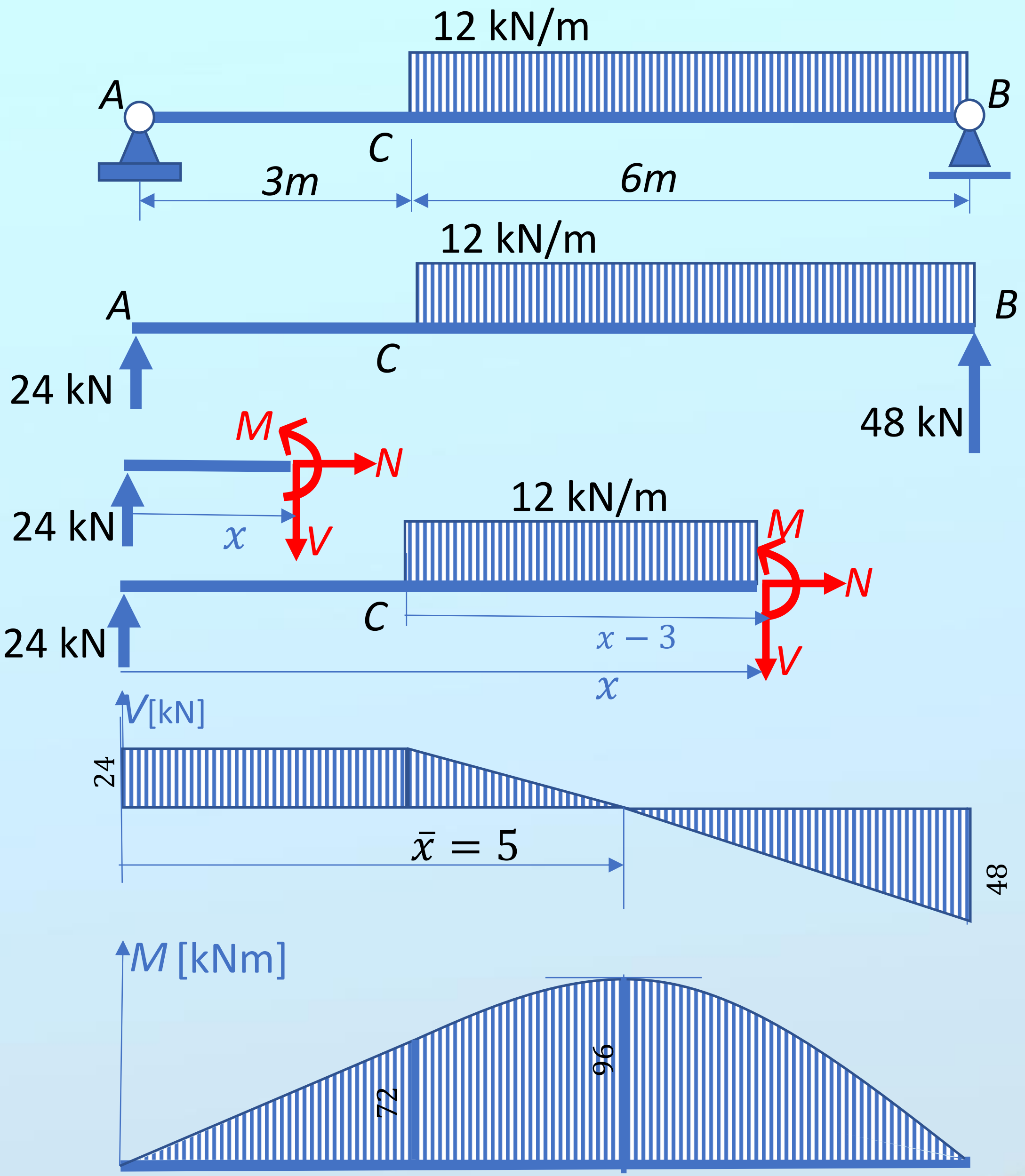
القص موجب فالعزم متزايد، القص سالب فالعزم متناقص، القص معدوم فالعزم عند نهاية حدية (كبرى أو صغرى).

إذا كانت الحمولة كثيرة حدود (معدومة، ثابتة، خطية....)

فالقص كثيرة حدود (ثابتة، خطية، درجة ثانية: قطع مكافئ...)

والعزم كثيرة حدود (خطية، درجة ثانية: مكافئ، درجة ثالثة...)

EXAMPLE 1. Draw the shear force and bending moment diagrams for the shown beam.



0. Reactions:

$\rightarrow: A_x = 0$

$\curvearrowleft_B: +3(6)(12) - 9A_z = 0 \Rightarrow A_z = 24 \text{ kN}$

$\curvearrowleft_A: -6(6)(12) + 9B_z = 0 \Rightarrow B_z = 48 \text{ kN}$

1. Cut: A…C, $0 < x < 3\text{m}$:

$\rightarrow: N = 0; \quad \uparrow: V = 24 \text{ kN}$

$\curvearrowleft_X: M - x(24) = 0 \Rightarrow M = 24x,$

$x = 0: M = 0; x = 3: M = 72 \text{ kNm}.$

2. Cut: C…B, $3 < x < 9\text{m}$:

$\rightarrow: N = 0;$

$\uparrow: V = 24 - 12(x-3) = -12x + 60$

$x = 3: V = 24 \text{ kN}, x = 9: V = -48 \text{ kN};$

$V = 0: \bar{x} = 60/12 = 5 \text{ m}$

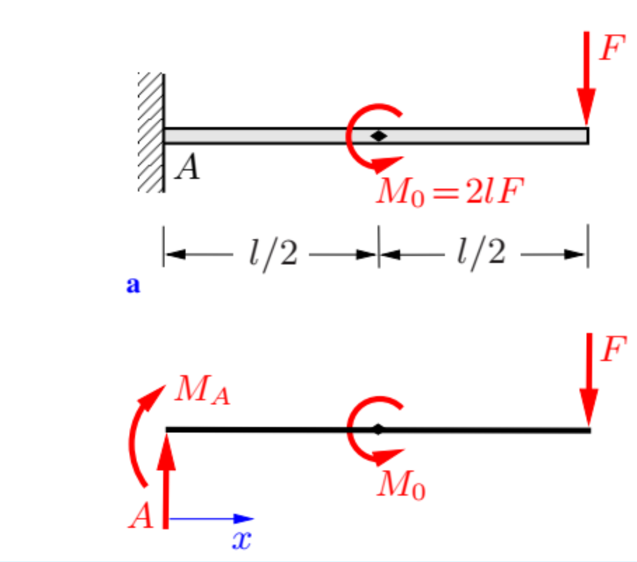
$\curvearrowleft_X: M - x(24) + \frac{1}{2}(x-3)(12)(x-3) = 0$

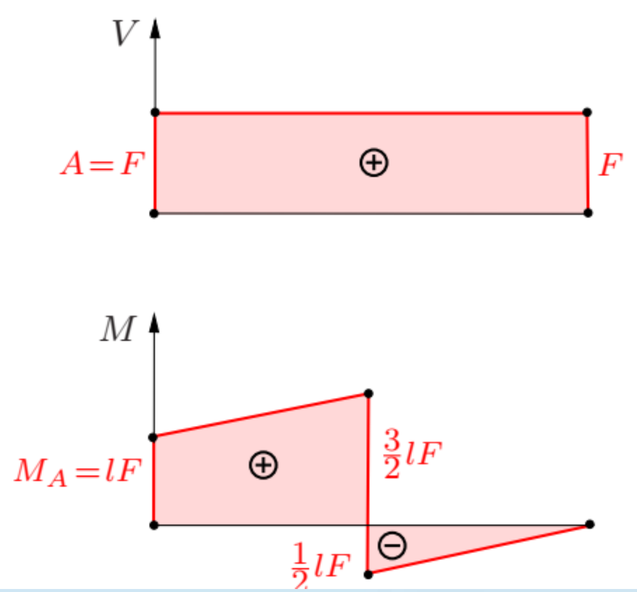
$\Rightarrow M = -6x^2 + 60x - 54,$

$x = 3: M = 72 \text{ kNm}; x = 9: M = 0.$

$M_{max} = M_{|V=0} = M_{|\bar{x}=5} = 96 \text{ kNm}.$

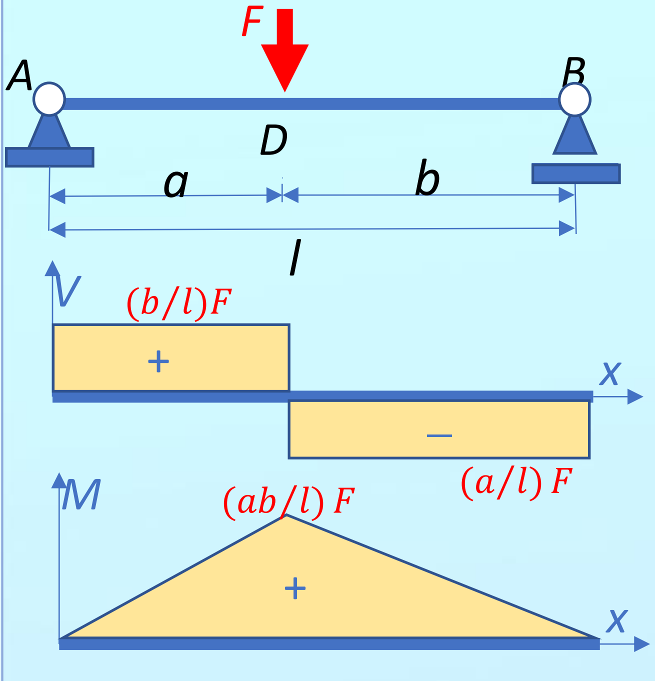
Example 2 Determine the shear-force and bending-moment diagrams for the cantilever beam shown in Fig.a.

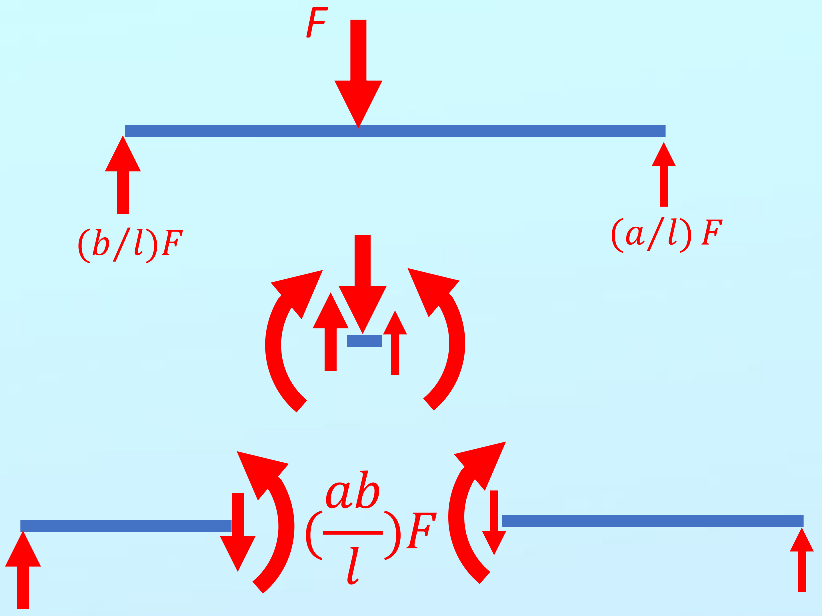




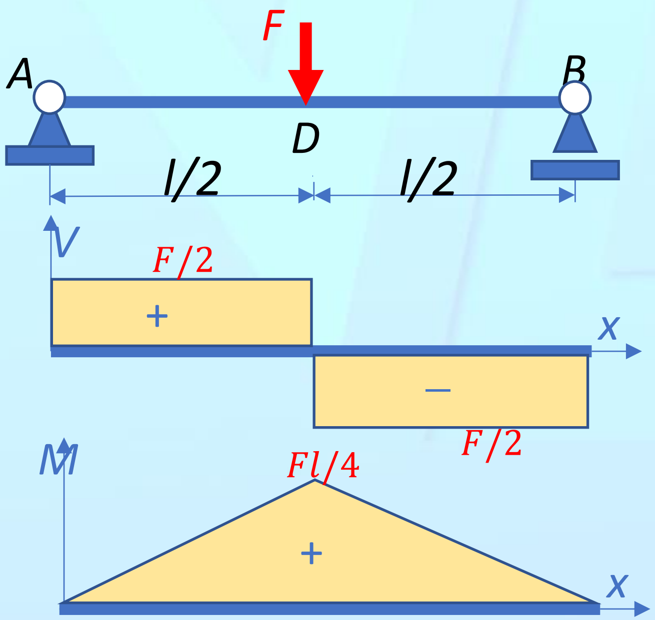
Solution:

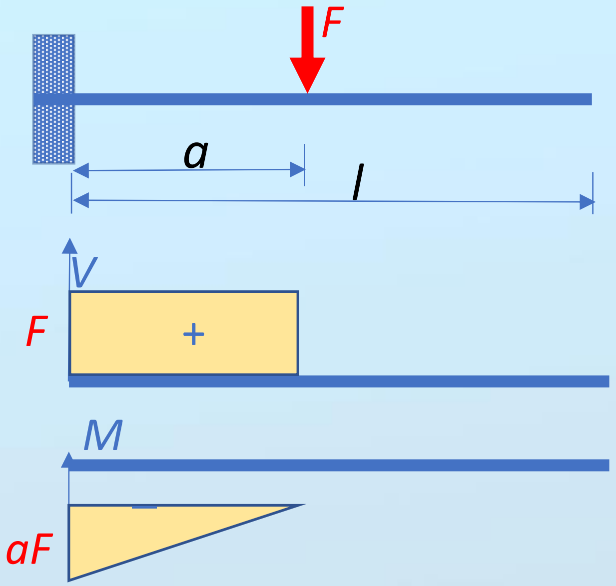
By Heart

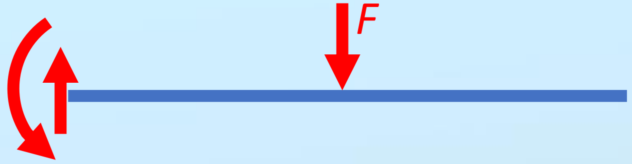




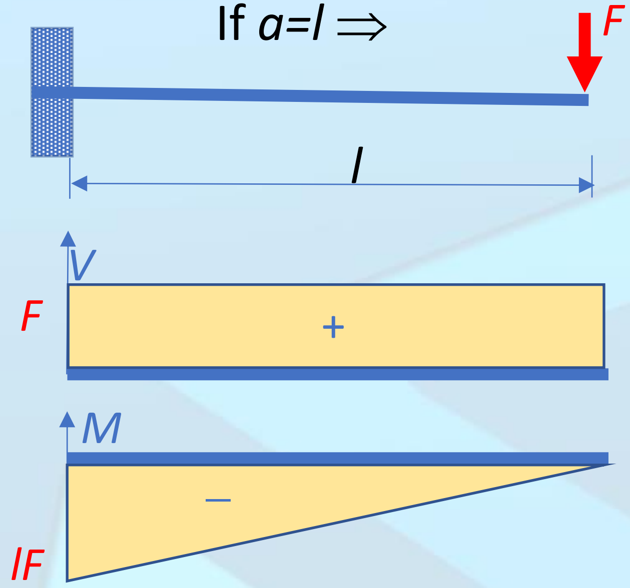
If $a=b=l/2 \Rightarrow$

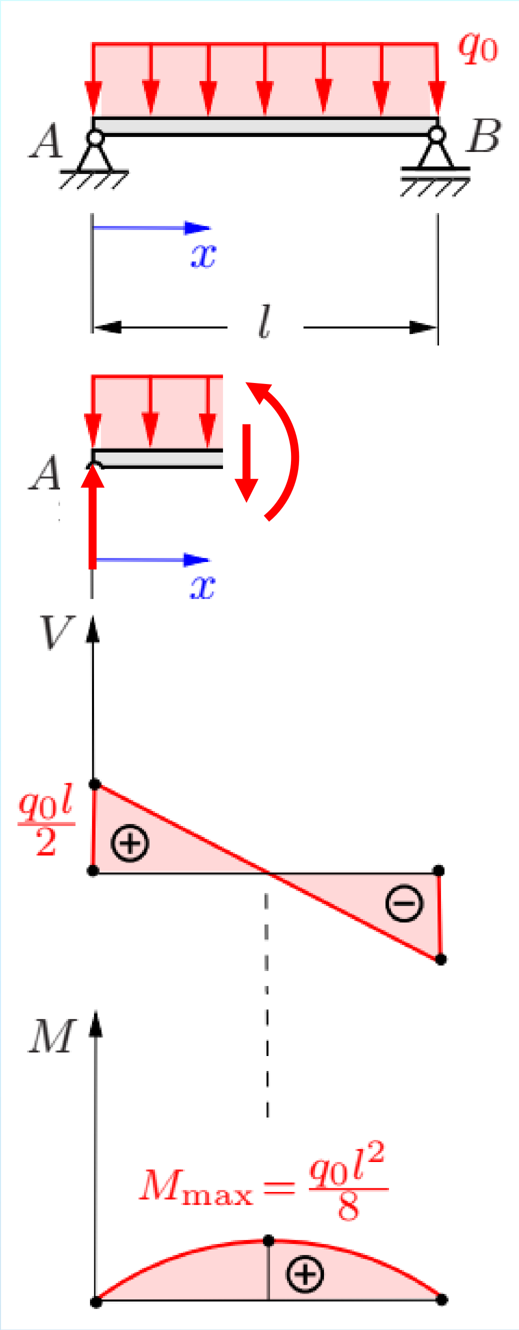






If $a=l \Rightarrow$





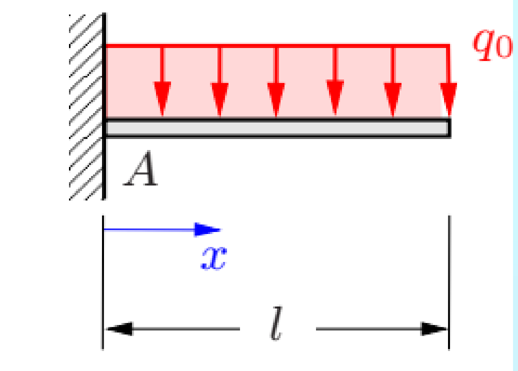
Example 3 Determine the shear-force and bending-moment diagrams for the beam shown in Fig. using the section method and the integration method.

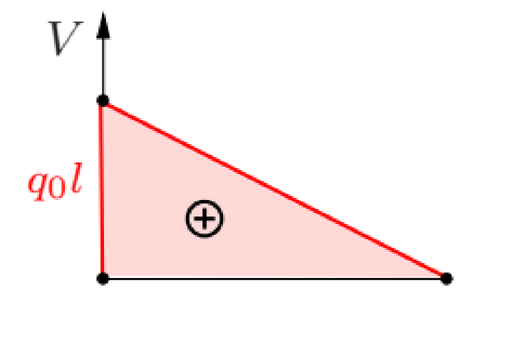
Solution:

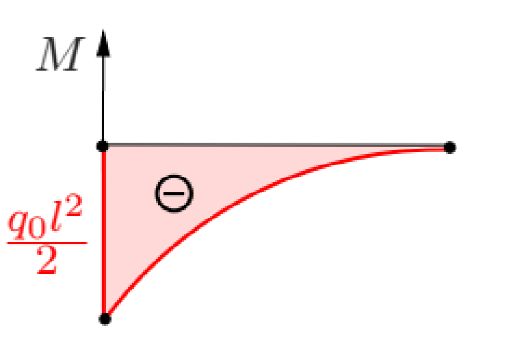
Reactions:

Cut or Cuts:

Example 4 Determine the shear-force and bending-moment diagrams for the cantilever beam shown in Fig. using the section method and the integration method.





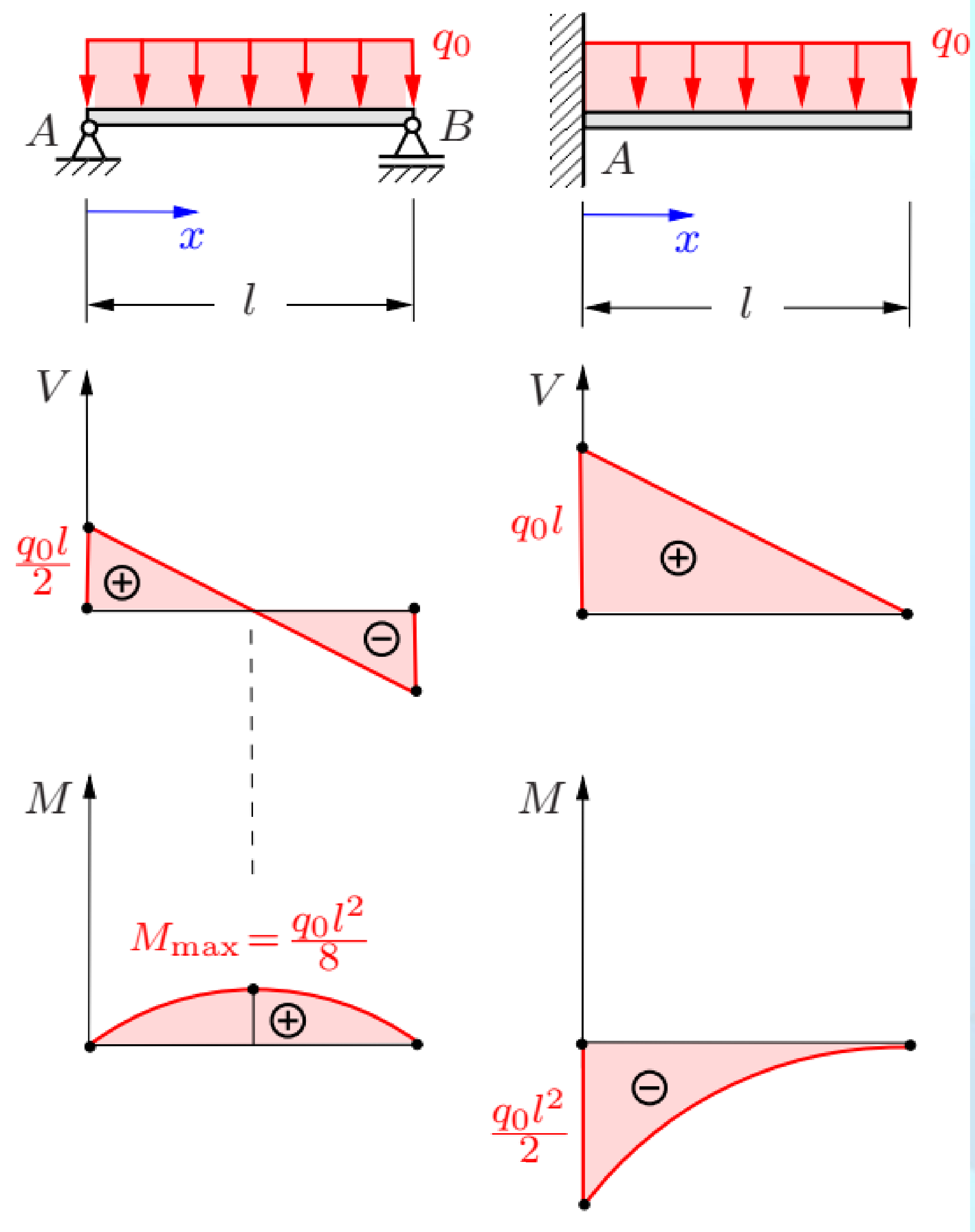


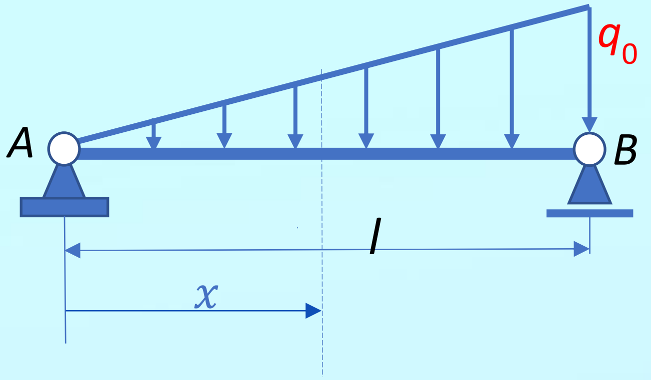
Solution:

Reactions:

Cut or Cuts:

By Heart





Example 5 Determine the shear-force & bending-moment diagrams for the shown simple beam. using the integration method.

Solution:

1) Find the function of the distributed load: $q(x) = \frac{q_0}{l}x$

2) Integrate twice the equation: $\frac{d^2M}{dx^2} = -q(x)$ To get:

$$\frac{d^2M}{dx^2} = -\frac{q_0}{l}x \quad \Rightarrow \quad V = \frac{dM}{dx} = -\frac{q_0}{2l}x^2 + C_1 \quad \Rightarrow \quad M = -\frac{q_0}{6l}x^3 + C_1x + C_2$$

3) Determine the two constants C_1 & C_2 from the two boundary conditions:

1- At x =0 (pin support at A) M = 0: $0 = -\frac{q_0}{6l}(0)^3 + C_1(0) + C_2 \Rightarrow C_2 = 0$

2- At x =l (roller support at B) M = 0: $0 = -\frac{q_0}{6}l^2 + C_1 l \Rightarrow C_1 = \frac{q_0}{6}l$

4) Write the final expressions of V & M as:

$$V = -\frac{q_0}{2l}x^2 + \frac{q_0}{6}l = \frac{q_0}{6l}(-3x^2 + l^2)$$

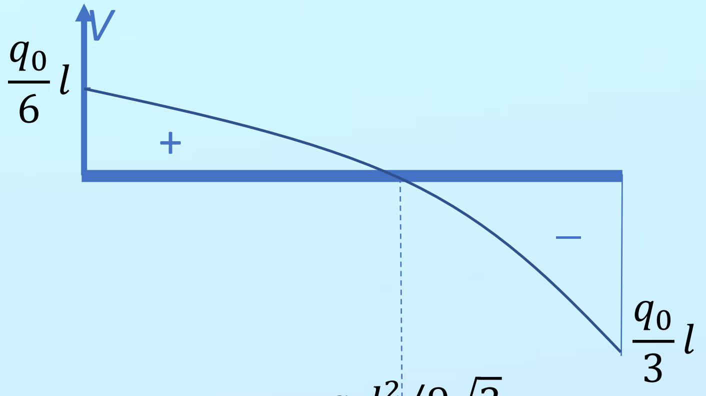
$$x = 0: V = \frac{q_0}{6}l \quad \& \quad x = l: V = -\frac{q_0}{3}l$$

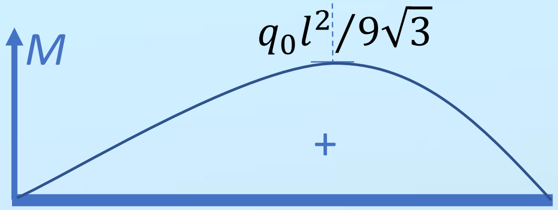
$$V = 0 \Rightarrow x = \frac{l}{\sqrt{3}} = 0.577l$$

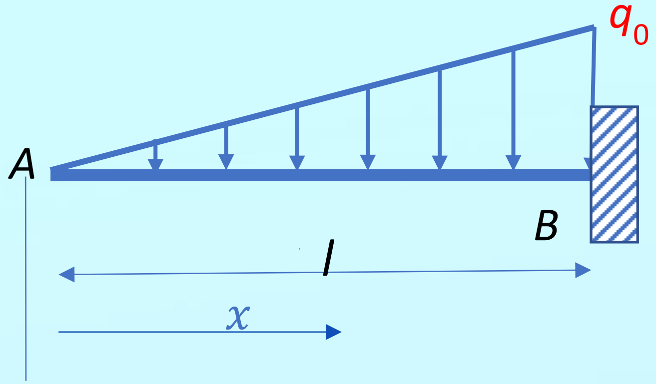
$$M = -\frac{q_0}{6l}x^3 + \frac{q_0}{6}lx = \frac{q_0}{6l}(-x^3 + l^2x)$$

$$x = 0: M = 0 \quad \& \quad x = l: M = 0$$

$$x = \frac{l}{\sqrt{3}} = 0.577l \Rightarrow M_{max} = \frac{q_0l^2}{9\sqrt{3}} = \frac{q_0l^2}{15.6}$$







Example 6. Determine the shear-force & bending-moment diagrams for the shown cantilever beam, using the integration method.

Solution:

Reactions: No need

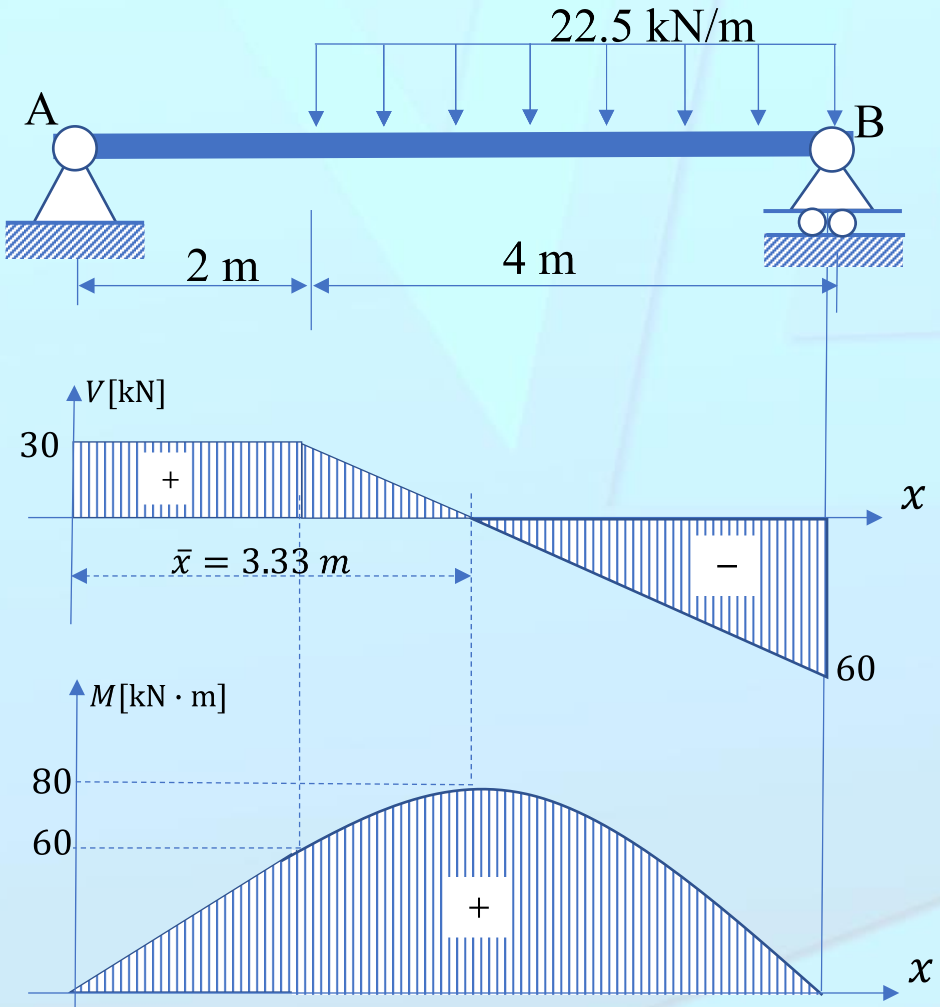
Cut or Cuts:

Problem 1. For the shown beam find

(a) Reactions in supports A and B.

(b) Shear force and bending moment diagrams.

(c) Determine the value and the location of the maximum bending moment.



Solution:

(a) Reactions:

$\sum M_A = 0: \; -22.5(4)(4) + 6B_y = 0 \; \Rightarrow B_y = \; 60 \text{ kN}(\uparrow)$

$\sum F_y = 0: \; A_y + 60 - 22.5(4) = 0 \; \Rightarrow A_y = \; 30 \text{ kN}(\uparrow)$

(b) Shear force and bending moment diagrams

For: $0 < x < 2: V(x) = 30,$

$$M = 30x, M(0) = 0, M(2) = 60 \; kNm$$

For: $2 < x < 6: V(x) = 30 - 22.5(x - 2) = -22.5x + 75, \quad V(2) = 30 \; kN, V(6) = -60 \; kN$

$$M(x) = 30x - 22.5\frac{(x-2)^2}{2} = -11.25x^2 + 75x - 45, \quad M(2) = 60 \; kNm, M(6) = 0$$

(c) Maximum bending moment:

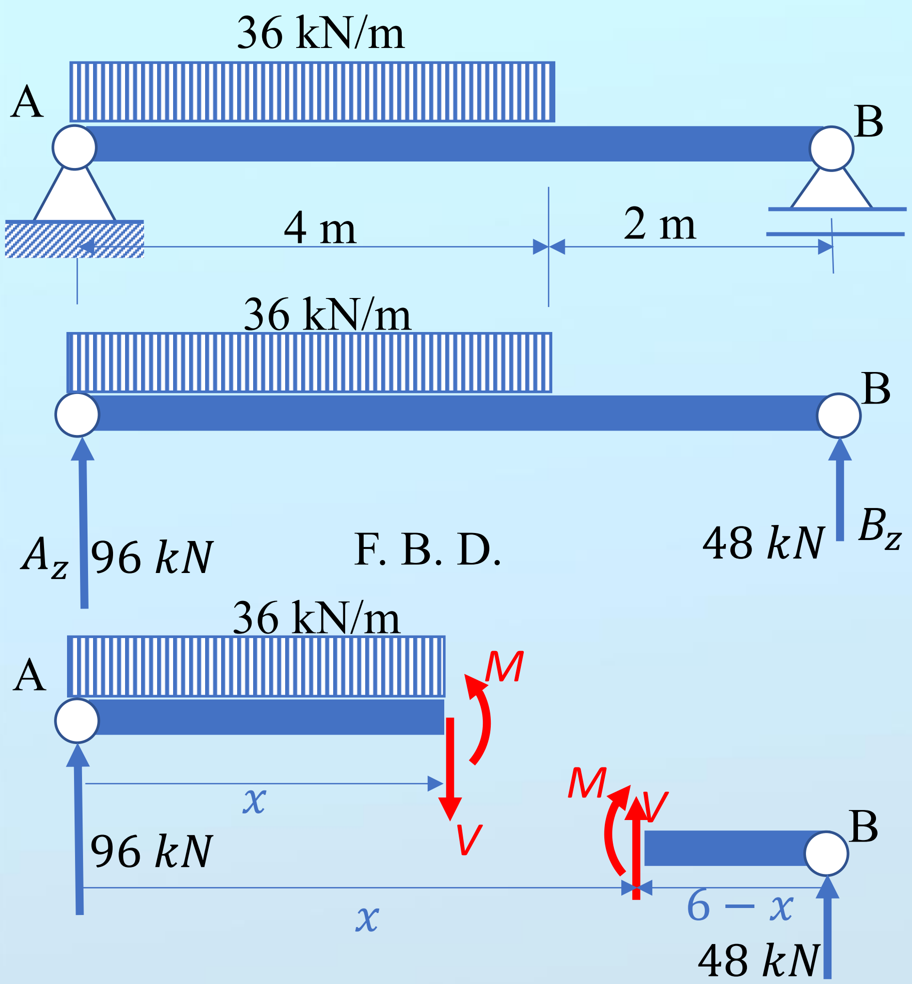
$$V(x) = 0 \; if \; x = 75 \div 22.5 = 10/3 = \bar{x}, \qquad M_{max} = M(10/3) = 80 \; kNm$$

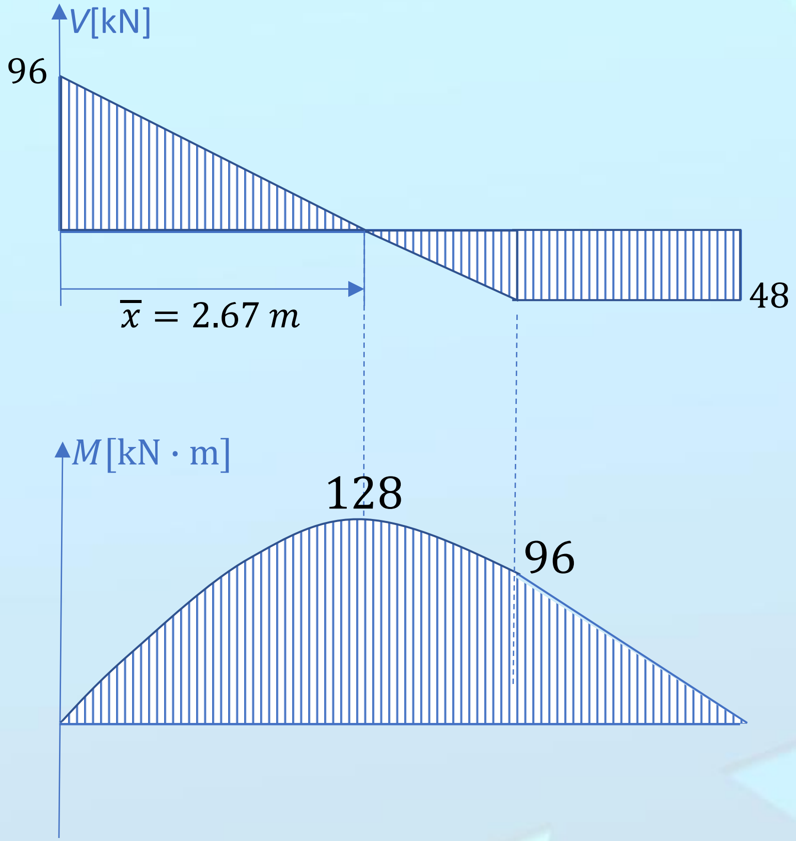
Problem 2. For the shown beam find

(a) Reactions in supports A and B.

(b) Shear force and bending moment diagrams.

(c) Determine the value and the location of the maximum bending moment.





Cut 1: $0 < x < 4$:

$V(x) = 96 - 36x.$

$x = 0: V = 96\ kN;$

$x = 4: V = -48\ kN;$

$V = 0: \bar{x} = 2.67\ m$

$M(x) = 96x - 18x^2.$

$x = 0: M = 0;$

$x = 4: M = 96 kN \cdot m;$

$\bar{x} = 2.67\ m: M_{max}$

$= 128\ kN \cdot m$

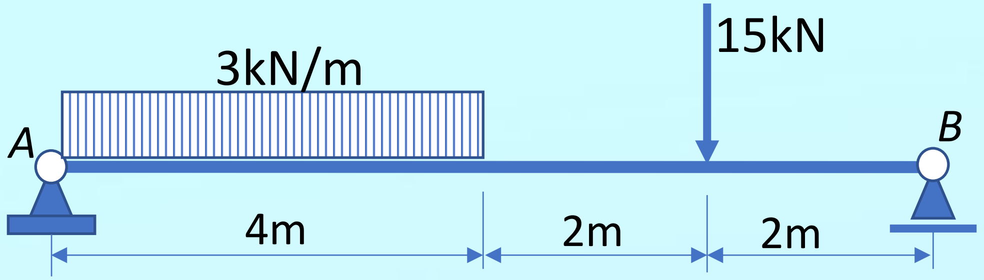
Cut 2: $4 < x < 6$:

$V = -48\ kN.$

$M(x) = 48(6 - x).$

$x = 4: M = 96 kN \cdot m.$

$x = 0: M = 0.$



Problem 3. Determine the shear-force and bending-moment diagrams for the simple beam shown in Fig. using the section method.

Problem 4. Determine the shear-force and bending-moment diagrams for the simple beam shown in Fig. using the section method. Then determine the location and the value of the maximum bending moment.