المحاضرة الثانية نظري- جزء 2

تحليل رياضي 2

جامعة المنارة

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Elementary Matrices المصفوفات الأولية

- **Row elementary matrix** العمليات الأولية على الأسطر :

An $n\times n$ matrix is called an elementary matrix if it can be obtained from the identity matrix I_n by a single elementary operation.

- **Three row elementary matrices:**

(1) $R_{ij} = r_{ij}(I)$ تبديل سطرين.

(2) $R_i^{(k)} = r_i^{(k)}(I) \qquad (k \neq 0)$ ضرب سطر بعدد غير صفري.

(3) $R_{ij}^{(k)} = r_{ij}^{(k)}(I)$ ضرب سطر بعدد غير صفري وجمعه بسطر آخر.

- **Note:**

Only do a single elementary row operation. مسموح بعملية أولية واحدة على المصفوفة الواحدية

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- Ex : (Elementary matrices and non elementary matrices)

$(a)\begin{bmatrix}1&0&0\\0&3&0\\0&0&1\end{bmatrix}$ Yes $(r_2^{(3)}(I_3))$

$(b)\begin{bmatrix}1&0&0\\0&1&0\end{bmatrix}$ No (not square)

$(c)\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}$ No (Row multiplication must be by a nonzero constant)

$(d)\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}$ Yes $(r_{23}(I_3))$

$(e)\begin{bmatrix}1&0\\2&1\end{bmatrix}$ Yes $(r_{12}^{(2)}(I_2))$

$(f)\begin{bmatrix}1&0&0\\0&2&0\\0&0&-1\end{bmatrix}$ No (Use two elementary row operations)

- **Theorem (Representing elementary row operations):**

Let E be the elementary matrix obtained by performing an elementary row operation on I_m. If that same elementary row operation is performed on an $m \times n$ matrix A, then the resulting matrix is given by the product EA.

$$r(I) = E$$
$$r(A) = EA$$

- **Notes:**

(1) $r_{ij}(A) = R_{ij}A$

(2) $r_i^{(k)}(A) = R_i^{(k)}A$

(3) $r_{ij}^{(k)}(A) = R_{ij}^{(k)}A$

- Ex : (Elementary matrices and elementary row operation)

$$(a)\ \begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}\begin{bmatrix}0&2&1\\1&-3&6\\3&2&-1\end{bmatrix}=\begin{bmatrix}1&-3&6\\0&2&1\\3&2&-1\end{bmatrix}\ (r_{12}(A)=R_{12}A)$$

$$(b)\ \begin{bmatrix}1&0&0\\0&\frac{1}{2}&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&-4&1\\0&2&6&-4\\0&1&3&1\end{bmatrix}=\begin{bmatrix}1&0&-4&1\\0&1&3&-2\\0&1&3&1\end{bmatrix}\ (r_2^{(\frac{1}{2})}(A)=R_2^{(\frac{1}{2})}A)$$

$$(c)\ \begin{bmatrix}1&0&0\\2&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&-1\\-2&-2&3\\0&4&5\end{bmatrix}=\begin{bmatrix}1&0&-1\\0&-2&1\\0&4&5\end{bmatrix}\ (r_{12}^{(2)}(A)=R_{12}^{(2)}A)$$

- Ex : (Using elementary matrices)

Find a sequence of elementary matrices that can be used to write the matrix A in row-echelon form.

$$A = \begin{bmatrix} 0 & 1 & 3 & 5 \\ 1 & -3 & 0 & 2 \\ 2 & -6 & 2 & 0 \end{bmatrix}$$

Sol:

$$A_1 = r_{12}(A) = E_1 A = \begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & 1 & 3 & 5 \\ 2 & -6 & 2 & 0 \end{bmatrix} \qquad E_1 = r_{12}(I_3) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A_2 = r_{13}^{(-2)}(A_1) = E_2 A_1 = \begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & -4 \end{bmatrix} \qquad E_2 = r_{13}^{(-2)}(I_3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

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$$A_3 = r_3^{(\frac{1}{2})}(A_2) = E_3 A_2 = \begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & -2 \end{bmatrix} \qquad E_3 = r_3^{(\frac{1}{2})}(I_3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}$$

$$B = E_3 E_2 E_1 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 3 & 5 \\ 1 & -3 & 0 & 2 \\ 2 & -6 & 2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & 1 & 3 & 5 \\ 2 & -6 & 2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & -4 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

- Ex :

Elementary Matrix / Inverse Matrix

$$E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = R_{12}$$

$$(R_{12})^{-1} = E_1^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = R_{12}$$

(Elementary Matrix)

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} = R_{13}^{(-2)}$$

$$(R_{13}^{(-2)})^{-1} = E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} = R_{13}^{(2)}$$

(Elementary Matrix)

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} = R_3^{(\frac{1}{2})}$$

$$(R_3^{(\frac{1}{2})})^{-1} = E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = R_3^{(2)}$$

(Elementary Matrix)

- Theorem (A property of invertible matrices):

A square matrix A is invertible if and only if it can be written as the product of elementary matrices.

- Ex : Find a sequence of elementary matrices whose product is

$$A = \begin{bmatrix} -1 & -2 \\ 3 & 8 \end{bmatrix}$$

Sol:

$$A = \begin{bmatrix} -1 & -2 \\ 3 & 8 \end{bmatrix} \xrightarrow{r_1^{(-1)}} \begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix} \xrightarrow{r_{12}^{(-3)}} \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$$

$$\xrightarrow{r_2^{(1/2)}} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \xrightarrow{r_{21}^{(-2)}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Therefore $R_{21}^{(-2)} R_2^{(\frac{1}{2})} R_{12}^{(-3)} R_1^{(-1)} A = I$

$$\text{Thus } A = (R_1^{(-1)})^{-1}(R_{12}^{(-3)})^{-1}(R_2^{(\frac{1}{2})})^{-1}(R_{21}^{(-2)})^{-1}$$

$$= R_1^{(-1)} R_{12}^{(3)} R_2^{(2)} R_{21}^{(2)}$$

$$= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

- Note:

If A is invertible then

$$E_k \cdots E_3 E_2 E_1 A = I$$

$$A^{-1} = E_k \cdots E_3 E_2 E_1$$

$$A = E_1^{-1} E_2^{-1} E_3^{-1} \cdots E_k^{-1}$$

- LU–factorization:

If the $n\times n$ matrix A can be written as the product of a lower triangular matrix L and an upper triangular matrix U, then $A = LU$ is an LU–factorization of A

- Note:

If a square matrix A can be row reduced to an upper triangular matrix U using only the row operation of adding a multiple of one row to another row below it, then it is easy to find an LU–factorization of A.

$$E_k \cdots E_2 E_1 A = U$$

$$A = E_1^{-1} E_2^{-1} \cdots E_k^{-1} U$$

$$A = LU \qquad (L = E_1^{-1} E_2^{-1} \cdots E_k^{-1})$$

- Ex :(LU –factorization)

$$(a)\ A=\begin{bmatrix}1&2\\1&0\end{bmatrix}\qquad (b)\ A=\begin{bmatrix}1&-3&0\\0&1&3\\2&-10&2\end{bmatrix}$$

Sol: (a)

$$A=\begin{bmatrix}1&2\\1&0\end{bmatrix}\xrightarrow{r_{12}^{(-1)}}\begin{bmatrix}1&2\\0&-2\end{bmatrix}=U$$

$$\Rightarrow R_{12}^{(-1)}A=U$$

$$\Rightarrow A=(R_{12}^{(-1)})^{-1}U=LU$$

$$\Rightarrow L=(R_{12}^{(-1)})^{-1}=R_{12}^{(1)}=\begin{bmatrix}1&0\\1&1\end{bmatrix}$$

(b)

$$A=\begin{bmatrix}1 & -3 & 0\\ 0 & 1 & 3\\ 2 & -10 & 2\end{bmatrix}\xrightarrow{r_{13}^{(-2)}}\begin{bmatrix}1 & -3 & 0\\ 0 & 1 & 3\\ 0 & -4 & 2\end{bmatrix}\xrightarrow{r_{23}^{(4)}}\begin{bmatrix}1 & -3 & 0\\ 0 & 1 & 3\\ 0 & 0 & 14\end{bmatrix}=U$$

$$\Rightarrow R_{23}^{(4)}R_{13}^{(-2)}A=U$$

$$\Rightarrow A=(R_{13}^{(-2)})^{-1}(R_{23}^{(4)})^{-1}U=LU$$

$$\Rightarrow L=(R_{13}^{(-2)})^{-1}(R_{23}^{(4)})^{-1}=R_{13}^{(2)}R_{23}^{(-4)}$$

$$=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 2 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & -4 & 1\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 2 & -4 & 1\end{bmatrix}$$

- Solving $A\mathbf{x} = \mathbf{b}$ with an LU–factorization of A

$A\mathbf{x} = \mathbf{b}$ If $A = LU$, then $LU\mathbf{x} = \mathbf{b}$

Let $\mathbf{y} = U\mathbf{x}$, then $L\mathbf{y} = \mathbf{b}$

- Two steps:

(1) Write $\mathbf{y} = U\mathbf{x}$, and solve $L\mathbf{y} = \mathbf{b}$ for $\mathbf{y}$

(2) Solve $U\mathbf{x} = \mathbf{y}$ for $\mathbf{x}$

- Ex : (Solving a linear system using LU–factorization)

$$\begin{aligned} x_1 - 3x_2 \qquad\qquad &= -5 \\ x_2 + 3x_3 &= -1 \\ 2x_1 - 10x_2 + 2x_3 &= -20 \end{aligned}$$

Sol:

$$A = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 2 & -10 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -4 & 1 \end{bmatrix} \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 14 \end{bmatrix} = LU$$

(1) Let $\boldsymbol{y} = U\boldsymbol{x}$, and solve for $L\boldsymbol{y} = \boldsymbol{b}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -4 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -5 \\ -1 \\ -20 \end{bmatrix} \Rightarrow \begin{aligned} y_1 &= -5 \\ y_2 &= -1 \\ y_3 &= -20 - 2y_1 + 4y_2 = -14 \end{aligned}$$

(2) Solve the following system $U\boldsymbol{x} = \boldsymbol{y}$

$$\begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 14 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -5 \\ -1 \\ -14 \end{bmatrix}$$

So $x_3 = -1$

$$x_2 = -1 - 3x_3 = -1 - (3)(-1) = 2$$

$$x_1 = -5 + 3x_2 = -5 + 3(2) = 1$$

Thus, the solution is

$$\boldsymbol{x} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$