

## المحاضرة الثالثة: المحددات

تحليل رياضي 2

جامعة المنارة

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# المحددات المحددات والعمليات الأولية خواص المحددات



## The Determinant of a Matrix محدد مصفوفة

$\det: M_n(K) \rightarrow K; A \mapsto \det(A) = |A|$  محدد مصفوفة عبارة عن تابع معرف بالشكل

$K: (R \text{ or } C)$  مجموعة عددية

- **The determinant of a  $2 \times 2$  matrix مرتبة من المصفوفة  $2 \times 2$ :**

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \Rightarrow \det(A) = |A| = a_{11}a_{22} - a_{21}a_{12}$$

- **Ex. 1:**

$$\begin{vmatrix} 2 & -3 \\ 1 & 2 \end{vmatrix} = 2(2) - 1(-3) = 4 + 3 = 7$$

$$\begin{vmatrix} 0 & 3 \\ 2 & 4 \end{vmatrix} = 0(4) - 2(3) = 0 - 6 = -6$$

- **ملاحظة:** محدد المصفوفة قد يكون موجب أو سالب أو صفر.



■ **مصغر العنصر  $a_{ij}$ : Minor of the entry  $a_{ij}$**

محدد المصفوفة الناتجة عن حذف السطر  $i$ - والعمود  $j$  من المصفوفة

$$M_{ij} = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1(j-1)} & a_{1(j+1)} & \dots & a_{1n} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ a_{(i-1)1} & a_{(i-1)2} & \dots & a_{(i-1)(j-1)} & a_{(i-1)(j+1)} & \dots & a_{(i-1)n} \\ a_{(i+1)1} & a_{(i+1)2} & \dots & a_{(i+1)(j-1)} & a_{(i+1)(j+1)} & \dots & a_{(i+1)n} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{n(j-1)} & a_{n(j+1)} & \dots & a_{nn} \end{vmatrix}$$

■ **المرافق الجبري لـ  $a_{ij}$ : Cofactor of  $a_{ij}$**

$$C_{ij} = (-1)^{i+j} M_{ij}$$



■ Ex 2:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\Rightarrow M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$$

$$\Rightarrow C_{21} = (-1)^{2+1} M_{21} = -M_{21}$$

$$M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$$

$$C_{22} = (-1)^{2+2} M_{22} = M_{22}$$



■ Ex 3: Find all the minors and cofactors of  $A$

أوجد كافة المصغرات والمرافقات الجبرية

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}$$

**Sol:** (1) All the minors of  $A$

$$M_{11} = \begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} = -1,$$

$$M_{12} = \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix} = -5,$$

$$M_{13} = \begin{vmatrix} 3 & -1 \\ 4 & 0 \end{vmatrix} = 4$$

$$M_{21} = \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} = 2,$$

$$M_{22} = \begin{vmatrix} 0 & 1 \\ 4 & 1 \end{vmatrix} = -4,$$

$$M_{23} = \begin{vmatrix} 0 & 2 \\ 4 & 1 \end{vmatrix} = -8$$

$$M_{31} = \begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} = 5,$$

$$M_{32} = \begin{vmatrix} 0 & 1 \\ 3 & 2 \end{vmatrix} = -3,$$

$$M_{33} = \begin{vmatrix} 0 & 2 \\ 3 & -1 \end{vmatrix} = -6$$



## (2) All the cofactors of $A$

$$C_{ij} = (-1)^{i+j} M_{ij}$$

$$C_{11} = + \begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} = -1,$$

$$C_{12} = - \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix} = 5,$$

$$C_{13} = + \begin{vmatrix} 3 & -1 \\ 4 & 0 \end{vmatrix} = 4$$

$$C_{21} = - \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} = -2,$$

$$C_{22} = + \begin{vmatrix} 0 & 1 \\ 4 & 1 \end{vmatrix} = -4,$$

$$C_{23} = - \begin{vmatrix} 0 & 2 \\ 4 & 1 \end{vmatrix} = 8$$

$$C_{31} = + \begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} = 5,$$

$$C_{32} = - \begin{vmatrix} 0 & 1 \\ 3 & 2 \end{vmatrix} = 3,$$

$$C_{33} = + \begin{vmatrix} 0 & 2 \\ 3 & -1 \end{vmatrix} = -6$$



■ **Theorem :**

محدد المصفوفة المربعة A يحسب بالشكل

$$(a) \det(A) = |A| = \sum_{j=1}^n a_{ij} C_{ij} = a_{i1} C_{i1} + a_{i2} C_{i2} + \cdots + a_{in} C_{in}$$

( $i$ -th row,  $i = 1, 2, \dots, n$  النشر وفق السطر)

or

$$(b) \det(A) = |A| = \sum_{i=1}^n a_{ij} C_{ij} = a_{1j} C_{1j} + a_{2j} C_{2j} + \cdots + a_{nj} C_{nj}$$

( $j$ -th column,  $j = 1, 2, \dots, n$  النشر وفق العمود)



■ **Ex 4: محدد مصفوفة من الحجم 3**

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{aligned} \Rightarrow \det(A) &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \\ &= a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23} \\ &= a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33} \\ &= a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31} \\ &= a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{32} \\ &= a_{13}C_{13} + a_{23}C_{23} + a_{33}C_{33} \end{aligned}$$



■ Ex 5: The determinant of a matrix of order 3

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{bmatrix}$$

From Ex 3:

$$\begin{aligned} C_{11} &= -1, & C_{12} &= 5, & C_{13} &= 4 \\ C_{21} &= -2, & C_{22} &= -4, & C_{23} &= 8 \\ C_{31} &= 5, & C_{32} &= 3, & C_{33} &= -6 \end{aligned}$$

**Sol:**

$$\begin{aligned} \Rightarrow \det(A) &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} = (0)(-1) + (2)(5) + (1)(4) = 14 \\ &= a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23} = (3)(-2) + (-1)(-4) + (2)(8) = 14 \\ &= a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33} = (4)(5) + (0)(3) + (1)(-6) = 14 \\ &= a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31} = (0)(-1) + (3)(-2) + (4)(5) = 14 \\ &= a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{32} = (2)(5) + (-1)(-4) + (0)(3) = 14 \\ &= a_{13}C_{13} + a_{23}C_{23} + a_{33}C_{33} = (1)(4) + (2)(8) + (1)(-6) = 14 \end{aligned}$$



■ Ex 6: The determinant of a matrix of order 3

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & -4 & 1 \end{bmatrix} \Rightarrow \det(A) = ?$$

**Sol:**

$$C_{11} = (-1)^{1+1} \begin{vmatrix} -1 & 2 \\ -4 & 1 \end{vmatrix} = 7$$

$$C_{12} = (-1)^{1+2} \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix} = (-1)(-5) = 5$$

$$C_{13} = (-1)^{1+3} \begin{vmatrix} 3 & -1 \\ 4 & -4 \end{vmatrix} = -8$$

$$\begin{aligned} \Rightarrow \det(A) &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \\ &= (0)(7) + (2)(5) + (1)(-8) = 2 \end{aligned}$$



- **Note:**

أفضل خيار لحساب المحدد هو النشر وفق السطر أو العمود الذي يحوي أكبر عدد من الأصفار

- **Ex 7: The determinant of a matrix of order 4**

$$A = \begin{bmatrix} 1 & -2 & 3 & 0 \\ -1 & 1 & 0 & 2 \\ 0 & 2 & 0 & 3 \\ 3 & 4 & 0 & -2 \end{bmatrix} \Rightarrow \det(A) = ?$$



**Sol:**

$$\det(A) = (3)(C_{13}) + (0)(C_{23}) + (0)(C_{33}) + (0)(C_{43}) = 3C_{13}$$

$$= 3(-1)^{1+3} \begin{vmatrix} -1 & 1 & 2 \\ 0 & 2 & 3 \\ 3 & 4 & -2 \end{vmatrix}$$

$$= 3 \left[ (0)(-1)^{2+1} \begin{vmatrix} 1 & 2 \\ 4 & -2 \end{vmatrix} + (2)(-1)^{2+2} \begin{vmatrix} -1 & 2 \\ 3 & -2 \end{vmatrix} + (3)(-1)^{2+3} \begin{vmatrix} -1 & 1 \\ 3 & 4 \end{vmatrix} \right]$$

$$= 3 \left[ 0 + (2)(1)(-4) + (3)(-1)(-7) \right]$$

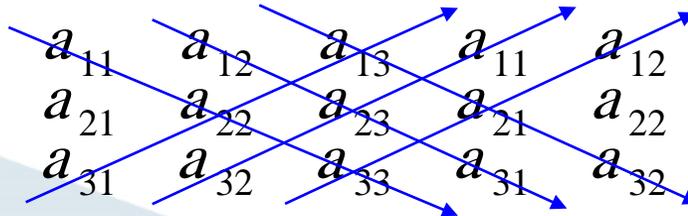
$$= (3)(13)$$

$$= 39$$



■ The determinant of a matrix of order 3 (Sarrus Rule)

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

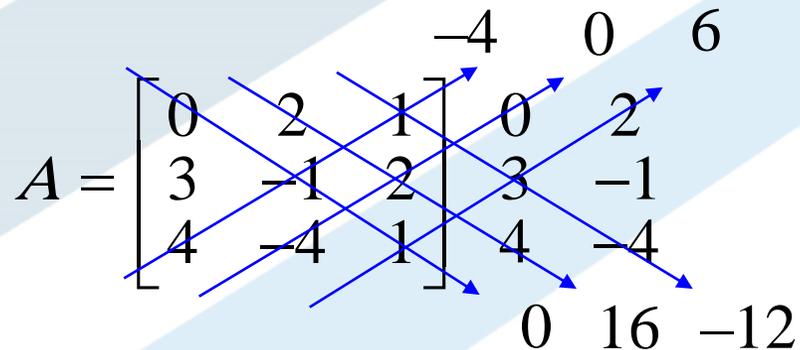


ثم طرح هذه الجداءات الثلاث

جمع هذه الجداءات الثلاث

$$\Rightarrow \det(A) = |A| = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}$$

■ Ex 8:



$$A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & -4 & 1 \end{bmatrix}$$

$$\Rightarrow \det(A) = |A| = 0 + 16 - 12 - (-4 + 0 + 6) = 2$$



- **المصفوفة المثلثية العليا: Upper triangular matrix**  
كافة العناصر الواقعة تحت القطر الرئيسي أصفار
- **المصفوفة المثلثية السفلى: Lower triangular matrix**  
كافة العناصر فوق القطر الرئيسي أصفار
- **المصفوفة القطرية: Diagonal matrix**  
كافة العناصر الواقعة تحت وفوق القطر الرئيسي أصفار



■ **Ex :**

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$$

**upper triangular**

$$\begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

**lower triangular**

$$\begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$$

**diagonal**

■ **Theorem :**

If  $A$  is an  $n \times n$  triangular matrix (upper triangular, lower triangular, or diagonal),

عندما تكون المصفوفة قطرية أو مثلثية عليا أو سفلى يكون المحدد جداء عناصر القطر الرئيسي

$$\det(A) = |A| = a_{11} a_{22} a_{33} \cdots a_{nn}$$



■ Ex :

$$(a) \quad A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 4 & -2 & 0 & 0 \\ -5 & 6 & 1 & 0 \\ 1 & 5 & 3 & 3 \end{bmatrix}$$

$$(b) \quad B = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix}$$

Sol:

$$(a) \quad |A| = (2)(-2)(1)(3) = -12$$

$$(b) \quad |B| = (-1)(3)(2)(4)(-2) = 48$$



## Evaluation of a determinant using elementary operations

### ■ Theorem : (العمليات الأولية على الأسطر والمحددات)

Let  $A$  and  $B$  be square matrices مصفوفتين مربعتين .

$$(a) B = r_{ij}(A) \Rightarrow \det(B) = -\det(A) \quad (\text{i.e. } |r_{ij}(A)| = -|A|)$$

$$(b) B = r_i^{(k)}(A) \Rightarrow \det(B) = k \det(A) \quad (\text{i.e. } |r_i^{(k)}(A)| = k|A|)$$

$$(c) B = r_{ij}^{(k)}(A) \Rightarrow \det(B) = \det(A) \quad (\text{i.e. } |r_{ij}^{(k)}(A)| = |A|)$$



■ Ex :

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 1 & 2 & 1 \end{bmatrix}, \quad \det(A) = -2$$

$$A_1 = \begin{bmatrix} 4 & 8 & 12 \\ 0 & 1 & 4 \\ 1 & 2 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 & 4 \\ 1 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & 2 & 3 \\ -2 & -3 & -2 \\ 1 & 2 & 1 \end{bmatrix}$$

$$A_1 = r_1^{(4)}(A) \Rightarrow \det(A_1) = \det(r_1^{(4)}(A)) = 4\det(A) = (4)(-2) = -8$$

$$A_2 = r_{12}(A) \Rightarrow \det(A_2) = \det(r_{12}(A)) = -\det(A) = -(-2) = 2$$

$$A_3 = r_{12}^{(-2)}(A) \Rightarrow \det(A_3) = \det(r_{12}^{(-2)}(A)) = \det(A) = -2$$



■ **Notes:**

$$\det(r_{ij}(A)) = -\det(A) \quad \Rightarrow \quad \det(A) = -\det(r_{ij}(A))$$

$$\det(r_i^{(k)}(A)) = k \det(A) \quad \Rightarrow \quad \det(A) = \frac{1}{k} \det(r_i^{(k)}(A))$$

$$\det(r_{ij}^{(k)}(A)) = \det(A) \quad \Rightarrow \quad \det(A) = \det(r_{ij}^{(k)}(A))$$



■ Ex :

$$A = \begin{bmatrix} 2 & -3 & 10 \\ 1 & 2 & -2 \\ 0 & 1 & -3 \end{bmatrix}, \quad \det(A) = ?$$

Sol:

$$\det(A) = \begin{vmatrix} 2 & -3 & 10 \\ 1 & 2 & -2 \\ 0 & 1 & -3 \end{vmatrix} \stackrel{r_{12}}{=} - \begin{vmatrix} 1 & 2 & -2 \\ 2 & -3 & 10 \\ 0 & 1 & -3 \end{vmatrix} \stackrel{r_{12}^{(-2)}}{=} - \begin{vmatrix} 1 & 2 & -2 \\ 0 & -7 & 14 \\ 0 & 1 & -3 \end{vmatrix}$$

$$\stackrel{r_2^{(-\frac{1}{7})}}{=} (-1) \left( \frac{1}{7} \right) \begin{vmatrix} 1 & 2 & -2 \\ 0 & 1 & -2 \\ 0 & 1 & -3 \end{vmatrix} \stackrel{r_{23}^{(-1)}}{=} (7) \begin{vmatrix} 1 & 2 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & -1 \end{vmatrix} \stackrel{r_3^{(-1)}}{=} (7)(-1) \begin{vmatrix} 1 & 2 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= (7)(-1)(1)(1)(1) = -7$$



## Determinants and elementary column operations

### ■ Theorem : (العمليات الأولية على الأعمدة والمحددات)

Let  $A$  and  $B$  be square matrices مصفوفتين مربعتين .

$$(a) B = c_{ij}(A) \Rightarrow \det(B) = -\det(A) \quad (\text{i.e. } |c_{ij}(A)| = -|A|)$$

$$(b) B = c_i^{(k)}(A) \Rightarrow \det(B) = k \det(A) \quad (\text{i.e. } |c_i^{(k)}(A)| = k|A|)$$

$$(c) B = c_{ij}^{(k)}(A) \Rightarrow \det(B) = \det(A) \quad (\text{i.e. } |c_{ij}^{(k)}(A)| = |A|)$$



■ Ex :

$$A = \begin{bmatrix} 2 & 1 & -3 \\ 4 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix}, \quad \det(A) = -8$$

$$A_1 = \begin{bmatrix} 1 & 1 & -3 \\ 2 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 4 & 1 \\ 0 & 0 & 2 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 2 & 1 & 0 \\ 4 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

$$A_1 = c_1^{(\frac{1}{2})}(A) \Rightarrow \det(A_1) = \det(c_1^{(\frac{1}{2})}(A)) = \frac{1}{2} \det(A) = \left(\frac{1}{2}\right)(-8) = -4$$

$$A_2 = c_{12}(A) \Rightarrow \det(A_2) = \det(c_{12}(A)) = -\det(A) = -(-8) = 8$$

$$A_3 = c_{23}^{(3)}(A) \Rightarrow \det(A_3) = \det(c_{23}^{(3)}(A)) = \det(A) = -8$$



■ **Theorem : (Conditions that yield a zero determinant)**

إذا كانت المصفوفة مربعة وتحققت إحدى الشروط التالية عندئذ  $\det(A) = 0$

1. سطر (عمود) كل عناصره أصفار.
2. سطرين أو عمودين متساويين.
3. سطر (عمود) مضاعف لسطر (عمود) آخر



■ Ex :

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 4 & 5 & 6 \end{vmatrix} = 0$$

$$\begin{vmatrix} 1 & 4 & 0 \\ 2 & 5 & 0 \\ 3 & 6 & 0 \end{vmatrix} = 0$$

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 4 & 5 & 6 \end{vmatrix} = 0$$

$$\begin{vmatrix} 1 & 4 & 2 \\ 1 & 5 & 2 \\ 1 & 6 & 2 \end{vmatrix} = 0$$

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ -2 & -4 & -6 \end{vmatrix} = 0$$

$$\begin{vmatrix} 1 & 8 & 4 \\ 2 & 10 & 5 \\ 3 & 12 & 6 \end{vmatrix} = 0$$



■ Ex : (Evaluating a determinant)

Sol:

$$A = \begin{bmatrix} -3 & 5 & 2 \\ 2 & -4 & -1 \\ -3 & 0 & 6 \end{bmatrix}$$

$$\det(A) = \begin{vmatrix} -3 & 5 & 2 \\ 2 & -4 & -1 \\ -3 & 0 & 6 \end{vmatrix} \stackrel{C_{13}^{(2)}}{=} \begin{vmatrix} -3 & 5 & -4 \\ 2 & -4 & 3 \\ -3 & 0 & 0 \end{vmatrix} = (-3)(-1)^{3+1} \begin{vmatrix} 5 & -4 \\ -4 & 3 \end{vmatrix} = (-3)(-1) = 3$$

$$\det(A) = \begin{vmatrix} -3 & 5 & 2 \\ 2 & -4 & -1 \\ -3 & 0 & 6 \end{vmatrix} \stackrel{r_{12}^{(4/5)}}{=} \begin{vmatrix} -3 & 5 & 2 \\ -2/5 & 0 & 3/5 \\ -3 & 0 & 6 \end{vmatrix} = (5)(-1)^{1+2} \begin{vmatrix} -2/5 & 3/5 \\ -3 & 6 \end{vmatrix} = (-5)(-\frac{3}{5}) = 3$$



■ Ex : (Evaluating a determinant)

Sol:

$$A = \begin{bmatrix} 2 & 0 & 1 & 3 & -2 \\ -2 & 1 & 3 & 2 & -1 \\ 1 & 0 & -1 & 2 & 3 \\ 3 & -1 & 2 & 4 & -3 \\ 1 & 1 & 3 & 2 & 0 \end{bmatrix}$$

$$\det(A) = \begin{vmatrix} 2 & 0 & 1 & 3 & -2 \\ -2 & 1 & 3 & 2 & -1 \\ 1 & 0 & -1 & 2 & 3 \\ 3 & -1 & 2 & 4 & -3 \\ 1 & 1 & 3 & 2 & 0 \end{vmatrix} \begin{matrix} r_{24}^{(1)} \\ = \\ r_{25}^{(-1)} \end{matrix} = \begin{vmatrix} 2 & 0 & 1 & 3 & -2 \\ -2 & 1 & 3 & 2 & -1 \\ 1 & 0 & -1 & 2 & 3 \\ 1 & 0 & 5 & 6 & -4 \\ 3 & 0 & 0 & 0 & 1 \end{vmatrix}$$

$$= (1)(-1)^{2+2} \begin{vmatrix} 2 & 1 & 3 & -2 \\ 1 & -1 & 2 & 3 \\ 1 & 5 & 6 & -4 \\ 3 & 0 & 0 & 1 \end{vmatrix}$$



$$c_{41}^{(-3)} = \begin{vmatrix} 8 & 1 & 3 & -2 \\ -8 & -1 & 2 & 3 \\ 13 & 5 & 6 & -4 \\ 0 & 0 & 0 & 1 \end{vmatrix} = (1)(-1)^{4+4} \begin{vmatrix} 8 & 1 & 3 \\ -8 & -1 & 2 \\ 13 & 5 & 6 \end{vmatrix} r_{21}^{(1)} = \begin{vmatrix} 0 & 0 & 5 \\ -8 & -1 & 2 \\ 13 & 5 & 6 \end{vmatrix}$$

$$= 5(-1)^{1+3} \begin{vmatrix} -8 & -1 \\ 13 & 5 \end{vmatrix}$$

$$= (5)(-27)$$

$$= -135$$



## Properties of Determinants

- **Theorem 3.6:** (محدد جداء مصفوفة)

$$\det (AB) = \det (A) \det (B)$$

- **Notes:**

$$(1) \det (EA) = \det (E) \det (A)$$

$$(2) \det (A + B) \neq \det (A) + \det (B)$$

$$(3) \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} + b_{21} & a_{22} + b_{22} & a_{23} + b_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ b_{21} & b_{22} & b_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$



■ **Ex :**

$$A = \begin{bmatrix} 1 & -2 & 2 \\ 0 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 & 1 \\ 0 & -1 & -2 \\ 3 & 1 & -2 \end{bmatrix}$$

Find  $|A|$ ,  $|B|$ , and  $|AB|$

**Sol:**

$$|A| = \begin{vmatrix} 1 & -2 & 2 \\ 0 & 3 & 2 \\ 1 & 0 & 1 \end{vmatrix} = -7, \quad |B| = \begin{vmatrix} 2 & 0 & 1 \\ 0 & -1 & -2 \\ 3 & 1 & -2 \end{vmatrix} = 11$$



$$AB = \begin{bmatrix} 1 & -2 & 2 \\ 0 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 0 & -1 & -2 \\ 3 & 1 & -2 \end{bmatrix} = \begin{bmatrix} 8 & 4 & 1 \\ 6 & -1 & -10 \\ 5 & 1 & -1 \end{bmatrix}$$

$$\Rightarrow |AB| = \begin{vmatrix} 8 & 4 & 1 \\ 6 & -1 & -10 \\ 5 & 1 & -1 \end{vmatrix} = -77$$

تحقق: Check:

$$|AB| = |A| |B|$$

$$-77 = -7 \times 11$$



■ **Theorem :** (محدد مصفوفة مضروبة بعدد)

If  $A$  is an  $n \times n$  matrix and  $c$  is a scalar, then

$$\det (cA) = c^n \det (A)$$

■ **Ex 2:**

$$A = \begin{bmatrix} 10 & -20 & 40 \\ 30 & 0 & 50 \\ -20 & -30 & 10 \end{bmatrix}, \quad \begin{vmatrix} 1 & -2 & 4 \\ 3 & 0 & 5 \\ -2 & -3 & 1 \end{vmatrix} = 5 \quad \text{Find } |A|$$

**Sol:**

$$A = 10 \begin{bmatrix} 1 & -2 & 4 \\ 3 & 0 & 5 \\ -2 & -3 & 1 \end{bmatrix} \Rightarrow |A| = 10^3 \begin{vmatrix} 1 & -2 & 4 \\ 3 & 0 & 5 \\ -2 & -3 & 1 \end{vmatrix} = (1000)(5) = 5000$$



■ **Theorem :** (محدد مصفوفة غير شاذة)

تكون المصفوفة المربعة قابلة للقلب إذا وفقط إذا كان  $\det(A) \neq 0$

■ **Ex 3:** حدد فيما إذا كانت المصفوفة شاذة أو لا

$$A = \begin{bmatrix} 0 & 2 & -1 \\ 3 & -2 & 1 \\ 3 & 2 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 2 & -1 \\ 3 & -2 & 1 \\ 3 & 2 & 1 \end{bmatrix}$$

**Sol:**

$$|A| = 0 \Rightarrow A \text{ شاذة (it is singular).}$$

$$|B| = -12 \neq 0 \Rightarrow B \text{ قابلة للقلب (it is nonsingular).}$$



■ **Theorem :** (محدد مقلوب مصفوفة)

If  $A$  is invertible, then  $\det(A^{-1}) = \frac{1}{\det(A)}$ .

■ **Theorem :** (محدد منقول مصفوفة)

If  $A$  is a square matrix, then  $\det(A^T) = \det(A)$ .

■ **Ex :**

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & -1 & 2 \\ 2 & 1 & 0 \end{bmatrix}$$

(a)  $|A^{-1}| = ?$       (b)  $|A^T| = ?$

**Sol:**

$$|A| = \begin{vmatrix} 1 & 0 & 3 \\ 0 & -1 & 2 \\ 2 & 1 & 0 \end{vmatrix} = 4 \Rightarrow |A^{-1}| = \frac{1}{|A|} = \frac{1}{4}, \quad |A^T| = |A| = 4$$



■ **Equivalent conditions for a nonsingular matrix:**

If  $A$  is an  $n \times n$  matrix, عندئذ الشروط التالية متكافئة.

(1)  $A$  قابلة للقلب.

(2)  $A\mathbf{x} = \mathbf{b}$  تملك حل وحيد  $\mathbf{x}$ .

(3)  $A\mathbf{x} = \mathbf{0}$  تملك الحل الصفري  $\mathbf{x} = \mathbf{0}$ .

(4)  $\det(A) \neq 0$



- Ex : Which of the following system has a unique solution?

$$(a) \quad \begin{aligned} 2x_2 - x_3 &= -1 \\ 3x_1 - 2x_2 + x_3 &= 4 \\ 3x_1 + 2x_2 - x_3 &= -4 \end{aligned}$$

$$(b) \quad \begin{aligned} 2x_2 - x_3 &= -1 \\ 3x_1 - 2x_2 + x_3 &= 4 \\ 3x_1 + 2x_2 + x_3 &= -4 \end{aligned}$$

**Sol:**

(a)  $Ax = b \Rightarrow |A| = 0$ . هذه الجملة لا تملك حل وحيد.

(b)  $Bx = b \Rightarrow |B| = -12 \neq 0$ . هذه الجملة تملك حل وحيد.

