كلية الصيدلة السنة الأولى

٢٠٢٥-٢٠٢٦

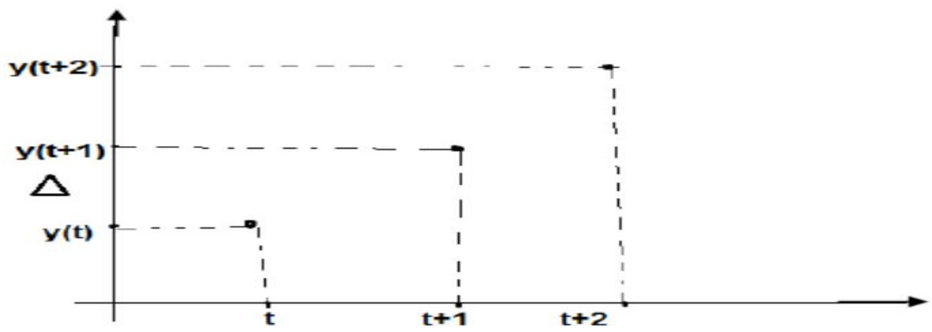
الرياضيات

المحاضرة الرابعة د. زياد اليوسف

توابـــع الفـــروق (Difference Functions)

تعريف(1-1):-

تعرف معادلات الفروق على أنها معادلة معرفة بشكل ترجيعي.



وتكتب معادلة الفروق للدالة $y(t)$ على الشكل الأتي:-

$$\Delta y(t) = y(t+1) - y(t)$$

ملاحظة(1-1):-

الرمز Δ يسمى بعامل الفرق (Difference Operator).

مثال(1-1):-

أحسب $\Delta y(t)$ إذا كان :

$$y(t) = t^2$$

الحل:-

$$\Delta y(t) = y(t+1) - y(t)$$

$$\Delta y(t) = (t+1)^2 - t^2$$

$$= t^2 + 2t + 1 - t^2$$

$$= 2t + 1$$

تعريف(1-2):-

إذا كانت لدينا $y(t,x)$ فإن :

1-) $\Delta_t y(t,x) = y(t+1,x) - y(t,x)$

2-) $\Delta_x y(t,x) = y(t,x+1) - y(t,x)$

حيث $\Delta_t y(t,x)$ هو الفرق الجزئي بالنسبة للمعامل t, و $\Delta_x y(t,x)$ هو الفرق الجزئي بالنسبة للمعامل x .

مثال(1-2):-

احسب الفرق بالنسبة للمعامل x وبالنسبة للمعامل t إذا كانت لدينا $y(t,x) = xt + t^2$.

الحل:-

$$\Delta_x y(t,x) = \left((x+1)t + t^2\right) - (xt + t^2)$$

$$= xt + t + t^2 - xt - t^2 = t$$

$$\therefore \Delta_x y(t,x) = t$$

$$\Delta_t y(t,x) = (x(t+1) + (t+1)^2) - (xt + t^2)$$

$$= xt + x + t^2 + 2t + 1 - xt - t^2$$

$$= 2t + x + 1$$

$$\therefore \Delta_t y(t,x) = 2t + x + 1$$

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فروق بعض الدوال العامة:-

1)- $\Delta a^t = (a-1)a^t \,,\; a \neq 1$

2)- $\Delta \sin(at) = 2\sin(\frac{a}{2})\cos(a(t+\frac{1}{2}))$

3)- $\Delta \cos(at) = -2\sin(\frac{a}{2})\sin(a\left(t+\frac{1}{2}\right))$

4)- $\Delta \log(at) = \log(1+\frac{1}{t})$

5)- $\Delta \log(\Gamma(t)) = \log(t)$

6)- $\Delta t^n = \sum_{k=0}^{n-1}\binom{n}{k}t^k$

مثال(1-3):-

احسب $\Delta y(t)$ إذا كانت $y(t) = 8^t$.

الحل:-

$$\Delta 8^t = (8-1)8^t = 7.8^t$$

مثال(1-4):-

احسب $\Delta y(t)$ إذا كانت $y(t) = \cos(\pi t)$.

الحل:-

$$\Delta\cos(\pi t) = -2\sin(\frac{\pi}{2})\sin(\pi\left(t+\frac{1}{2}\right))$$

تعريف(1-3):-

يعرف عامل التغير**(shift operator)** للدالة $y(t)$ والذي يرمز له بالرمز (E) على الشكل التالي:-

$$Ey(t) = y(t+1)$$

مثال(1-5):-

أحسب $Ey(2)$ إذا كانت $y(t) = 3t + 1$.

الحل:-

$$Ey(t) = 3(t+1) + 1$$

في حالة $t=2$ لدينا:

$$Ey(2) = 3(2+1) + 1 = 10$$

خاصية(1-1):-

لتكن لدينا الدالة $y(t)$, فإن :

$$\Delta Ey(t) = E\Delta y(t)$$

البرهان:-

$$\Delta Ey(t) = \Delta y(t+1)$$

$$= y(t+2) - y(t+1)$$

$$= Ey(t+1) - Ey(t)$$

$$= E\big(y(t+1) - y(t)\big)$$

$$= E\Delta y(t)$$

$$\therefore \Delta Ey(t) = E\Delta y(t)$$

تعريف(1-4):-

يعرف العامل المحايد**(Identity operator)** للدالة $y(t)$ والذي يرمز له بالرمز (I) علي الشكل التالي:

$$Iy(t) = y(t)$$

مثال(1-6):-

احسب $Iy(t)$ إذا كانت $y(t) = 3t$.

الحل:-

$$Iy(t) = 3t$$

خاصية(1-2):-

$$\Delta y(t) = Ey(t) - Iy(t)$$

البرهان:-

حسب تعريف (1-1) لدينا :

$$\Delta y(t) = y(t+1) - y(t)$$

وحسب التعريفان (1-3), (1-4) نستنتج أن :

$$\Delta y(t) = Ey(t) - Iy(t)$$

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نظرية(1-1):-

ليكن لدينا دالتين $z(t), y(t)$ حيث $t \in \mathbb{N}$ وليكن k ثابت, فإن:

-(1 $\Delta(ky(t)) = k\Delta y(t)$

البرهان:-

حسب التعريف (1-1) نجد أن :

$$\Delta(ky(t)) = ky(t+1) - ky(t)$$
$$= k\Delta y(t)$$
$$\therefore \Delta(ky(t)) = k\Delta y(t)$$

مثال(1-7):-

إذا كانت $y(t) = t^2$ حيث $t \in \mathbb{Z}_+$ وكانت $k = 3$, أحسب $\Delta(ky(t))$.

الحل:-

$$\because \Delta(ky(t)) = k\Delta y(t)$$

$$\therefore \Delta(3(t^2)) = 3\Delta(t^2)$$

$$= 3((t+1)^2 - t^2)$$

$$= 3(t^2 + 2t + 1 - t^2)$$

$$= 3t^2 + 6t + 3 - 3t^2$$

$$= 6t + 3$$

$$\Delta\big(y(t)+z(t)\big)=\Delta y(t)+\Delta z(t)$$ -(2

البرهان:-

حسب التعريف (1-1) نجد أن :

$$\Delta\big(y(t)+z(t)\big)=y(t+1)+z(t+1)-\big(y(t)+z(t)\big)$$

$$=y(t+1)-y(t)+z(t+1)-z(t)$$

$$=\Delta y(t)+\Delta z(t)$$

$$\therefore\ \Delta\big(y(t)+z(t)\big)=\Delta y(t)+\Delta z(t)$$

مثال(1-8):-

إذا كانت $y(t)=2t$, $z(t)=t+2$,حيث $t\in\mathbb{Z}_+$, أحسب $\Delta\big(y(t)+z(t)\big)$.

الحل:-

$$\because\ \Delta\big(y(t)+z(t)\big)=\Delta y(t)+\Delta z(t)$$

$$\therefore\ \Delta\big((2t)+(t+1)\big)=\Delta(2t)+\Delta(t+1)$$

$$=(2(t+1)-2t)+((t+2)-(t+1))$$

$$=3$$

$$\Delta\big(y(t)z(t)\big)=y(t)\Delta z(t)+Ez(t)\Delta y(t)$$ -(3

البرهان:-

حسب التعريف (1-1) نجد أن :

$$\Delta\big(y(t)z(t)\big)=y(t+1)z(t+1)-y(t)z(t)$$

$$=y(t+1)z(t+1)-y(t)z(t+1)+y(t)z(t+1)-y(t)z(t)$$

$$=z(t+1)\big(y(t+1)y(t)\big)+y(t)(z(t+1)-z(t))$$

$$=z(t+1)\Delta y(t)+y(t)\Delta z(t)$$

$$=Ez(t)\Delta y(t)+y(t)\Delta z(t)$$

$$\therefore\ \Delta\big(y(t)z(t)\big)=y(t)\Delta z(t)+Ez(t)\Delta y(t)$$

أخذنا فكرة عن النهايات فقط

النهايات (Limits)

2.1 Limits:

لذا سنترك هذا البحث للمحاضرة الخامسة

If x becomes arbitrarily close to a number a as approaches from either sides, the limit of $f(x)$ is written as:

$$\lim_{x \to a} f(x) = L .$$

And say "the limit of $f(x)$, as x approaches to a, equals L"

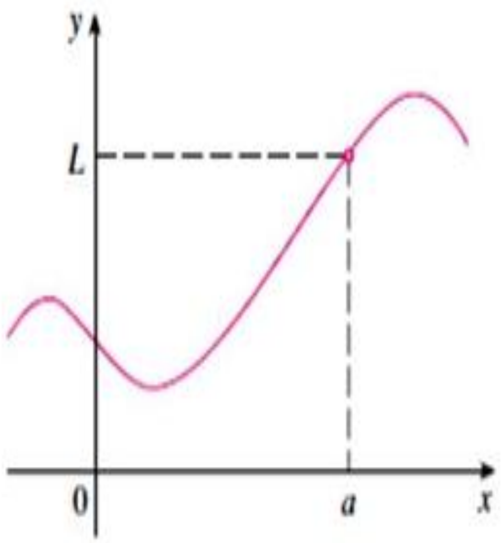

Figure 2.1

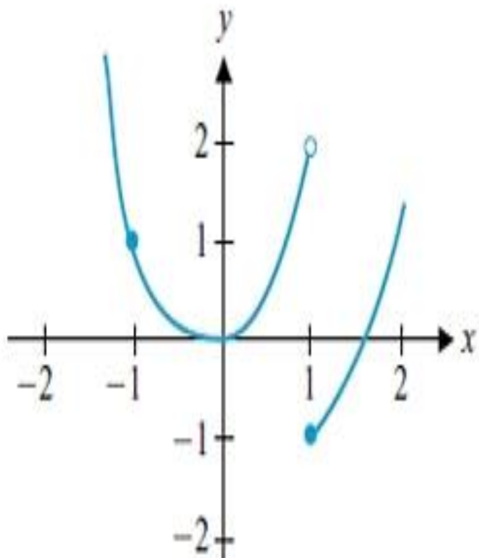

Figure 2.2

Existing of Limit:

If $\lim_{x\to a^-} f(x) = L$ and $\lim_{x\to a^+} f(x) = L$, then $\lim_{x\to a} f(x) = L\,(Exist)$ and vice versa

For example : $\lim_{x\to -1^-} f(x) = 1$, and $\lim_{x\to -1^+} f(x) = 1$, then $\lim_{x\to -1} f(x) = 1\,(Exist)$. see Figure

But If $\lim_{x\to a^-} f(x) \neq$ and $\lim_{x\to a^+} f(x)$, then $\lim_{x\to a} f(x)$ *Does Not Exist (DNE)*.

$\lim_{x\to 1^-} f(x) = 2$, and $\lim_{x\to 1^+} f(x) = -1$, then $\lim_{x\to 1} f(x)$ *Does Not Exist (DNE)*. see Figure

2.2 Computing Limits Graphically

The limit of *f(x)* as *x* approaches the value of *a* form the right is written as:

$$\lim_{x\to a^+} f(x)$$

The limit of *f(x)* as *x* approaches the value of *a* form the left is written as:

$$\lim_{x\to a^-} f(x)$$

Let's explore these ideas with the graph of *f(x)* in Figure 2.3.

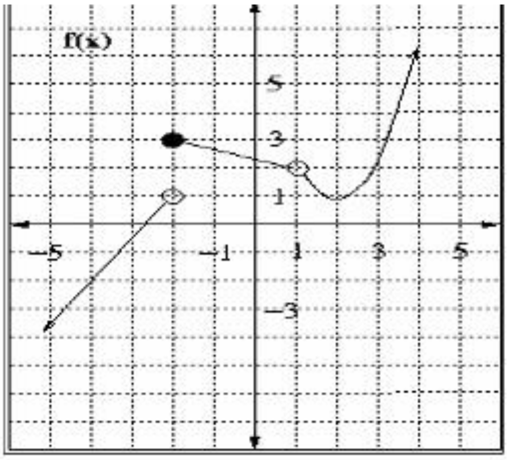

Figure 2.3

Looking at $f(x)$ when $x = -2$, you notice there is a "jump" in the function. However, if you approach $x = -2$ "from the left" (Figure 2.4a) you can see that the function values are getting closer and closer to 1. On the other hand, if we approach $x = -2$ "from the right" (Figure 2.4b) you can see that the function values are getting closer and closer to 3, Therefore, the following statements are true:

$\lim_{x \to -2^+} f(x) = 3$ $\lim_{x \to -2^-} f(x) = 1$ **, then** $\lim_{x \to -2} f(x)$ *is Does Not Exist* (*DNE*)

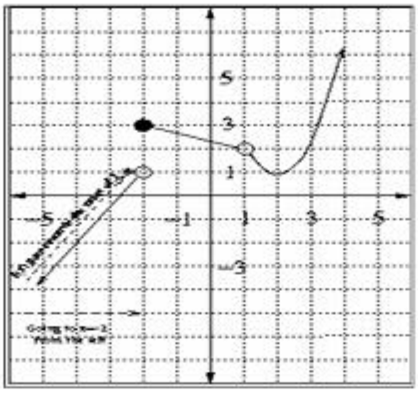

Figure 2.4a

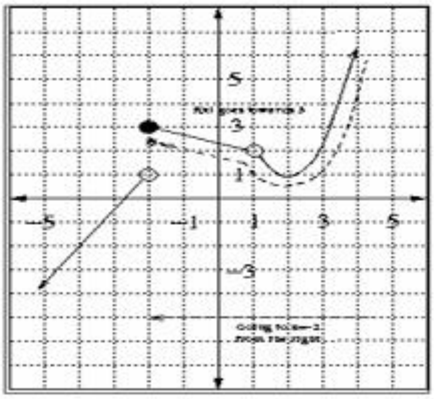

Figure 2.4b

Now Look at *f(x)* when *x = 1*, you notice there is a hole in the function. If we approach *f(x)* from the left *or* from the right (Figure 2.5), you can see that the function values are getting closer and closer to 2.

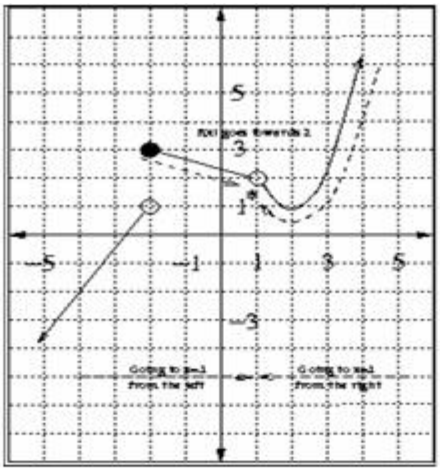

Figure 2.5

Therefore,

$\lim_{x\to 1^+} f(x) = 2$ $\lim_{x\to 1^-} f(x) = 2$ then $\lim_{x\to 1} f(x) = 2$

Note that $f(1)$ is undefined ,and we say that $f(x)$ is Discontinuous at $x = 1$

Example 1:

Using the given graph of *g(x)*, find the following left- and right-hand limits.

a. $\lim_{x\to 0^-} g(x)$

b. $\lim_{x\to 0^+} g(x)$

c. $\lim_{x\to 1^-} g(x)$

d. $\lim_{x\to 1^+} g(x)$

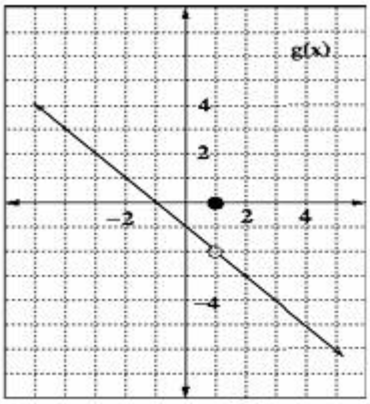

Figure 2.6

Solution :

a. This asks us to look at the graph of *g(x)* as *x* approaches *0* from the left. You can see that the function values are getting closer and closer to *-1*. So,

$$\lim_{x\to 0^-} g(x) = -1$$

b. This asks us to look at the graph of *g(x)* as *x* approaches *0* from the right. You can see that the function values are getting closer and closer to *-1*. So,

$$\lim_{x \to 0^+} g(x) = -1$$

Then : $\lim_{x \to 0} g(x) = -1$ (Exist)

c. This asks us to look at the graph of *g(x)* as *x* approaches *1* from the left. You can see that the function values are getting closer and closer to *-2*. So,

$$\lim_{x \to 1^-} g(x) = -2$$

d. This asks us to look at the graph of *g(x)* as *x* approaches *1* from the right. You can see that the function values are getting closer and closer to *-2*. So,

$$\lim_{x \to 1^+} g(x) = -2$$

Then $\lim_{x \to 1} g(x) = -2$ (Exist),

But $g(1) = 0$, we say that $g(x)$ is Discontinuous at $x = 1$

Note that in the solutions to parts (c) and (d) above, the function value $g(1) = 1$ does not play a role in determining the values of the limits. A limit is strictly the behavior of a function "near" a point.

Example 2 :

Using the graph of *h(x)* below, find the following left- and right-hand limits.

a. $\lim_{x \to 4^-} h(x)$

b. $\lim_{x \to 4^+} h(x)$

c. $\lim_{x \to 4} h(x)$

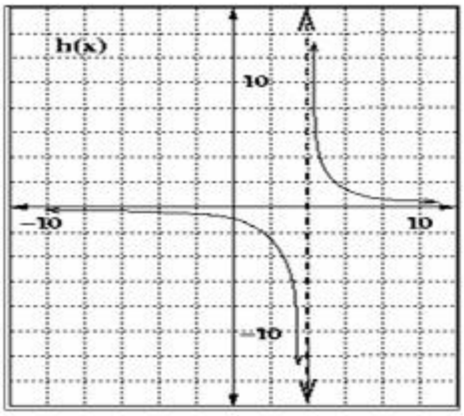

Figure 2.7

Solution:

a. Looking at the graph of $h(x)$, as x approaches 4 from the left, you can see that the function values keep getting more and more negative, without end. Thus, we say that the function values approach negative infinity, written:

$$\lim_{x \to 4^-} h(x) = -\infty$$

b. Looking at the graph of $h(x)$, as x approaches 4 from the right, you can see that the function values keep getting more and more positive without end. Thus, we say that the function values approach positive infinity, written

$$\lim_{x \to 4^+} h(x) = \infty$$

By considering both the left- and right-hand limits of a function as you approach a particular value of x, you can determine whether or not the limit of the function at that point exists.

c. $\lim_{x \to 4} h(x)$ *Does Not Exist*

Therefore, if the left-hand limit does not equal the right-hand limit as x approaches a, then the limit as x approaches a is does not exist.

Example 3:

Using the graph of *f(x)* below, find the following limits.

a. $\lim_{x \to 1} f(x)$

b. $\lim_{x \to -2} f(x)$

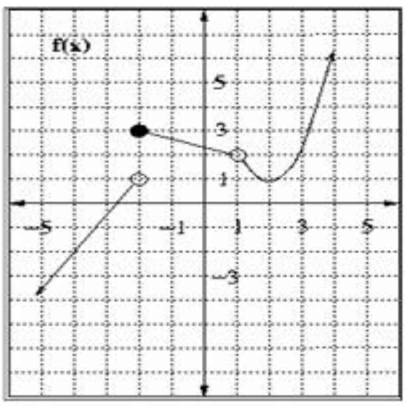

Figure 2.8

a. In previous investigations of this function, we found that

$$\lim_{x\to 1^-} f(x) = 2 \text{ and } \lim_{x\to 1^+} f(x) = 2.$$

Therefore

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = 2, \text{ then } \lim_{x\to 1} f(x) = 2.$$

It is important to notice that this limit exists even though $f(1)$ does not exist.

b. In previous investigations of this function, we found that

$$\lim_{x\to -2^-} f(x) = 1 \text{ and } \lim_{x\to -2^+} f(x) = 3.$$

Therefore, by definition, since

$$\lim_{x\to -2^-} f(x) \neq \lim_{x\to -2^+} f(x),$$

then

$$\lim_{x\to -2} f(x) \text{ does not exist (DNE)}.$$

Example 4 :

Using the given graph of $g(x)$, find $\lim_{x\to 1} g(x)$.

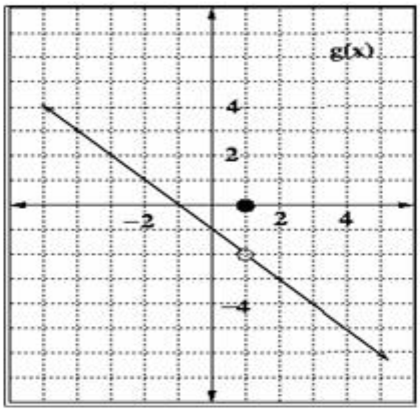

Figure 2.9

Solution:

In previous investigations of this function, we found that $\lim_{x\to 1^-} g(x) = -2$ and $\lim_{x\to 1^+} g(x) = -2$. Therefore, by definition,

Since

$$\lim_{x\to 1^-} g(x) = \lim_{x\to 1^+} g(x) = -2, \text{ then } \lim_{x\to 1} g(x) = -2.$$

Note that $f(1) = 0$.

Exercises 2.2

Use the given graph of $f(x)$ to answer Exercises 1 to 13.

(1) Evaluate $\lim_{x \to 1^-} f(x)$

(2) Evaluate $\lim_{x \to 1^+} f(x)$

(3) Evaluate $\lim_{x \to 1} f(x)$

(4) Find $f(1)$

(5) Evaluate $\lim_{x \to -3^+} f(x)$

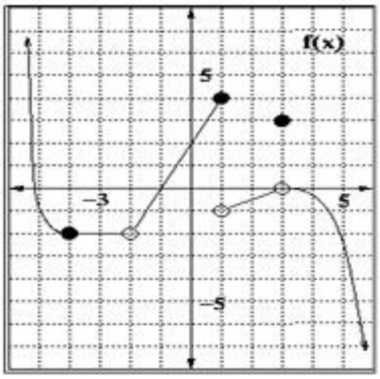

Figure 2.10

(6) Evaluate $\lim_{x \to -3^-} f(x)$

(7) Evaluate $\lim_{x \to -3} f(x)$

(8) Find $f(-3)$

(9) Evaluate $\lim_{x \to 3^+} f(x)$

(10) Evaluate $\lim_{x \to 3^-} f(x)$

(11) Evaluate $\lim_{x \to 3} f(x)$

(12) Find $f(3)$

(13) For what values of x is $f(x)$ discontinuous?

2.3 Limits at Infinity

So far we have been focusing on what is happening with functions at particular values of x by looking at what is happening to the function values corresponding to values very near to the x value. Let's now explore what happens to the function values when we allow x to approach positive and negative infinity.

In Figure 2.11a, you can see that if you move to the right on the graph and allow x to continually become larger (approach infinity), the function values also become larger and larger. If you move to the left on the graph and allow x to become more and more negative (approach negative infinity), you can see that the function values are again becoming larger and larger. Thus, we have

$$\lim_{x\to\infty} k(x) = \infty \qquad \text{and} \qquad \lim_{x\to-\infty} k(x) = \infty$$

In Figure 2.11b, you can see that if you allow x to approach infinity, the function values go towards negative infinity. If you allow x to approach negative infinity, you can see that the function values go towards positive infinity. Thus, we have

$$\lim_{x\to\infty} m(x) = -\infty \qquad \text{and} \qquad \lim_{x\to-\infty} m(x) = \infty$$

In Figure 2.11c, you can see that if you allow x to approach either positive or negative infinity, the function values approach zero. Thus, we have

$$\lim_{x\to\infty} p(x) = 0 \qquad \text{and} \qquad \lim_{x\to-\infty} p(x) = 0$$

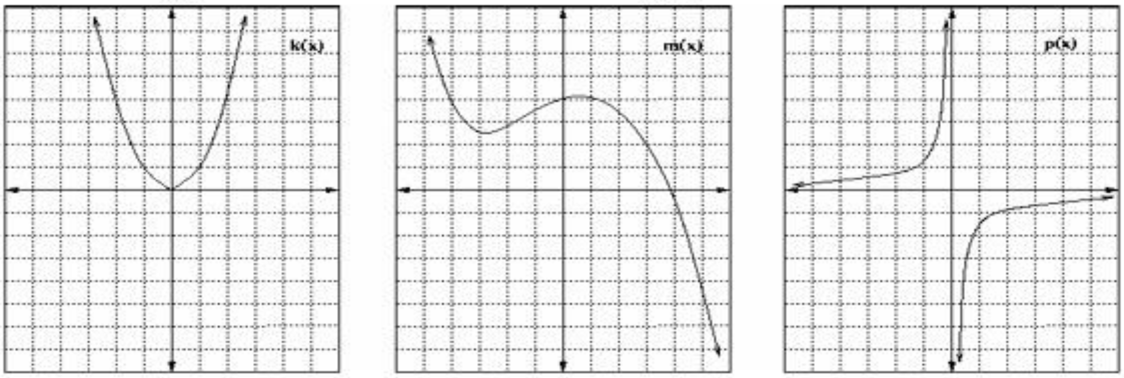

Figure 2.11a **Figure 2.11b** **Figure 2.11c**

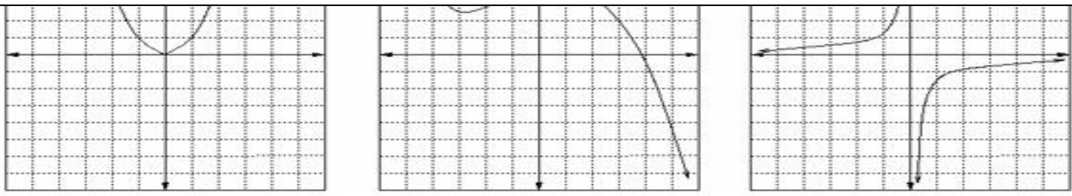

Figure 2.11a **Figure 2.11b** **Figure 2.11c**

When a function approaches a numerical value, say L, as $x \to \infty$ or as $x \to -\infty$, we say that the function has a **horizontal asymptote** *at* $y = L$. Thus, we have just found that $p(x)$ has a horizontal asymptote at $y = 0$, while $k(x)$ and $m(x)$ have no horizontal asymptotes.

(Note: You can only approach positive infinity from the left and you can only approach negative infinity from the right so there is no discussion of left- and right-hand limits at infinity.)

Example 1:

Using the graph of $f(x)$ Figure 2.12 ,to find the following limits.

a. $\lim_{x \to -5} f(x)$

b. $\lim_{x \to -\infty} f(x)$

c. $\lim_{x \to 0^-} f(x)$

d. $\lim_{x \to \infty} f(x)$

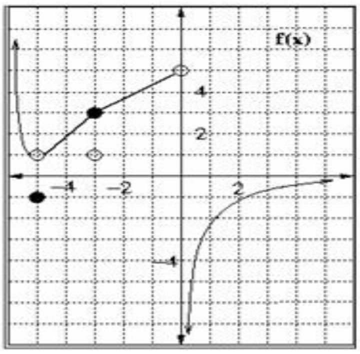

Figure 2.12

Solution :

a. We need to find and compare the left- and right-hand limits of $f(x)$ at $x = -5$. As x approaches -5 from the left, $f(x)$ approaches 1 and as x approaches -5 from the right, $f(x)$ also approaches 1. Therefore,

$$\lim_{x \to -5} f(x) = 1$$

b. As $x \to -\infty$, the function values get more and more positive without end. Therefore,

$$\lim_{x \to -\infty} f(x) = \infty$$

c. As x approaches zero from the left, $f(x)$ approaches 5. Therefore,

$$\lim_{x \to 0^-} f(x) = 5$$

d. As $x \to \infty$, the function values get closer and closer to zero. Therefore

$$\lim_{x \to \infty} f(x) = 0$$

Exercises 2.3

(1) Use the given graph of $f(x)$ to find:

(i) Evaluate $\lim_{x\to\infty} f(x)$

(ii) Evaluate $\lim_{x\to-\infty} f(x)$

(iii) Is there exist a Horizontal asymptote?

(iv) For what values of x is $f(x)$ discontinuous?

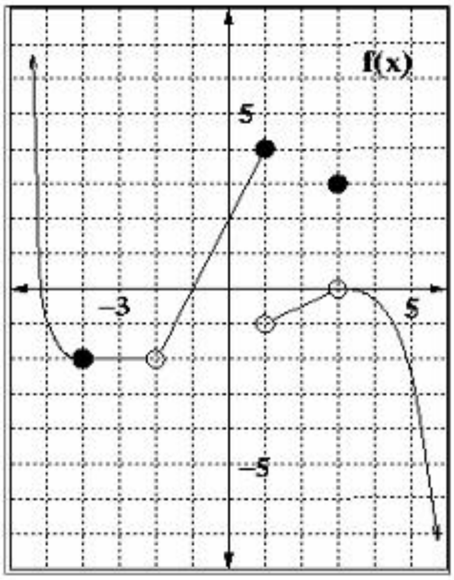

Figure 2.13

(2) Use the given graph of $h(x)$ to find:

(i) Evaluate $\lim_{x\to\infty} h(x)$

(ii) Evaluate $\lim_{x\to-\infty} h(x)$

(iii) Is there exist a Horizontal asymptote?

(iv) Is there exist a vertical asymptote?

(v) For what values of x is $h(x)$ discontinuous?

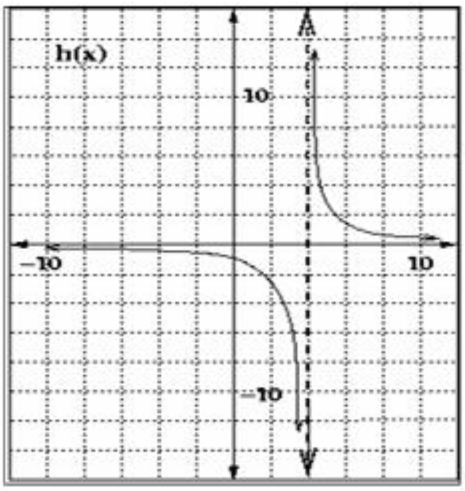

Figure
2.14

2.4 Computing Limits Numerically

While almost all limits can be found graphically, as we have been discussing, it is not always practical or necessary if the function is defined algebraically.

To evaluate $\lim_{x \to a} f(x)$ we can find both the left- and right-hand limits by using tables, By choosing x values that get closer and closer to $x = a$ from both sides, we can anticipate the behavior of $f(x)$.

Example 1: Using tables, find $\lim_{x \to 3} \frac{x^2 - 9}{x - 3}$

Solution:

we can find both the left- and right-hand limits by using tables. By choosing x values that get closer and closer to $x = 3$ from both sides, we can analyze the behavior of $f(x)$.

Table 2.1

	Limit from the left →				← Limit from the right		
x	2.99	2.999	2.9999	3	3.0001	3.001	3.01
$f(x)$	5.99	5.999	5.9999	?	6.0001	6.001	6.01

Notice that when we chose values on either side of $x = 3$, they were values that were very close to $x = 3$. It seems that as x approaches 3 from either side, the function values are approaching 6. Therefore, it seems reasonable to conclude that:

$$\lim_{x \to 3} \frac{x^2 - 9}{x - 3} = 6$$

Example 2:

Using tables, find the following limits given that $f(x)=\begin{cases}4, & x\le 1\\ x^2, & x>1\end{cases}$

a. $\lim_{x\to 2} f(x)$

b. $\lim_{x\to 1} f(x)$

Solution:

a. We need to construct a table with x-values approaching 2 from both sides. Since all of these x −values are in the domain of $x > 1$, we will use the part of the function defined by x^2 to determine the function values in our table.

Table 2.2

	Limit from the left →				← Limit from the right		
x	1.99	1.999	1.9999	2	2.0001	2.001	2.01
$f(x)$	3.96	3.996	3.9996	?	4.0004	4.004	4.04

Approaching $x = 2$ from both the left and the right sides shows that the function values are approaching 4. Thus, $\lim_{x\to 2} f(x) = 4$.

b. We need to construct a table with x-values approaching 1 from both sides. All x − values approaching $x = 1$ from the left are in the domain $x < 1$, so we will be using the part of the function defined by 4 when finding these function values. All x − values approaching $x = 1$ from the right are in the domain $x > 1$, so we will use x^2 to find these function values in our table.

Table 2.3

	Limit from the left →				← Limit from the right		
x	0.99	0.999	0.9999	1	1.0001	1.001	1.01
$f(x)$	4	4	4	?	1.0002	1.002	1.0201

As $x \to 1$ from the left, $f(x)$ seems to be approaching 4, while as $x \to 1$ from the right, $f(x)$ seems to be approaching *1*. Since these are not equal, by definition, $\lim_{x \to 1} f(x)$ does not exist.

Making tables can still be as time-consuming as graphing, so we will use the following rules to algebraically evaluate limits more efficiently. Most of these rules can intuitively be verified from looking at the previously worked examples.

Exercise 2.4

(1) Fill in the given table and use it to find $\lim_{x \to 2} f(x)$.

x	1.99	1.999	1.9999	2	2.0001	2.001	2.01
$f(x) = \dfrac{\lvert x-2 \rvert}{x-2}$				?			

(2) Fill in the given table and use it to find $\lim_{x\to 0} g(x)$.

x	-0.01	-0.001	-0.0001	0	0.0001	0.001	0.01
$g(x)=\begin{cases} x+1, & x<0 \\ 3, & x=0 \\ x^2+1, & x>0 \end{cases}$							

(3) Using tables, find $\lim_{x\to 2} \dfrac{x^2-4}{x-2}$

(4) Using tables, find the following limits given that $f(x)=\begin{cases} 3x & , x\le 2 \\ x^2+2, & x>2 \end{cases}$

a. $\lim_{x\to 2} f(x)$

b. $\lim_{x\to 1} f(x)$

2.5 Rules of Limit

If a, c, and n, are real numbers, then

1) $\lim_{x \to a} c = c$ (The limit of a constant real number is that number.)

2) $\lim_{x \to a} p(x) = p(a)$ where $p(x)$ is any polynomial

(The limit value of a polynomial is the function value at that point.)

3) $\lim_{x \to a} [c \cdot f(x)] = c \cdot \lim_{x \to a} f(x)$

(The limit of the product of a constant and a function equals the constant times the limit of the function.)

4) $\lim_{x \to a} [f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$

(The limit of the sum or difference of two functions equals the sum or difference of the limits of the functions.)

5) $\lim_{x \to a} [f(x) \cdot g(x)] = \left[\lim_{x \to a} f(x)\right] \cdot \left[\lim_{x \to a} g(x)\right]$

(The limit of the product of two functions is the product of the limits of the functions.)

6) $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$, if $\lim_{x \to a} g(x) \neq 0$

(The limit of a quotient is the quotient of the limits of the numerator and denominator if the limit of the denominator is not zero.)

7) $\lim_{x \to a} [f(x)]^n = \left[\lim_{x \to a} f(x)\right]^n$ (provided this is defined)

(The limit of a function raised to a power equals the limit of the function raised to the power provided the math makes sense.)

Example 1 Evaluate:

a. $\lim_{x \to -1} x^2 + 2x - 4$

b. $\lim_{x \to 3} \frac{x^2 - 9}{x - 3}$

c. $\lim_{x \to 4} \frac{1}{x - 4}$

Solution:

a. $\lim_{x \to -1} x^2 + 2x - 4 = (-1)^2 + 2(-1) - 4 = -5$ (Rule 2)

b. $\lim_{x \to 3} \frac{x^2 - 9}{x - 3} = \frac{\lim_{x \to 3} x^2 - 9}{\lim_{x \to 3} x - 3}$ (Rule 6)

$= \frac{0}{0}$ Indeterminate form (Rule 2)

When you get $\frac{0}{0}$ it is called an **indeterminate form** and you must try other techniques to determine the limit. In this case, factor both the numerator and denominator and cancel common terms to remove the zero in the denominator. Then, apply the limit rules to the simplified expression.

$\lim_{x \to 3} \frac{x^2 - 9}{x - 3} = \lim_{x \to 3} \frac{(x+3)(x-3)}{(x-3)}$ (Factor)

$= \lim_{x \to 3} x + 3$ (Cancel common terms)

$= 3 + 3 = 6$ (Rule 2)

c. $\lim_{x \to 4^+} \frac{1}{x - 4} = \frac{\lim_{x \to 4^+} 1}{\lim_{x \to 4^+} (x - 4)} =$ (Rule 6)

$= \frac{1}{0^+} = \infty$ (Rules 1 and 2)

$\lim_{x \to 4^-} \frac{1}{x - 4} = \frac{\lim_{x \to 4^-} 1}{\lim_{x \to 4^-} (x - 4)} =$

$$=\frac{1}{0^-}=-\infty$$

$$\text{Thus : } \lim_{x\to 4}\frac{1}{x-4}=DNE$$

This is not defined and whenever you get a result of a non-zero number over zero, there are no common factors in the numerator and denominator which can be cancelled. Therefore, there is no way to rid the denominator of its zero term, meaning that the limit does not exist. Looking at the graph of the function near $x = 4$, we can see what is happening. Notice this is the same function we analyzed when finding limits graphically. There, we also found that the limit does not exist.

While these rules also apply when looking for limits at infinity (or negative infinity), it is almost always necessary to algebraically manipulate the expression of the function to determine the limit.

Example 2: Evaluate $\lim_{x\to\infty}\frac{2x^2-7}{6-3x^2}$

Solution: $$\lim_{x\to\infty}\frac{2x^2-7}{6-3x^2}=\frac{\lim_{x\to\infty}2x^2-7}{\lim_{x\to\infty}6-3x^2}=\frac{\infty}{-\infty}$$

$\frac{\infty}{-\infty}$ or $\frac{\infty}{\infty}$ or $\frac{-\infty}{\infty}$ are all also known as indeterminate forms. When this form occurs when finding limits at infinity (or negative infinity) with rational functions, divide every term in the numerator and denominator by the highest power of x in the denominator to determine the limit. Since x^2 is the highest power of x in the denominator of our function, we have

$$\lim_{x\to\infty}\frac{2x^2-7}{6-3x^2}=\frac{\lim_{x\to\infty}\left(2x^2-7\right)\div x^2}{\lim_{x\to\infty}\left(6-3x^2\right)\div x^2}$$

$$=\frac{\lim_{x\to\infty}\left(2-\frac{7}{x^2}\right)}{\lim_{x\to\infty}\left(\frac{6}{x^2}-3\right)}$$

$$=\frac{2-0}{0-3}=\frac{-2}{3}$$

Example 3: Find $\lim_{x\to 9}\frac{3-\sqrt{x}}{x-9}$ if it exist

Solution:

$$\lim_{x\to 9}\frac{3-\sqrt{x}}{x-9}=\frac{0}{0}\text{ Indeterminate form,}$$

Multiply by the denominator conjugate, we have:

$$\lim_{x\to 9}\frac{3-\sqrt{x}}{x-9}\cdot\frac{3+\sqrt{x}}{3+\sqrt{x}}=$$

$$\lim_{x\to 9}\frac{(9-x)}{(x-9)(3+\sqrt{x})}=$$

$$\lim_{x\to 9}\frac{-1}{(3+\sqrt{x})}=\frac{-1}{9}$$

Exercise 2.5

Find each Limit if it exist :

(1) $\lim_{x \to 3} x^2 - 4x - 21$

(2) $\lim_{x \to 1} (12x^3 + x^2 - 1)$

(3) $\lim_{x \to -5} (3(x-1))$

(4) $\lim_{x \to 5} \frac{x+1}{x+2}$

(5) $\lim_{x \to 4^+} \frac{2}{x-4}$

(6) $\lim_{x \to 4^-} \frac{2}{x-4}$

(7) $\lim_{x \to 4} \frac{2}{x-4}$

(8) $\lim_{x \to -1} \frac{x^2 - 1}{x+1}$

(9) $\lim_{x \to -1} \frac{2x^2 - x - 3}{x+1}$

(10) $\lim_{x \to 3} \frac{2x+1}{x-3}$

(11) $\lim_{x \to 3^+} \frac{x+1}{x+2}$

(12) $\lim_{x \to 4^-} \frac{\sqrt{x} - 2}{x-4}$

(13) $\lim_{x \to 1} \frac{x^2 - 1}{1 - \sqrt{x}}$

(14) $\lim_{x \to 3} \frac{\sqrt{x+1} - 2}{x-3}$

(15) $\lim_{x \to 9} \frac{3 - \sqrt{x}}{x-9}$

(16) $\lim_{x \to 4} \frac{4-x}{2 - \sqrt{x}}$

(17) $\lim_{x \to 3} \frac{x^2 - 5x + 6}{x-3}$

(18) $\lim_{x \to -7} \frac{x+7}{x^2 - 49}$

(19) $\lim_{x \to -3} \frac{x^2 + 2x - 3}{3x + 9}$

(20) $\lim_{x \to 3} \frac{x^2 - 4x + 3}{x^2 - 9}$

(21) $\lim_{x \to 1} \frac{x-1}{x^2 - 2x + 1}$

(22) $\lim_{x \to 0} \frac{(x-6)^2 - 36}{x}$

(23) $\lim_{x \to 0} \frac{x}{x^2 - 3x}$

(24) $\lim_{x \to 1^-} \frac{x^2 - 2x + 1}{x-1}$

$$(25)\ \lim_{x\to 4}\frac{2x-8}{x^2-16}$$

$$(26)\ \lim_{x\to\infty}\frac{x^2+2x-x^3}{3x^4-7}$$

$$(27)\ \lim_{x\to\infty}\frac{x^2-4x-21}{x^2-5x-14}$$

$$(28)\ \lim_{x\to-\infty}\frac{x-1}{x^2-2x+1}$$

$$(29)\ \lim_{x\to\infty}\frac{x^2+2x-x^3}{3x^4-7}$$

$$(30)\ \lim_{x\to\infty}\frac{x^2-4x-21}{x^2-5x-14}$$

2.6 Trigonometric Limits

When working with limits of trigonometric functions it might be helpful to:

1. Remember that all previous algebraic "rules for limits" are still valid
2. Try to "plug in" the "target value" – watch for undefined terms and "$\frac{0}{0}$".
3. Write all functions in terms of $\sin\theta$ or $\cos\theta$.
4. Algebraically manipulate the function to look for a "replacement function".
5. Work towards the 3 "special limits"

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1 , \qquad \lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1 \text{ , and } \qquad \lim_{\theta \to 0} \frac{1-\cos\theta}{\theta} = 0$$

, *θ is measured by Radians*

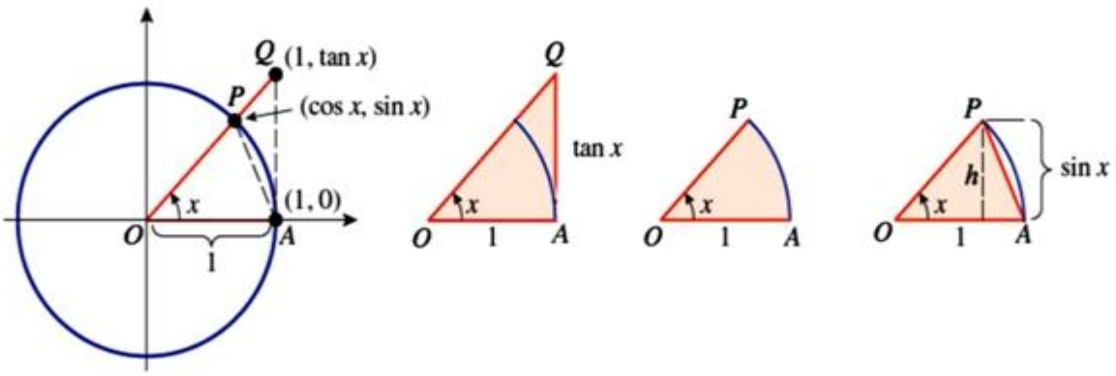

Figure 2.15

Examples 1: evaluate $\lim_{\theta\to 0}\frac{\tan\theta}{\sin\theta}$

Solution: direct we get indeterminate form , divided by the angle ,

$$\lim_{\theta\to 0}\frac{\tan\theta}{\sin\theta}=\lim_{\theta\to 0}\frac{\frac{\tan\theta}{\theta}}{\frac{\sin\theta}{\theta}}=\frac{\lim_{\theta\to 0}\frac{\tan\theta}{\theta}}{\lim_{\theta\to 0}\frac{\sin\theta}{\theta}}=\frac{1}{1}=1$$

Examples 2: evaluate $\lim_{\theta\to 0}\frac{1-\cos\theta}{\sin\theta}$

Solution: $\lim_{\theta\to 0}\frac{1-\cos\theta}{\sin\theta}=\lim_{\theta\to 0}\frac{1-\cos\theta}{\sin\theta}.\frac{1+\cos\theta}{1+\cos\theta}$ Multiplied by the conjugate,

$$=\lim_{\theta\to 0}\frac{1-\cos^2\theta}{\sin\theta(1+\cos\theta)}=\lim_{\theta\to 0}\frac{sin^2\theta}{\sin\theta(1+\cos\theta)}$$

$$=\lim_{\theta\to 0}\frac{sin\ \theta}{(1+\cos\theta)}=\frac{\lim_{\theta\to 0} sin\ \theta}{\lim_{\theta\to 0}(1+\cos\theta)}=\frac{0}{1+1}=\frac{0}{2}=0$$

Examples 3: evaluate $\lim_{\theta\to 0}\frac{\cot\theta\sin\theta}{2\sec\theta}$

Solution:

$$\lim_{\theta\to 0}\frac{\cot\theta\sin\theta}{2\sec\theta}=\lim_{\theta\to 0}\frac{\frac{1}{\tan\theta}\sin\theta}{2.\frac{1}{\cos\theta}}$$

$$=\lim_{\theta\to 0}\frac{\cos\theta\sin\theta}{2\tan\theta}=\frac{\lim_{\theta\to 0}\cos\theta\sin\theta}{\lim_{\theta\to 0}2\tan\theta}$$

$$=\lim_{\theta\to 0}\cos\theta.\lim_{\theta\to 0}\frac{\sin\theta}{2\tan\theta}=\cos 0.\frac{1}{2}\lim_{\theta\to 0}\frac{\sin\theta}{\tan\theta}=1.\frac{1}{2}.1=\frac{1}{2}$$

Examples 4: evaluate $\lim_{\theta\to 0}\frac{\sin 5\theta}{3\theta}$

Solution:

$$\lim_{\vartheta\to 0}\frac{\sin 5\vartheta}{3\vartheta}=\frac{1}{3}\lim_{\vartheta\to 0}\frac{\sin 5\vartheta}{\vartheta}=\frac{5}{3}\lim_{5\vartheta\to 0}\frac{\sin 5\vartheta}{5\vartheta}=\frac{5}{3}.1=\frac{5}{3}$$

Examples 5: evaluate $\lim_{\theta\to\frac{\pi}{4}}\frac{\sin\theta-\cos\theta}{\cos 2\theta}$

Solution:

$$\lim_{\theta \to \frac{\pi}{4}} \frac{\sin\theta - \cos\theta}{\cos 2\theta} = \lim_{\theta \to \frac{\pi}{4}} \frac{\sin\theta - \cos\theta}{\cos^2\theta - \sin^2\theta}$$

$$= \lim_{\theta \to \frac{\pi}{4}} \frac{-(\cos\theta - \sin\theta)}{(\cos\theta - \sin\theta)(\cos\theta + \sin\theta)}$$

$$= \lim_{\theta \to \frac{\pi}{4}} \frac{-1}{(\cos\theta + \sin\theta)} = \frac{-1}{\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}} = \frac{-1}{\sqrt{2}} = \frac{-\sqrt{2}}{2}$$

Corollary

$$\lim_{\theta \to 0} \frac{\sin(a\theta)}{b\theta} = \frac{a}{b}, \quad \lim_{\theta \to 0} \frac{\tan(a\theta)}{b\theta} = \frac{a}{b}, \text{ and } \lim_{\theta \to 0} \frac{1}{\sin\theta} = 1$$

$, \theta$ *is measured by Radians*

Examples 6: let $f(x) = \begin{cases} \frac{\tan(ax)}{x} & , x < 0 \\ 2(x-1) + a^2 & , x \geq 0, \end{cases}$

Find the value of a such that $\lim_{x \to 0} f(x)$ *is Exist*

Solution: $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} 2(x-1) + a^2 = -2 + a^2$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} \frac{\tan(ax)}{x} = a$$

Since $\lim_{x \to 0} f(x)$ *is Exist ,then :*

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^-} f(x)$$

$$-2 + a^2 = a$$

$$a^2 - a - 2 = 0$$

$$a = -1 , or \ a = 2$$

Exercise 2.6

Evaluate the following.

1. $\lim_{x\to 0}\frac{\sin 10x}{3x}=$

2. $\lim_{x\to 0}\frac{\sin 6x}{2x}=$

3. $\lim_{x\to 0}\frac{\sin 3x}{12x\cos x}=$

4. $\lim_{x\to 0}\frac{\tan 4x}{x}=$

5. $\lim_{x\to 0}\frac{7x}{\sin 3x}=$

6. $\lim_{x\to 0}\frac{\sin 10x}{\sin 2x}=$

7. $\lim_{x\to 0}\frac{5x^2-x}{\sin x}=$

8. $\lim_{x\to 0}\frac{\sin^2 x-\sin x\cos x}{x\cos x}=$

9. $\lim_{x\to 0}\frac{\sin 2x-3x^2+2x}{6x}=$

10. $\lim_{x\to \infty}\cos\frac{1}{x}=$

11. $\lim_{x\to \pi}\frac{x-\pi}{\sin(x-\pi)}=$

12. $\lim_{x\to 0}\frac{\sin 10x}{\tan 3x}=$

13. $\lim_{x\to 0}\frac{1-\cos^2 x}{3x^2}=$

14. $\lim_{x\to 0}\frac{\tan 4x-\sin x}{3x}=$

==

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