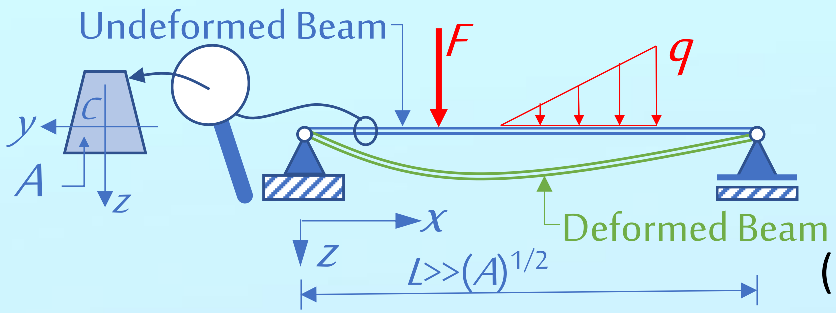
Bending of Beams

إنعطاف الجيزان

1 Introduction

1 المقدمة



(Fig.a).

Beams are among the most important elements in structural engineering.

A beam is straight bar with the dimensions of its cross-sectional area *A* are much smaller than its length *L*.

However, in contrast to the members of a truss it is loaded by forces which are perpendicular to its axis. Then, the originally straight beam deforms (Fig.a). This is referred to as the ***bending of the beam***.

تعد جيزان الإنعطاف وهي عناصر مستقيمة نحيلة أبعاد مقطعها العرضي *A* صغيرة أمام طولها *L*، من أهم العناصر الإنشائية، فهي وعلى خلاف عناصر الجيزان الشبكية تتلقى حمولات عمودية أو مائلة على محورها و في كافة نقاطها.

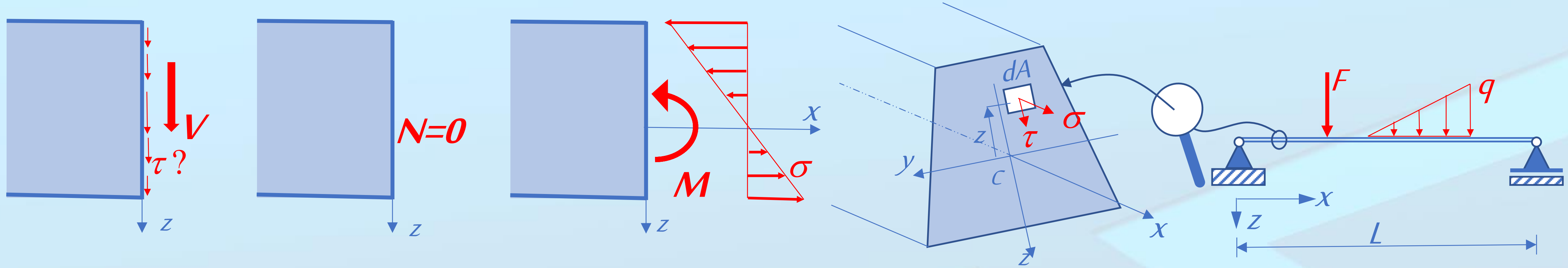
تتشوه هذه الجيزان المستقيمة عند تحميلها آخذةً أشكالاً منحنية ذات أنصاف قطر انحناء كبيرة لذلك يقال بأنها في حالة انعطاف bending . انظر الشكل (Fig.a).

Bending of Beams

إنعطاف الجيزان

رأينا سابقاً أن مقاومة الجائز للحمولات غير المحورية ونقل تأثيرها إلى مسانده تكون عبر محصلات إجهاد تسمى قوى داخلية هي عزم الانعطاف M، وقوة القص V ، والقوة الناظمية N التي سنعتبرها هنا غائبة لأننا درسنا الإجهادات والتشوهات الناتجة عنها في بحث سابق.

ينشأ عزم الانعطاف في مقطع ما كمحصلة لإجهاد ناظمي يتوزع على كامل نقاط هذا المقطع بشكل خطي كما سنرى في درسنا اليوم. بينما تنشأ قوة القص عن إجهاد مماسي للمقطع يتوزع على نقاطه بشكل غير خطي سيُدرس لاحقاً.
تشكل دراسة توزع هذين الإجهادين جزءً مما يعرف بنظرية إنعطاف الجيزان *Beam bending theory* .



$$V = \int_A \tau dA$$

$$N = \int_A \sigma dA = 0$$

$$M = \int_A z\sigma dA$$

$$\frac{dV}{dx} = -q(x)$$

$$\frac{dM}{dx} = V(x)$$

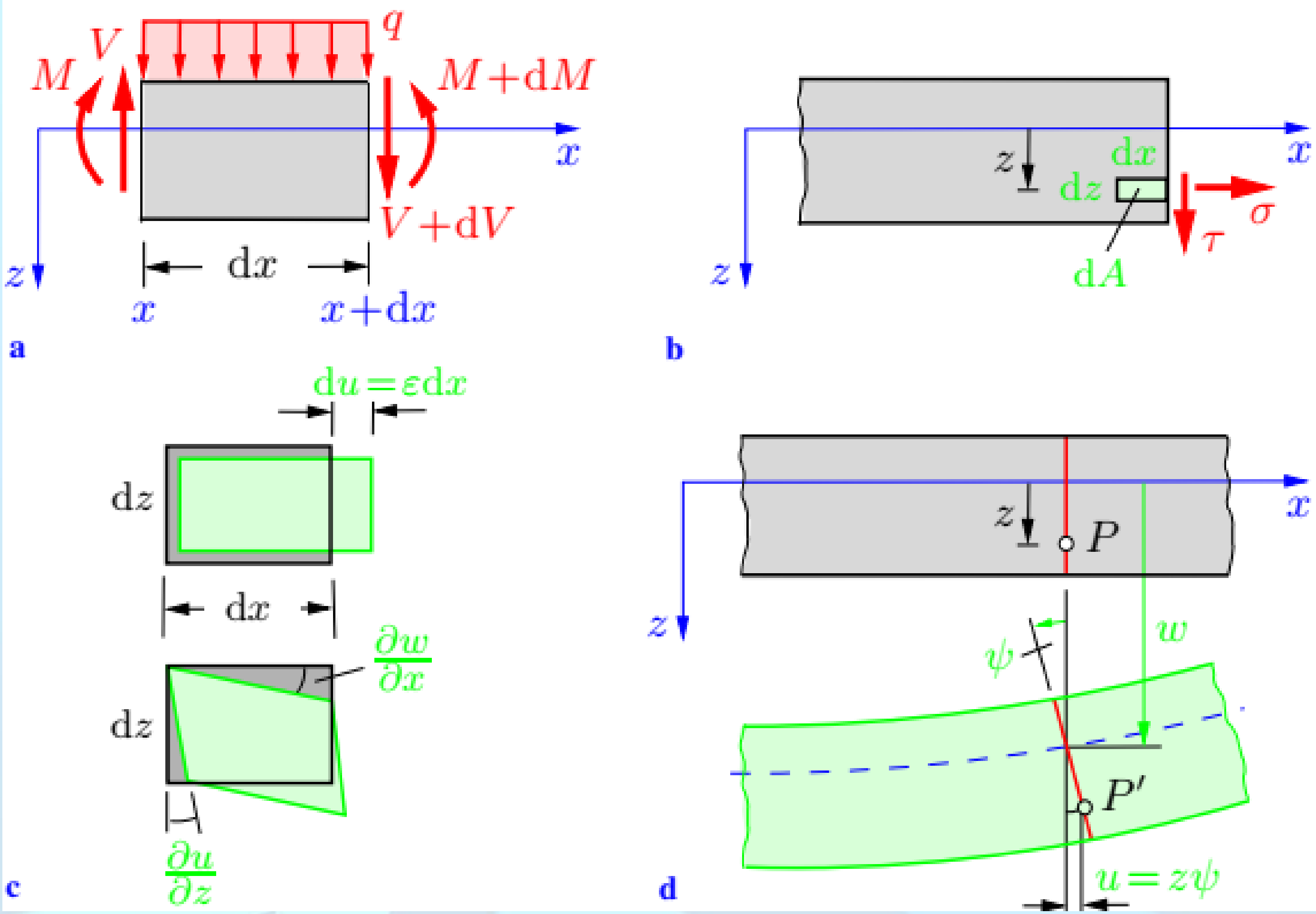
$$\frac{d^2M}{dx^2} = -q(x)$$

2 Basic Equations of Ordinary Bending Theory (Simple Beam Theory)

سنقوم في هذا الدرس باشتقاق المعادلات التي تحدد الإجهادات والتشوهات الناتجة عن انعطاف الجيزان.

سنكتفي بحالة الانعطاف العادي (أحادي المحور)، أي أننا سنفرض أن المحور Z هو محور تناظر للمقطع وأن الحمولات تقع في المستوي $Z - X$.

$\frac{dV}{dx} = -q(x)$	$\frac{dM}{dx} = V(x)$	$\frac{d^2M}{dx^2} = -q(x)$
$V = \int_A \tau dA$	$N = \int_A \sigma dA = 0$	$M = \int_A z\sigma dA$



بالإضافة إلى العلاقات الستاتيكية السابقة، سنستخدم في استنتاج المعادلات المطلوبة قانون هوك (Hooke’s law) والعلاقات الكينماتيكية (kinematic) التي تصف التغيرات الجيومترية في شكل الجائز اثناء الانعطاف.

وسنعتمد كفرضية مقبولة في حالة الجيزان النحيلة أن الإجهادين الناظميين σ_y & σ_z مهملان مقارنة بالإجهاد الناظمي σ_x . عندئذ يكون قانون هوك مبسطاً على النحو:

$$\sigma_x = \sigma = E\,\varepsilon_x = E\varepsilon$$

$$\tau_{zx} = \tau = G\gamma_{zx} = G\,\gamma$$

additional Kinematic assumptions
فرضيات كينماتيكية إضافية

$\frac{dV}{dx} = -q(x)$	$\frac{dM}{dx} = V(x)$	$\frac{d^2M}{dx^2} = -q(x)$
$V = \int_A \tau dA$	$N = \int_A \sigma dA = 0$	$M = \int_A z\sigma dA$

$$\sigma_x = \sigma = E\,\varepsilon_x = E\varepsilon \;\;\&\;\; \tau_{zx} = \tau = G\gamma_{zx} = G\,\gamma$$

a) الانتقال w في اتجاه z مستقل عن z. أي أن:

$$w = w(x)$$

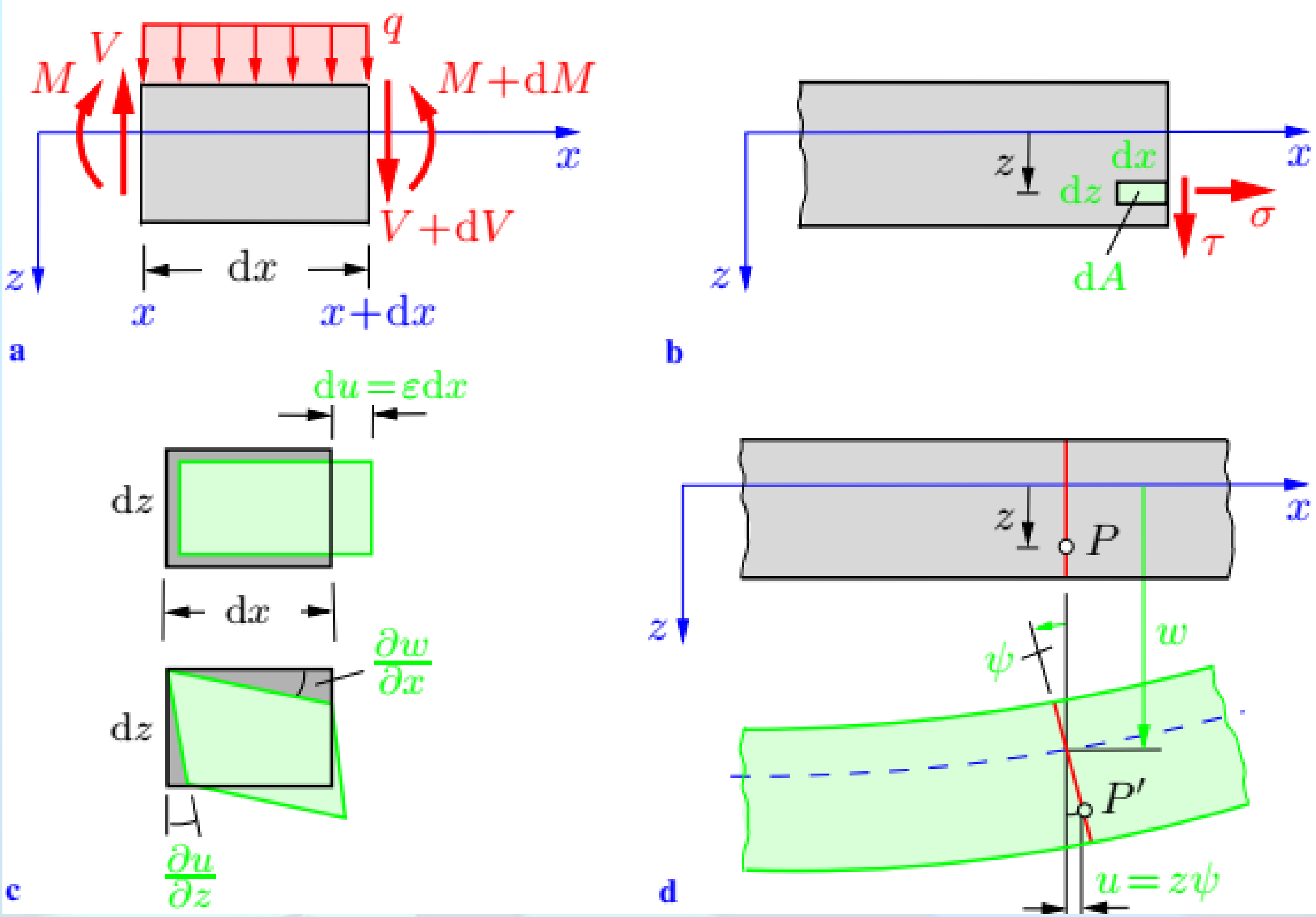
وهذا يكافئ اعتبار ارتفاع مقطع الجائز ثابت في طوله أي أن التشوه في اتجاه z معدوم: $\varepsilon_z = \partial w/\partial z = 0$.

b) تبقى المقاطع المستوية في الجائز أي المقاطع الناظمية على محوره، **مستويةً وناظميةً** على المحور x اثناء الانعطاف. ويدور كلٌ من هذه المقاطع حول المحور y بزاوية صغيرة ψ متغيرة مع x أي من مقطع إلى آخر. ، وتعتبر هذه الزاوية موجبة عندما يكون الدوران من المحور z إلى المحور x. ونكتب: $\psi = \psi(x)$

فيكون الانتقال u للنقطة P من المقطع والواقعة على بعد z من المحور ، معطى بالعلاقة

$$u(x,z) = \psi(x)\, z$$

كما تكون زاوية دوران المقطع مساوية ومعاكسة بالإشارة لميل التابع w، أي أن: $\psi(x) = -w'$



$V = \int_A \tau dA$ | $N = \int_A \sigma dA = 0$ | $M = \int_A z\sigma dA$ | $\frac{dV}{dx} = -q(x)$ | $\frac{dM}{dx} = V(x)$ | $\frac{d^2M}{dx^2} = -q(x)$

$u(x,z) = \psi(x)\, z$ | $w = w(x)$ | $\sigma_x = \sigma = E\,\varepsilon_x = E\varepsilon \;\&\; \tau_{zx} = \tau = G\gamma_{zx} = G\,\gamma$

Kinematic relations into Hooke's Law

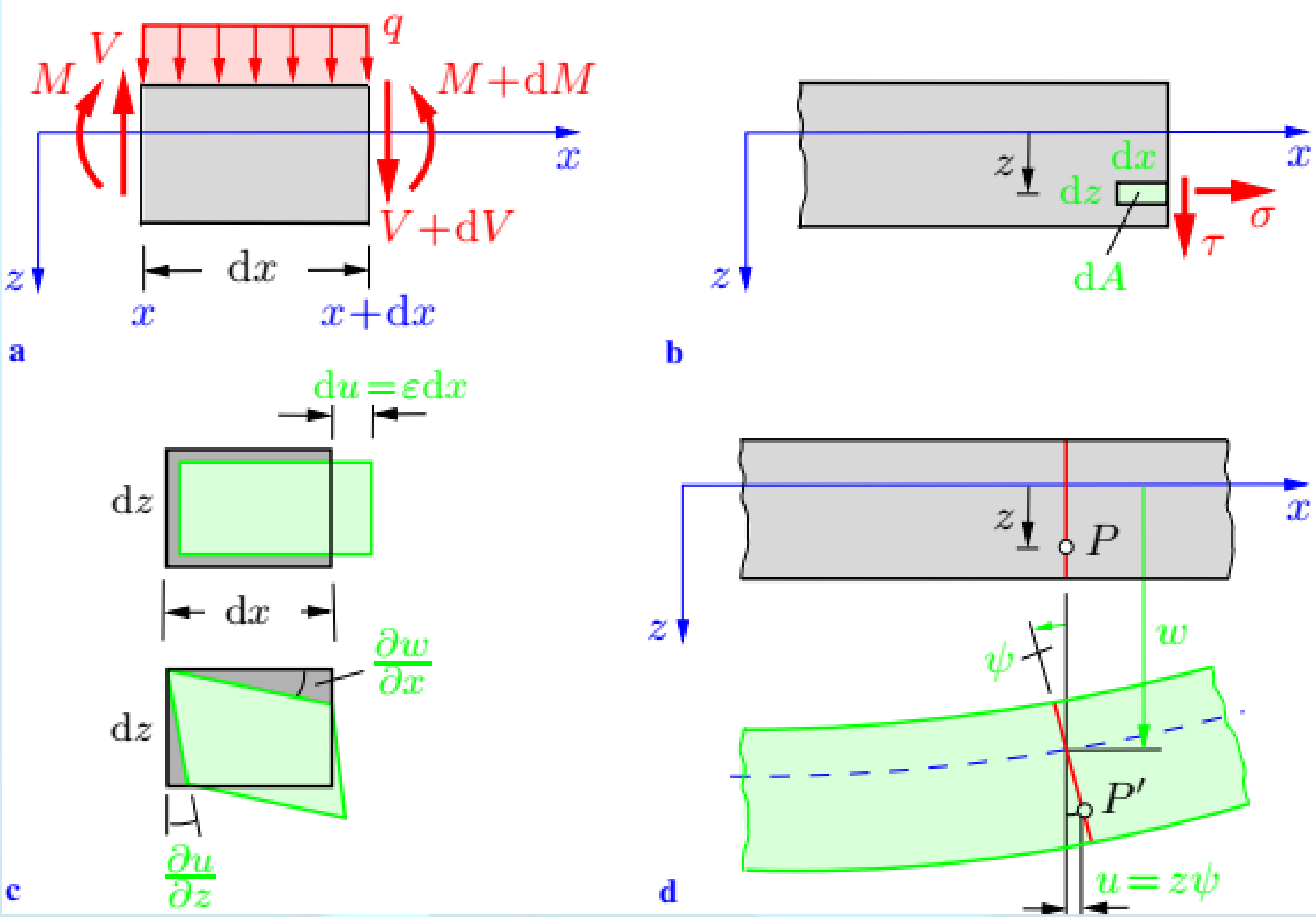
$$\sigma = E\varepsilon = E\frac{\partial u}{\partial x} = E\frac{d\psi}{dx}z = E\psi' z$$

where $d(\,)/dx = (\,)'$ and w' represents the slope of the deformed axis of the beam.

$$N = \int_A \sigma dA = 0 = E\psi' \int_A z dA = 0$$

which implies that the y-axis has to be a centroidal axis: C is the centroid of the section.

$$M = \int_A z\sigma dA = E\psi' \int_A z^2 dA = EI_y\psi'$$



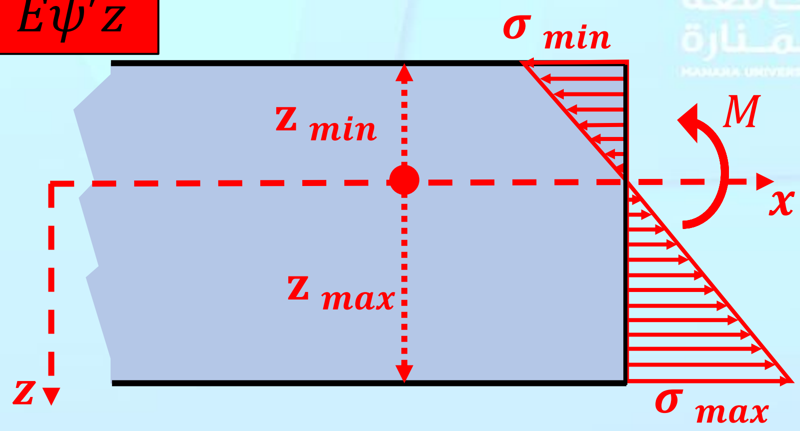
Where $I_y = \int_A z^2 dA$ is the second moment of area about y.

3 Normal Stresses in Bending Beams

4.3 الإجهاد الناظمي في انعطاف الجيزان:

$$M = \int_A z\sigma dA = E\psi' \int_A z^2 dA = EI_y\psi' \Rightarrow E\psi' = \frac{M}{I_y}$$ Sub. into $\sigma = E\psi' z$

$\Rightarrow$ Bending formula $\sigma = \frac{M}{I_y} z$ I_y is [L^4] Compare with $\sigma = \frac{N}{A}$



It shows that the normal stresses, which are referred to as the *flexural* or *bending stresses* (إجهاد الانعطاف), are linearly distributed in z-direction as shown in Fig. If the bending moment M is positive, the stresses are positive (tensile stresses) for $z > 0$ and they are negative (compressive stresses) for $z < 0$. For $z = 0$ (i.e., in the x, y-plane) we have $\sigma = 0$. Since $\varepsilon = \sigma / E$, the strain ε is also zero in the x, y plane: the fibers in this plane do not undergo any elongation or contraction. Therefore, this plane is called the *neutral surface* of the beam. The intersection of a cross section of the beam with the neutral surface (i.e., the y-axis) is called the *neutral axis* (المحور السليم) . The bending stresses (tensile or compressive) attain their maximum values at the extreme fibers. With the notation z_{max} for the maximum value of z (often also denoted by c) and : $\sigma_{max} = \frac{M}{I_y} z_{max} = \frac{M}{W}$.

Where $W = \frac{I_y}{z_{max}}$, is [L^3] (often also denoted by S) and called the *section modulus* (معامل المقطع).

If the state of stress in a beam is investigated, it often suffices to determine only the normal stresses since the shear stresses are usually negligibly small (slender beams!).

There are several different types of problems arising in this context.

If, for example, the bending moment M, the section modulus W and the allowable stress σ_{allow} are known, one has to verify that the maximum stress $\sigma_{\max}$ satisfies the requirement

$\sigma_{\max} \leq \sigma_{allow} \rightarrow \frac{M}{W} \leq \sigma_{allow}$ this is called stress check. تحقيق الإجهادات

On the other hand, if M and σ_{allow} are given, the required section modulus can be calculated from

$W_{req} = \frac{M}{\sigma_{allow}}$ This is referred to as the *design* of a beam. تصميم الجائز

Finally, if W and σ_{allow} are given, the allowable load can be calculated from the condition that the maximum bending moment M_{max} must not exceed the allowable moment $M_{allow} = W\sigma_{allow}$:

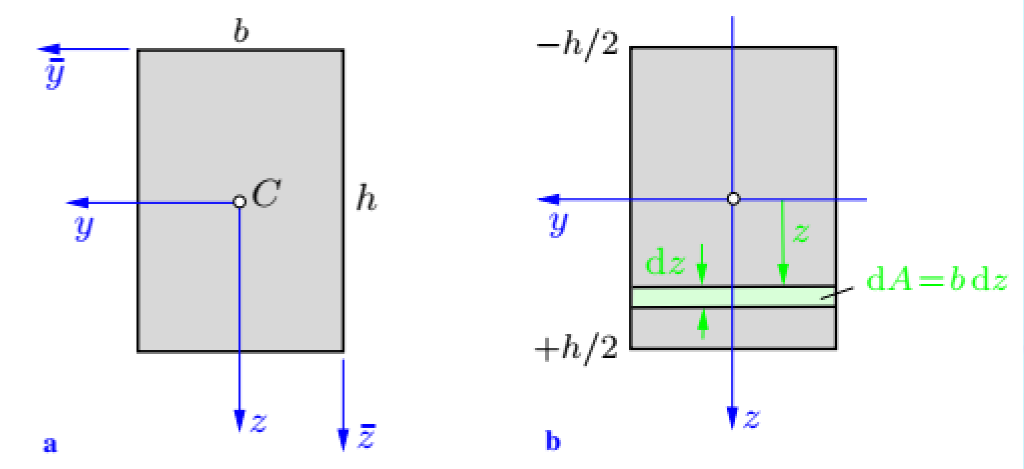
$$M_{max} \leq W\,\sigma_{allow}$$

العزم الأعظمي

Ex. 1 As a first example we consider a rectangular area (width b, height h). The coordinate system with the origin at the centroid C is given; (Fig. a).

In order to determine I_y, we select an infinitesimal area $dA = bdz$ according to (Fig. b) Then every point of the element has the same distance z from the y-axis. Thus, we obtain

$$I_y = \int z^2 dA = \int_{-h/2}^{+h/2} z^2 (bdz) = \frac{b}{3}[z^3]_{-h/2}^{+h/2} = \frac{bh^3}{12}$$



Ex. 2 In a second example we calculate the moments of inertia of a circular area (radius R)

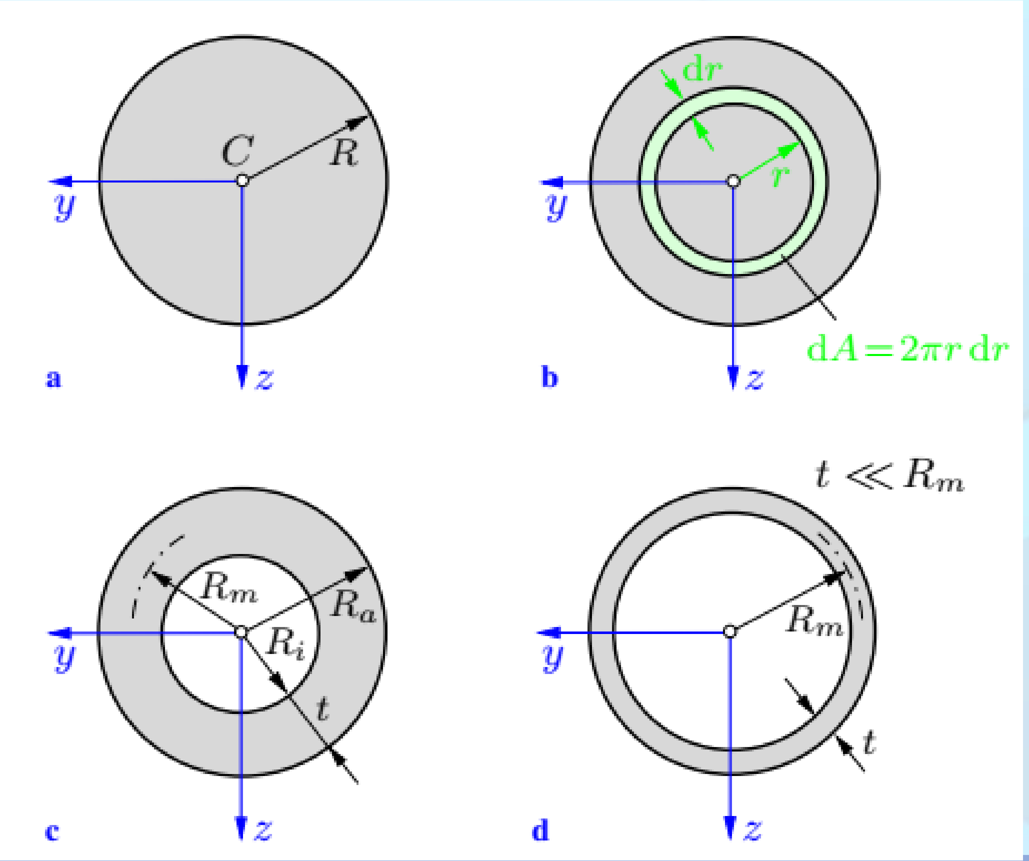
$$I_y = I_z = \frac{1}{2}\int r^2 dA = \frac{1}{2}\int_0^R r^2 (2\pi r dr) = \frac{\pi}{4}R^4$$

Ex. 3 In a third example we calculate the moments of inertia of a ring area (inner radius R_i and outer radius R_a)

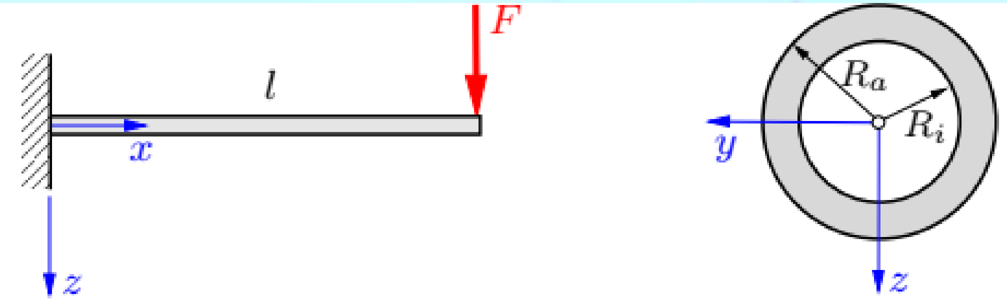
$$I_y = I_z = \frac{\pi}{4}R_a^4 - \frac{\pi}{4}R_i^4 = \pi t R_m (R_m^2 + \tfrac{1}{4}t^2)$$

For the thin ring: $t \ll R_m$

$$I_y = I_z = \pi t R_m^3$$



Example 1 The cross section of a cantilever beam ($l = 3\ m$) consists of a circular ring ($R_i = 4\ cm, Ra = 5\ cm$) The allowable stress is given by $\sigma_{allow} = 150\ MPa$. Determine the allowable value of the load F.



Solution:

$$W = \frac{I_y}{z_{max}} = \frac{\frac{\pi}{4}\left(R_a^4 - R_i^4\right)}{R_a} = \frac{\pi(5^4 - 4^4)}{4(5)} = 57.96\ cm^3 = 57960\ mm^3$$

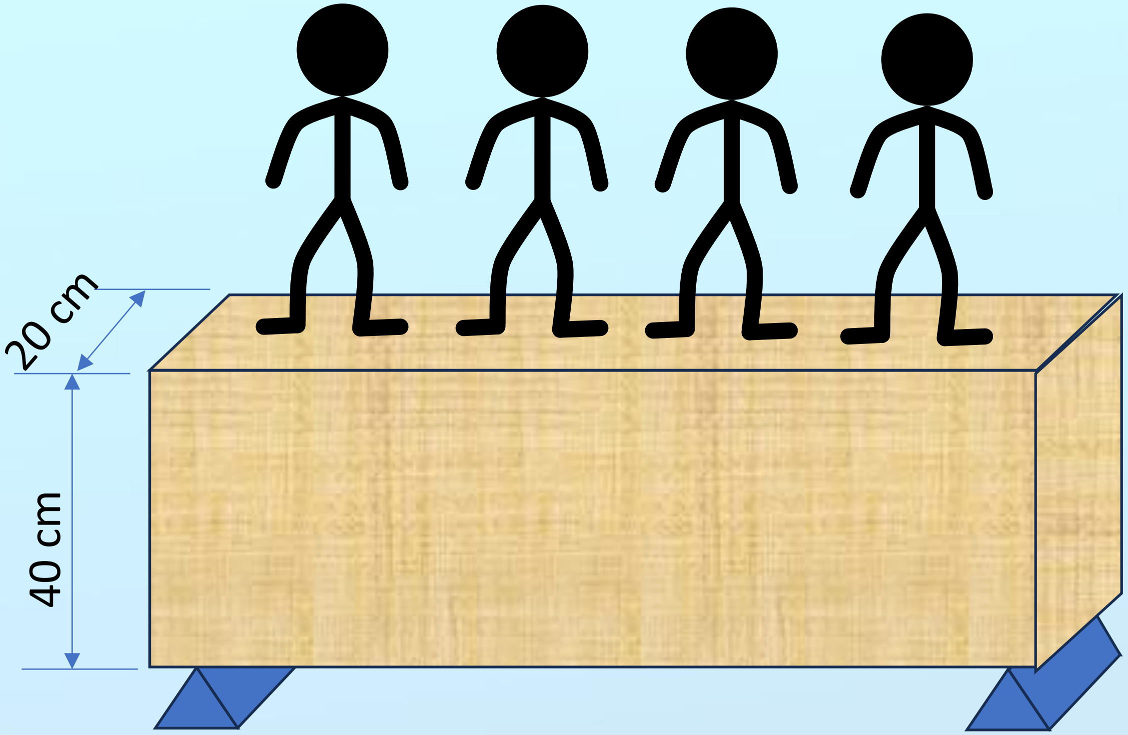
$$M_{allow} = W\sigma_{allow} = 57960 \times 150 = 8694000\ N \cdot mm = 8.694\ kN \cdot m$$

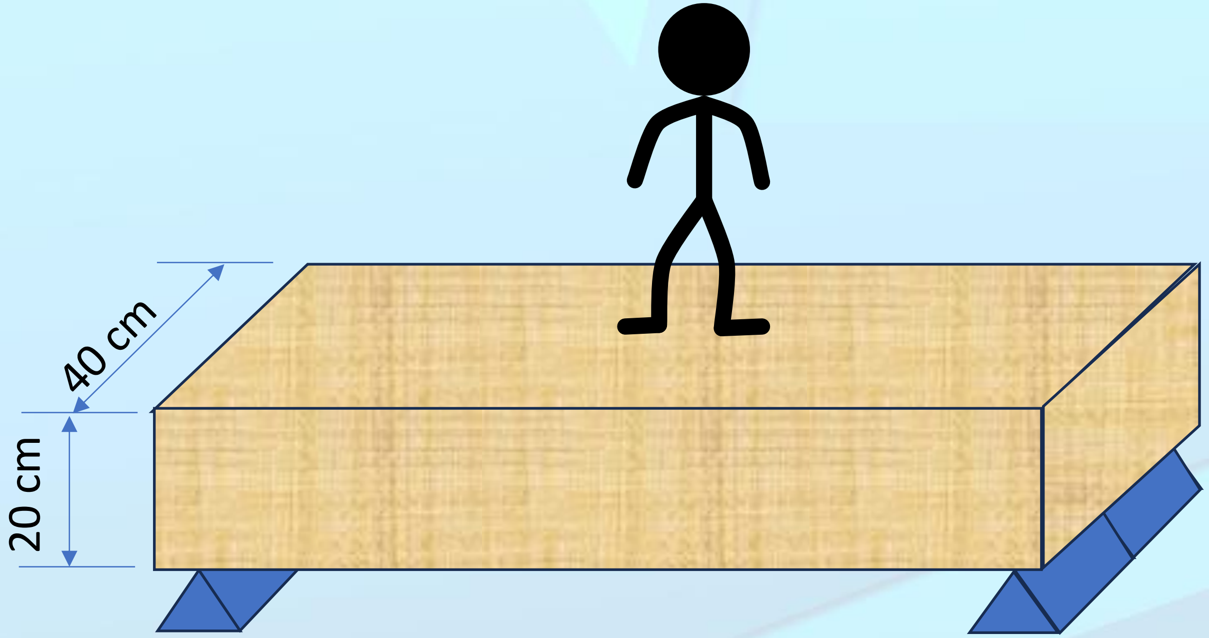
$$M_{max} = Fl \le M_{allow} = 8.694\ kN \cdot m$$

$$F_{allow} l = M_{allow} = 8.694\ kN \cdot m$$

$$F_{allow} = \frac{M_{allow}}{3} = 2.9\ kN$$

$$\sigma = \frac{M}{I_y} z$$





$$I_y = \frac{20 \times 40^3}{12}$$

$$I_y = \frac{40 \times 20^3}{12}$$

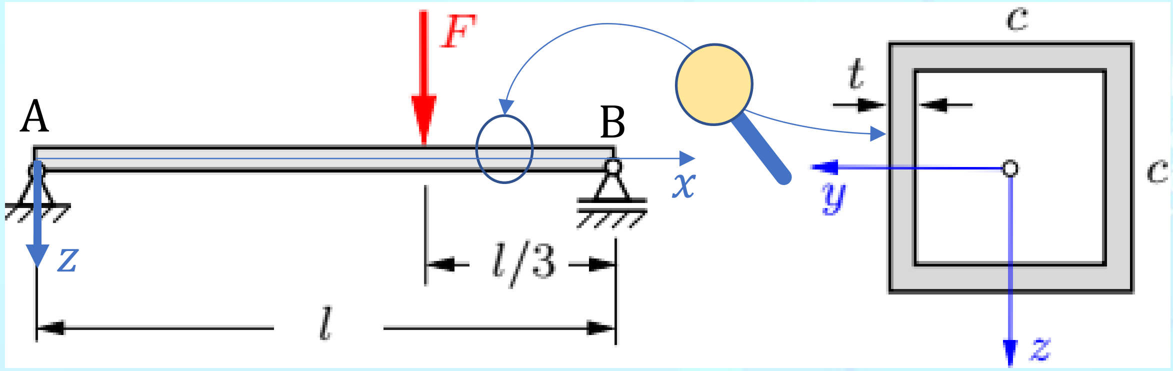
Example 2

The simply supported beam (length $l = 10\ m$) carries the force $F = 200$ kN.

a) Draw the shear force and bending moment diagrams

b) Find the required side length c of the thin-walled quadratic cross section such that the allowable stress $\sigma_{allo} = 200\ MPa$ is not exceeded. The thickness $t = 15\ mm$ of the profile is given.

c) Draw the maximum bending stress diagram



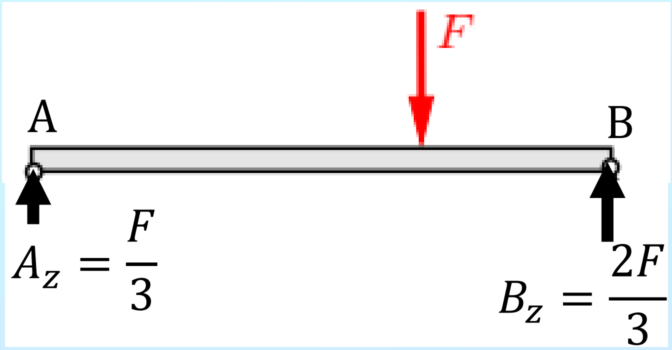
Solution:

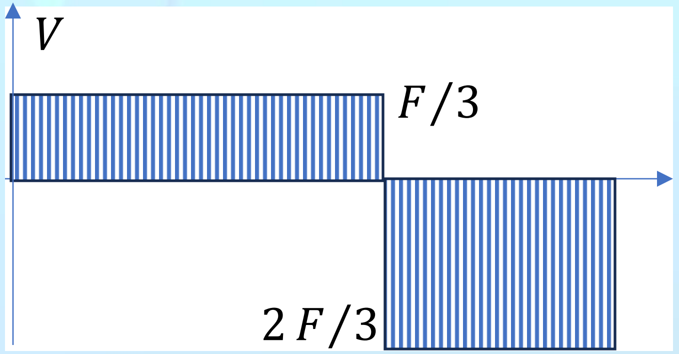
a) Shear force and bending moment diagrams:

a-1) Reactions:

$$\sum M_B = 0: \ -lA_z + (l/3)F = 0 \ \Rightarrow \ A_z = \frac{F}{3}$$

$$\sum M_A = 0: \ lB_z - (2\,l/3)F = 0 \ \Rightarrow \ B_z = \frac{2F}{3}$$





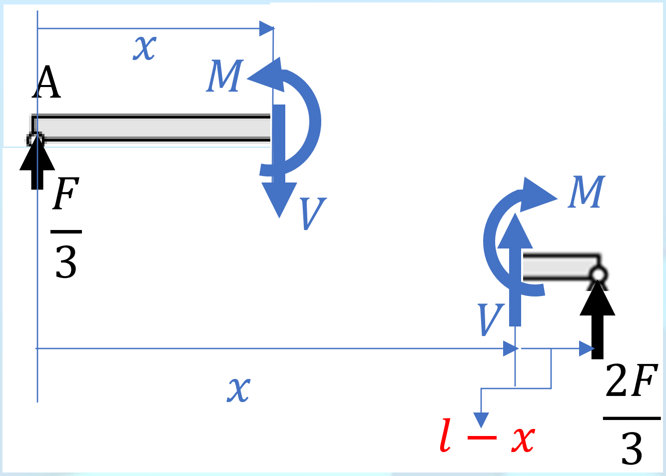
a-2) Cut: $0 < x < 2l/3$: $V = F/3\,,\ M = (F/3)x$

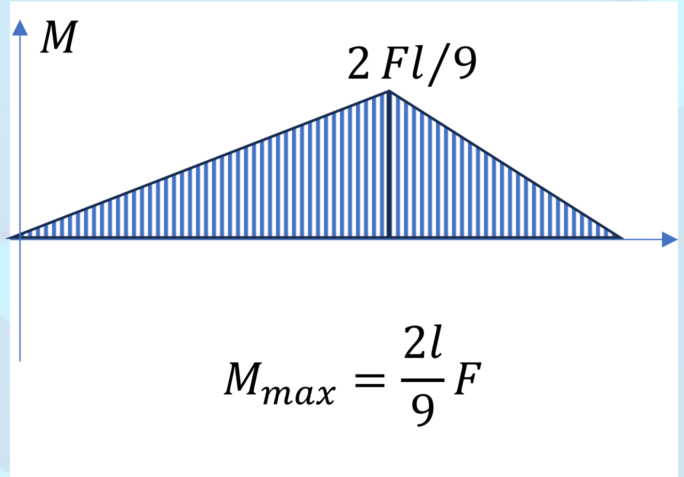
$$x = 0, M = 0; x = 2l/3\,, M = (2l/9)F$$

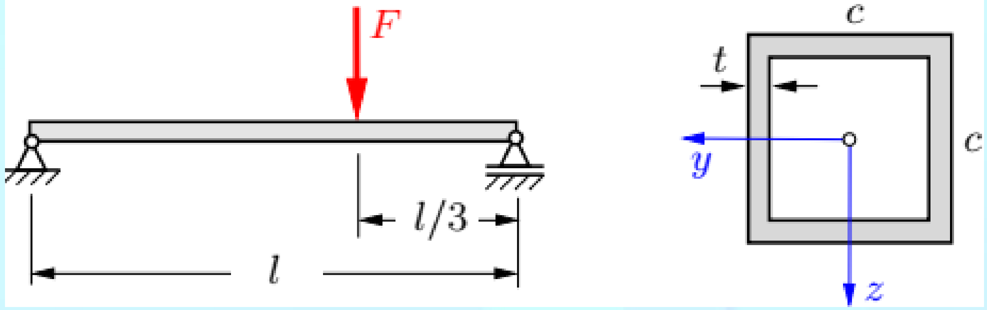
a-3) Cut: $2l/3 < x < l$:

$$V = -2\,F/3\,, M = (2\,F/3)(l - x)$$

$$x = 2l/3\,, M = (2l/9)F; x = l, M = 0$$







b) From the bending moment diagram:

$$M_{max} = \frac{\left(\frac{2l}{3}\right)\left(\frac{l}{3}\right)}{l}F = \left(\frac{2l}{9}\right)F = 444.4\ kN \cdot m$$

$$= 444.4 \times 10^6 N \cdot mm$$

The value of required section modulus is: $W_{req} = \frac{M_{max}}{\sigma_{allo}} = \frac{444.4\times10^6}{200} = 2.222 \times 10^6\ mm^3$

From the shape given in the figure, the section modulus as function of C is: $W = \frac{I_y}{c/2} = \frac{2I_y}{c}$ But for the hollow square section

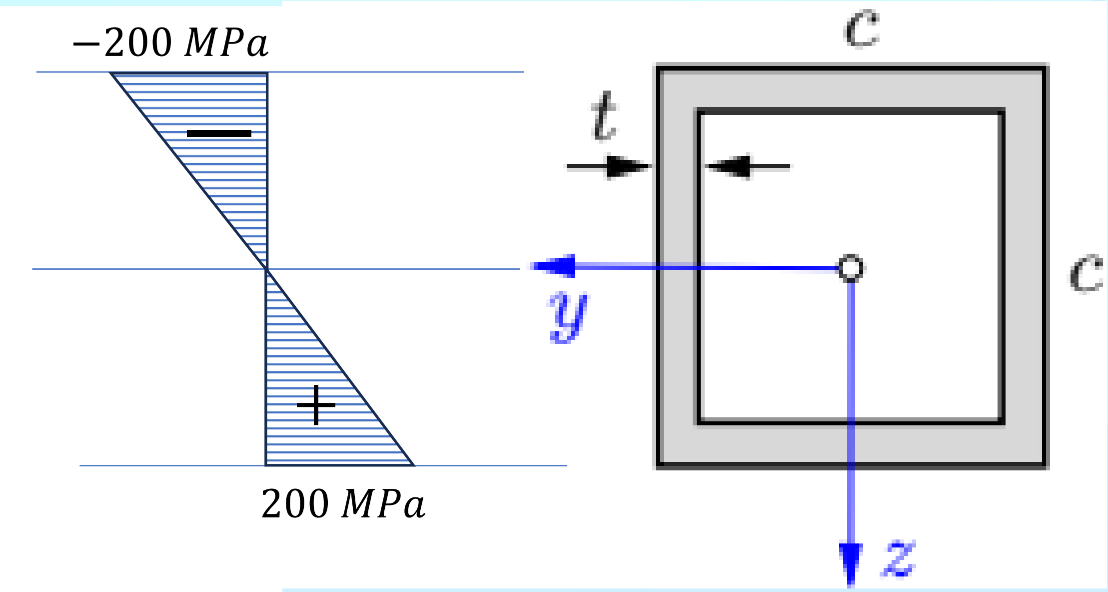
$$I_y = \frac{c^4 - (c-2t)^4}{12} = \frac{[c^2-(c-2t)^2][c^2+(c-2t)^2]}{12} = \frac{(2t)(2c-2t)(2c^2-4ct+4t^2)}{12} = \frac{2t(c-t)(c^2-2ct+2t^2)}{3}$$

$$I_y = \frac{2t(c^3-3c^2t+4ct^2-2t^3)}{3} \Rightarrow W = \frac{4t(c^3-3c^2t+4ct^2-2t^3)}{3c} = \frac{60(c^3-45c^2+900c-6750)}{3c} = 2.222\times10^6$$

$$\Rightarrow c^3 - 45c^2 + 900c - 6750 = \left(\frac{2.222\times10^6}{20}\right)c \quad \Rightarrow c^3 - 45c^2 - 110211c - 6750 = 0$$

$$\Rightarrow c_1 = -310, c_2 = 355, c_3 = -0.061 \qquad \Rightarrow c = 355\ mm$$

C) Diagram of maximum Stress distribution :



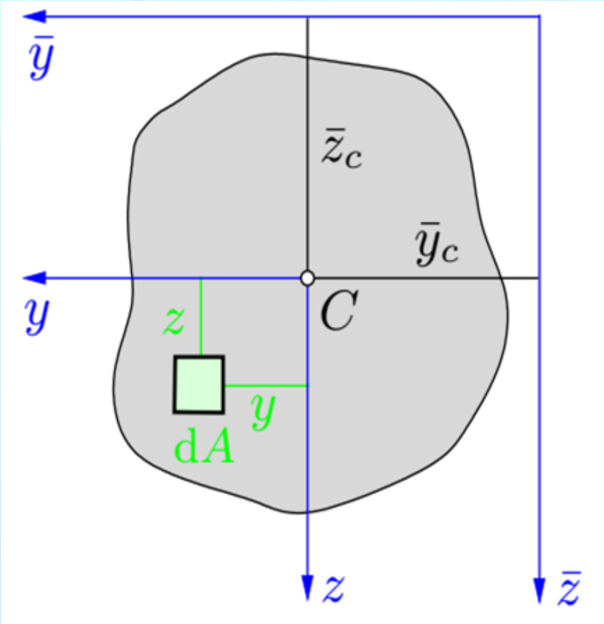
Second Moments of Area

1. Definitions:

The shown coordinate system $\bar{y}, \bar{z}$ is arbitrary. The centroid C is given by

$$\bar{y}_C = \frac{1}{A}\int_A y dA \,, \bar{z}_C = \frac{1}{A}\int_A z dA$$

لتحديد مركز مقطع ما، نبدأ من جملة
إحداثيات اختيارية ثم نستخدم العلاقتين:



First moments of area (Static moments of area) in $\bar{y}, \bar{z}$ system

$$\bar{S}_y = \int_A \bar{z} dA \quad , \quad \bar{S}_z = \int_A \bar{y} dA$$

يُعرف العزمان الأوليان لمساحة المقطع حول المحورين
الكيفيين، ويدعيان أيضاً العزمان الستاتيكيان بالعلاقتين

First moments of area (Static moments of area) in y, z system are null.

$$S_y = \int_A y dA = 0, \qquad S_z = \int_A z dA = 0$$

ينعدم هذان العزمان حول المحورين المركزيين

Second moments of area (Inertia moments of area)

يُعرف العزمان الثانيان حول المحورين المركزيين والعزم القطبي، وجداء العطالة بالعلاقات التالية على الترتيب

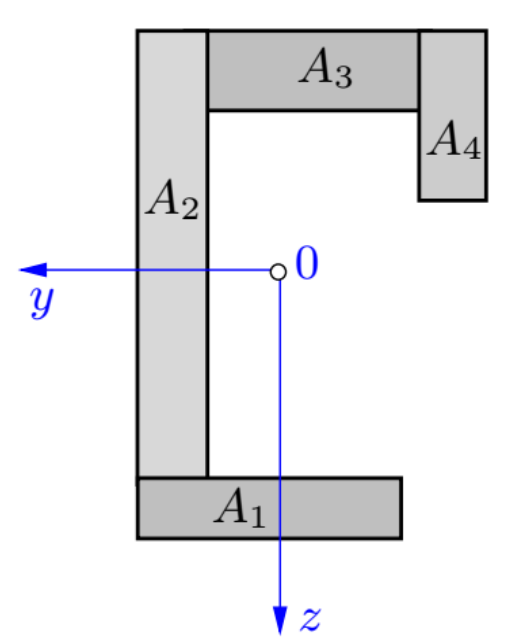
$$I_y = \int_A z^2 dA \qquad I_z = \int_A y^2 dA \qquad I_P = \int_A (y^2 + z^2) dA = I_y + I_z \qquad I_{yz} = I_{zy} = -\int_A yz dA$$

Radii of gyration (Radii plural of radius)

وأخيراً تُعرف أنصاف أقطار العطالة بالعلاقات:

$$r_{gy} = \sqrt{I_y / A} \qquad r_{gz} = \sqrt{I_z / A} \qquad r_{gP} = \sqrt{I_P / A}$$

Frequently, an area A is composed of several parts A_i the moments of inertia of which are known (Fig.). In this case, the moment of inertia about the y-axis, for example, is obtained as the sum of the moments of inertia I_{y_i} of the individual parts about the *same axis*:



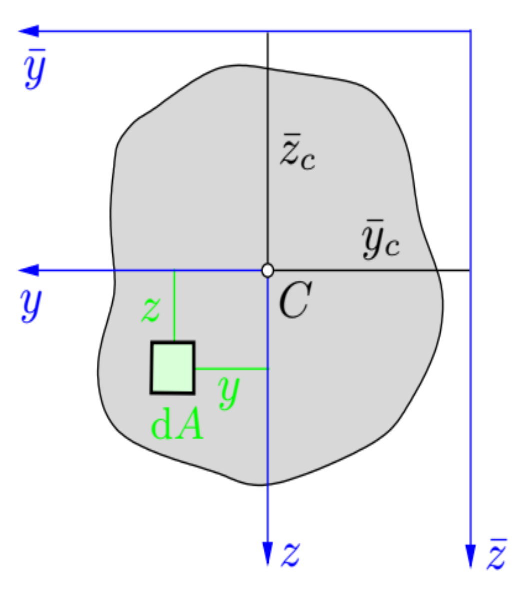
$$I_y = \int_A z^2 dA = \int_{A_1} z^2 dA + \int_{A_2} z^2 dA + \cdot = \sum I_{y_i}$$

$$I_z = \sum I_{z_i}$$

$$I_{yz} = \sum (I_{yz})_i$$

2. Parallel-Axis Theorem

$$\bar{y} = y + \bar{y}_C \qquad \bar{z} = z + \bar{z}_C$$

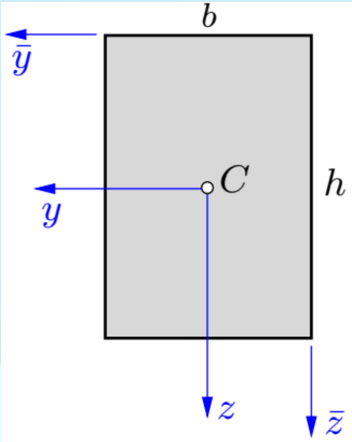


$$I_{\bar{y}} = \int \bar{z}^2 dA = \int (z + \bar{z}_C)^2 dA = \int z^2 dA + 2\bar{z}_C \int z dA + \bar{z}_C^2 \int dA$$

$$I_{\bar{y}} = \int z^2 dA + 2\bar{z}_C(0) + \bar{z}_C^2 A = I_y + \bar{z}_C^2 A$$

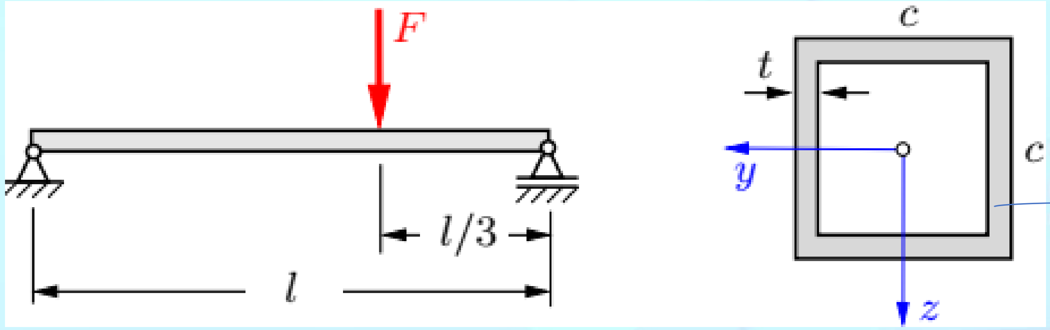
$$I_{\bar{y}} = I_y + \bar{z}_C^2 A \qquad I_{\bar{z}} = I_z + \bar{y}_C^2 A \qquad I_{\bar{y}\bar{z}} = I_{yz} - \bar{y}_C \bar{z}_C A$$

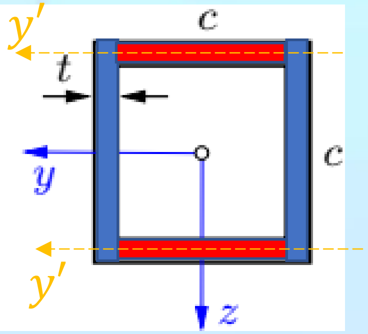
Ex. Determine the moment of inertia with respect to the $\bar{y}$ axis for the shown rectangle.



$$I_{\bar{y}} = \frac{bh^3}{12} + \left(\frac{h}{2}\right)^2 (bh) = \frac{bh^3}{3}$$

Example 2 The simply supported beam (length $l = 10\ m$) carries the force $F = 200\ kN$. Find the required side length c of the thin-walled quadratic cross section such that the allowable stress $\sigma_{allo} = 200\ MPa$ is not exceeded. The thickness $t = 15\ mm$ of the profile is given





Solution:

From the bending moment diagram: $M_{max} = \frac{\left(\frac{2l}{3}\right)\left(\frac{l}{3}\right)}{l} F = \left(\frac{2l}{9}\right) F = 444.4 \times 10^6 N \cdot mm$

The value of required section modulus is: $W_{req} = \frac{M_{max}}{\sigma_{allo}} = \frac{444.4 \times 10^6}{200} = 2.222 \times 10^6\ mm^3$

From the shape given in the figure, the section modulus as function of c is: $W = \frac{I_y}{c/2} = \frac{2I_y}{c}$

$$I_y = \frac{2t(c^3 - 3c^2t + 4ct^2 - 2t^3)}{3}$$

$$I_y = \frac{2t(c^3 - 3c^2t + 4ct^2 - 2t^3)}{3}$$

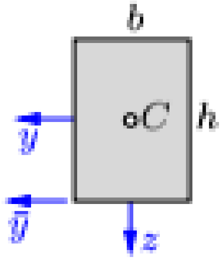
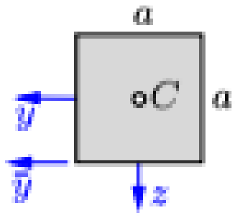
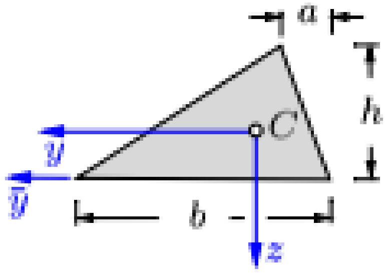
But the inertia moment for the thin-walled section can be simplified as:

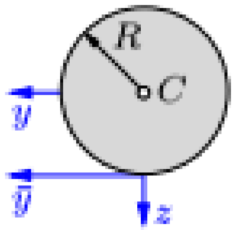
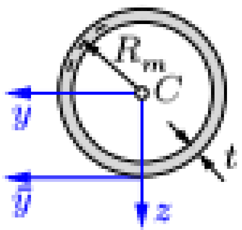
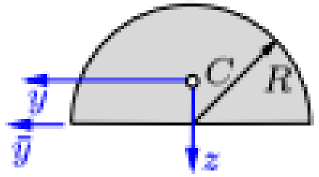
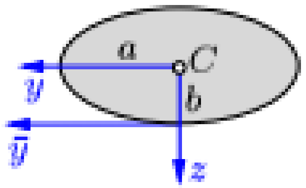
$$I_y = 2\frac{tc^3}{12} + 2tc\left(\frac{c-2t}{2}\right)^2 + 2\frac{(c-2t)t^3}{12} \approx \frac{tc^3}{6} + \frac{tc^3}{2} = \frac{4tc^3}{6} = \frac{2tc^3}{3}$$

$$\Rightarrow W = \frac{2I_y}{c} = \frac{4tc^3}{3c} = \frac{4tc^2}{3} = 2.222 \times 10^6\ mm^3$$

Take $t = 15\ mm$ to get: $20c^2 = 2.222 \times 10^6$

$$\Rightarrow c = \sqrt{0.1111 \times 10^6} = 333\ mm \approx 355\ mm$$

Area	I_y	I_z	I_{yz}	I_p	$I_{\bar{y}}$
Rectangle 	$\frac{b\,h^3}{12}$	$\frac{h\,b^3}{12}$	0	$\frac{b\,h}{12}(h^2+b^2)$	$\frac{b\,h^3}{3}$
Square 	$\frac{a^4}{12}$	$\frac{a^4}{12}$	0	$\frac{a^4}{6}$	$\frac{a^4}{3}$
Triangle 	$\frac{b\,h^3}{36}$	$\frac{b\,h}{36}(b^2-b\,a+a^2)$	$-\frac{b\,h^2}{72}(b-2\,a)$	$\frac{b\,h}{36}(h^2+b^2-b\,a+a^2)$	$\frac{b\,h^3}{12}$

Circle 	$\frac{\pi R^4}{4}$	$\frac{\pi R^4}{4}$	0	$\frac{\pi R^4}{2}$	$\frac{5\pi}{4} R^4$
Thin Circular Ring $t \ll R_m$ 	$\pi R_m^3 t$	$\pi R_m^3 t$	0	$2 \pi R_m^3 t$	$3 \pi R_m^3 t$
Semi-Circle 	$\frac{R^4}{72 \pi}(9 \pi^2 - 64)$	$\frac{\pi R^4}{8}$	0	$\frac{R^4}{36 \pi}(9 \pi^2 - 32)$	$\frac{\pi R^4}{8}$
Ellipse 	$\frac{\pi}{4} a b^3$	$\frac{\pi}{4} b a^3$	0	$\frac{\pi a b}{4}(a^2 + b^2)$	$\frac{5\pi}{4} a b^3$

Ex. 2 Determine the moments of inertia for the I-profile shown in Fig. a. Simplify the results for $d, t \ll b, h$.

Solution We consider the area to be composed of three rectangles (Fig. b).

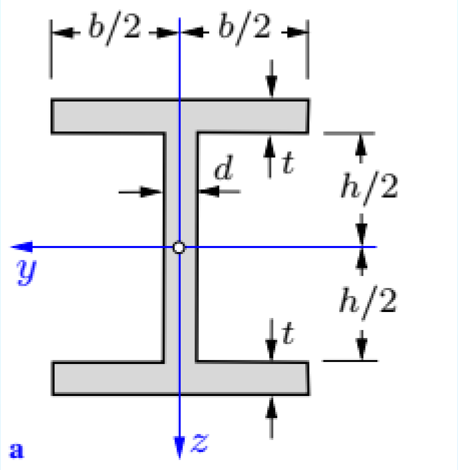
$$I_y = \frac{dh^3}{12} + 2\left[\frac{bt^3}{12} + \left(\frac{t}{2} + \frac{h}{2}\right)^2 bt\right]$$

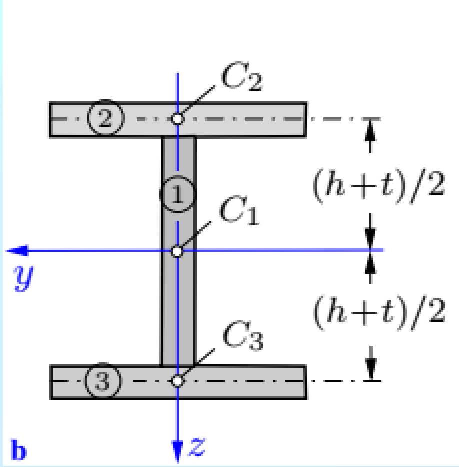
$$= \frac{dh^3}{12} + \frac{bt^3}{6} + \frac{bt^3}{2} + bht^2 + \frac{h^2bt}{2}$$

$$I_y = \frac{dh^3}{12} + \frac{2bt^3}{3} + bht^2 + \frac{h^2bt}{2}$$

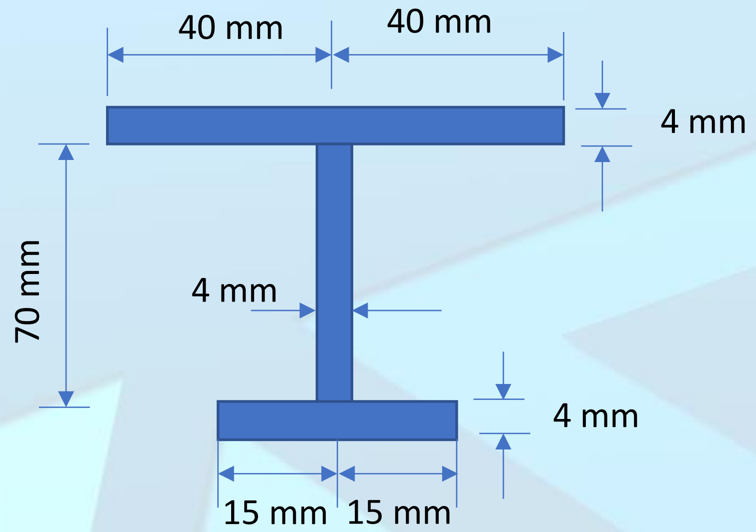
$$\approx \frac{dh^3}{12} + \frac{h^2bt}{2} = \frac{dh^3}{12} + 2\left[\left(\frac{h}{2}\right)^2 bt\right]$$

$$I_z = \frac{ht^3}{12} + 2\frac{tb^3}{12} \approx \frac{tb^3}{6}$$



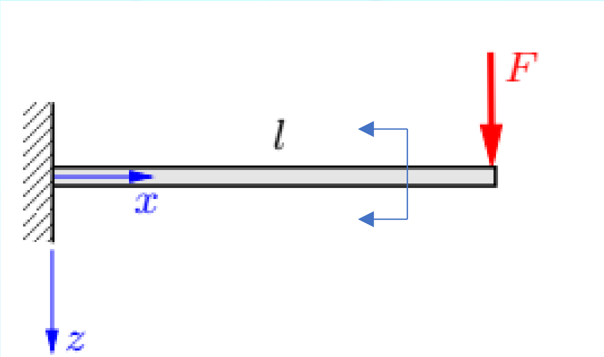


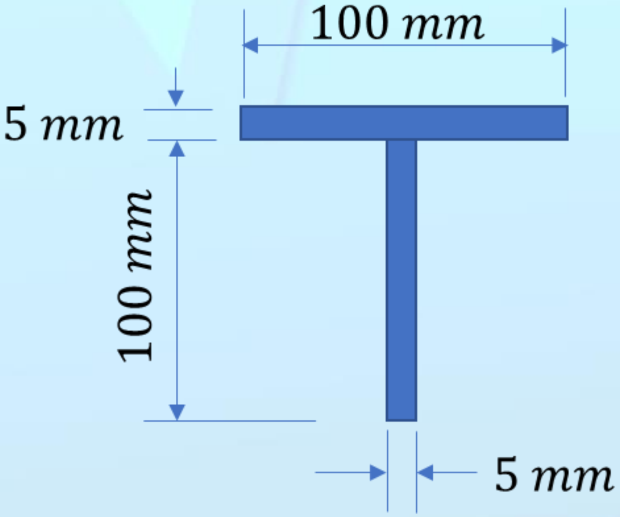
Ex. 3 **Find the centroid then** Determine the moments of inertia for the profile shown in the Fig.



Ex. 3. The cross section of a cantilever beam $l = 2m$, consists of a thin-walled "T" profile. The allowable stress is given by $\sigma_{allow} = 150\ MPa$. Determine

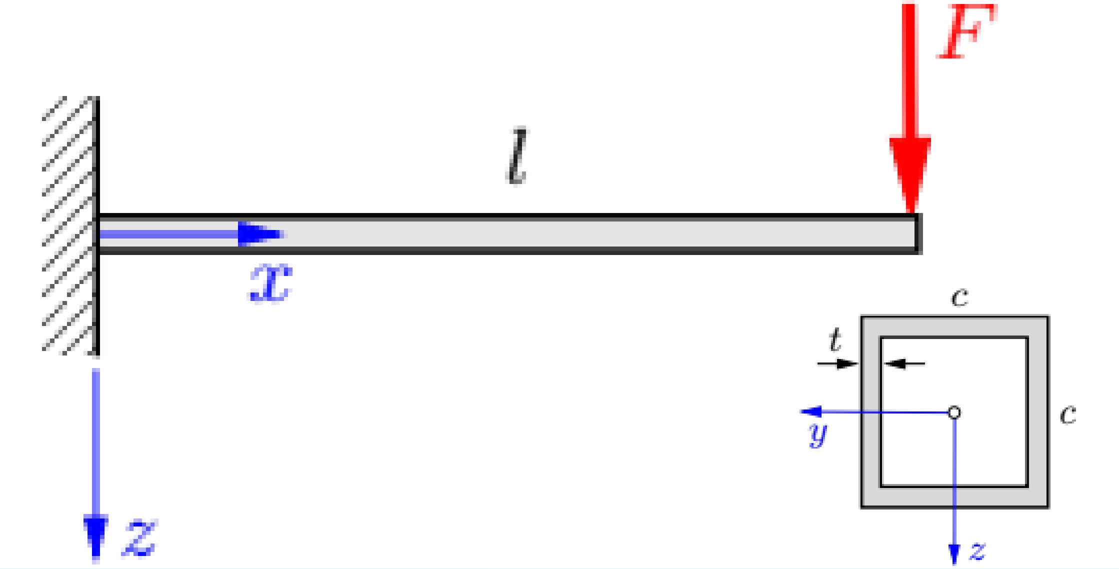
1) The centroid of the section
2) The second moment of the section about the y axis passing through the centroid.
3) The allowable load F.





Ex.4 The cross section of a cantilever beam $l = 2\ m$, consists of a thin-walled square profile with $c = 100\ mm\ \&\ t = 5\ mm$.

1) Find the maximum stress in the beam if the is $F = 4\ kN$.
2) If the allowable stress is $\sigma_{allwa} = 150\ MPa$ Determine the allowable value of F.



Ex. 5. A cantilever beam with the depicted cross section (constant wall thickness $t, t \ll a$, is subjected to a concentrated force F at one end. Determine the maximum stress in the cross section at the support.