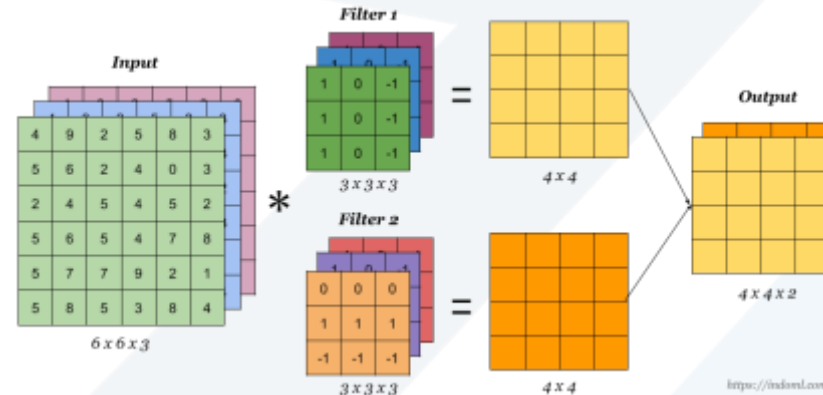


CECC102, CECC122 & CEDC102 : Linear Algebra (and Matrix Theory)

Exercises 3: Determinants



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Use expansion by cofactors to find the determinant of the matrix

$$\textcircled{1} \begin{bmatrix} 1 & 4 & -2 \\ 3 & 2 & 0 \\ -1 & 4 & 3 \end{bmatrix}$$

$$\textcircled{2} \begin{bmatrix} 2 & 4 & 6 \\ 0 & 3 & 1 \\ 1 & 1 & -5 \end{bmatrix}$$

$$\textcircled{3} \begin{bmatrix} x & y & 1 \\ 2 & 3 & 1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\begin{vmatrix} 1 & 4 & -2 \\ 3 & 2 & 0 \\ -1 & 4 & 3 \end{vmatrix} = -3 \begin{vmatrix} 4 & -2 \\ 4 & 3 \end{vmatrix} + 2 \begin{vmatrix} 1 & -2 \\ -1 & 3 \end{vmatrix} - 0 \begin{vmatrix} 1 & 4 \\ -1 & 4 \end{vmatrix} = -3(20) + 2(1) = -58$$

$$\begin{vmatrix} 2 & 4 & 6 \\ 0 & 3 & 1 \\ 1 & 1 & -5 \end{vmatrix} = 2 \begin{vmatrix} 3 & 1 \\ 1 & -5 \end{vmatrix} - 0 \begin{vmatrix} 4 & 6 \\ 1 & -5 \end{vmatrix} + 1 \begin{vmatrix} 4 & 6 \\ 3 & 1 \end{vmatrix} = 2(-15 - 1) + 1(4 - 18) = -46$$



$$\begin{vmatrix} x & y & 1 \\ 2 & 3 & 1 \\ 0 & -1 & 1 \end{vmatrix} = -(-1) \begin{vmatrix} x & 1 \\ 2 & 1 \end{vmatrix} + 1 \begin{vmatrix} x & y \\ 2 & 3 \end{vmatrix} = x - 2 + 3x - 2y = 4x - 2y - 2$$

Use elementary row or column operations to find the determinant

① $\begin{bmatrix} 1 & 7 & -3 \\ 1 & 3 & 1 \\ 4 & 8 & 1 \end{bmatrix}$

② $\begin{bmatrix} 2 & -1 & -1 \\ 1 & 3 & 2 \\ 1 & 1 & 3 \end{bmatrix}$

③ $\begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & -2 \\ 1 & -2 & -1 \end{bmatrix}$

④ $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \\ 0 & 3 & -1 \end{bmatrix}$

$r_{12}^{(-1)}, r_{13}^{(-4)}, r_{23}^{(-5)}$

$$\begin{vmatrix} 1 & 7 & -3 \\ 1 & 3 & 1 \\ 4 & 8 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 7 & -3 \\ 0 & -4 & 4 \\ 4 & 8 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 7 & -3 \\ 0 & -4 & 4 \\ 0 & -20 & 13 \end{vmatrix} = \begin{vmatrix} 1 & 7 & -3 \\ 0 & -4 & 4 \\ 0 & 0 & -7 \end{vmatrix} = (1)(-4)(-7) = 28$$



$$r_{12}, r_{12}^{(-2)}, r_{13}^{(-1)}$$

$$\begin{vmatrix} 2 & -1 & -1 \\ 1 & 3 & 2 \\ 1 & 1 & 3 \end{vmatrix} = - \begin{vmatrix} 1 & 3 & 2 \\ 2 & -1 & -1 \\ 1 & 1 & 3 \end{vmatrix} = - \begin{vmatrix} 1 & 3 & 2 \\ 0 & -7 & -5 \\ 0 & -2 & 1 \end{vmatrix} = -(1) \begin{vmatrix} -7 & -5 \\ -2 & 1 \end{vmatrix} = -(1)(-7 - 10) = 17$$

$$c_{31}^{(1)}$$

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & -2 \\ 1 & -2 & -1 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & -2 & -1 \end{vmatrix} = (2)^{1+1} \begin{vmatrix} -1 & -2 \\ -2 & -1 \end{vmatrix} = (2)(1 - 4) = -6$$

$$r_{23}$$

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \\ 0 & 3 & -1 \end{vmatrix} = - \begin{vmatrix} 1 & 2 & 3 \\ 0 & 3 & -1 \\ 0 & 0 & 4 \end{vmatrix} = -(1)(3)(4) = -12$$



Solve the system of linear equations by each of the methods listed below.

(a) Gaussian elimination with back-substitution

(b) Gauss-Jordan elimination

(c) Cramer's Rule

$$\begin{aligned} x_1 + 2x_2 - x_3 &= -7 \\ \textcircled{1} \quad 2x_1 - 2x_2 - 2x_3 &= -8 \\ -x_1 + 3x_2 + 4x_3 &= 8 \end{aligned}$$

$$\text{(a)} \quad \begin{bmatrix} 1 & 2 & -1 & -7 \\ 2 & -2 & -2 & -8 \\ -1 & 3 & 4 & 8 \end{bmatrix} \xrightarrow{r_{12}^{(-2)}, r_{13}^{(1)}} \begin{bmatrix} 1 & 2 & -1 & -7 \\ 0 & -6 & 0 & 6 \\ 0 & 5 & 3 & 1 \end{bmatrix} \xrightarrow{r_2^{(-1/6)}} \begin{bmatrix} 1 & 2 & -1 & -7 \\ 0 & 1 & 0 & -1 \\ 0 & 5 & 3 & 1 \end{bmatrix} \xrightarrow{r_{23}^{(-5)}}$$

$$\begin{bmatrix} 1 & 2 & -1 & -7 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 3 & 6 \end{bmatrix} \xrightarrow{r_2^{(\frac{1}{3})}} \begin{bmatrix} 1 & 2 & -1 & -7 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\begin{aligned} \text{So, } x_3 &= 2, \quad x_2 = -1, \quad \text{and } x_1 = -7 + 1(2) \\ &\quad - 2(-1) = -3 \end{aligned}$$



$$(b) \begin{bmatrix} 1 & 2 & -1 & -7 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{r_{21}^{(-2)}} \begin{bmatrix} 1 & 0 & -1 & -5 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{r_{31}^{(1)}} \begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

So, $x_1 = -3$, $x_2 = -1$, and $x_3 = -3$

$$(c) \quad A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & -2 & -2 \\ -1 & 3 & 4 \end{bmatrix} \Rightarrow |A| = -18 \quad A_1 = \begin{bmatrix} -7 & 2 & -1 \\ -8 & -2 & -2 \\ 8 & 3 & 4 \end{bmatrix} \Rightarrow |A_1| = 54$$

$$A_2 = \begin{bmatrix} 1 & -7 & -1 \\ 2 & -8 & -2 \\ -1 & 8 & 4 \end{bmatrix} \Rightarrow |A_2| = 18 \quad A_3 = \begin{bmatrix} 1 & 2 & -7 \\ 2 & -2 & -8 \\ -1 & 3 & 8 \end{bmatrix} \Rightarrow |A_3| = -36$$

$$\text{So, } x_1 = \frac{54}{-18} = -3, \quad x_2 = \frac{18}{-18} = -1, \quad x_3 = \frac{-36}{-18} = 2$$



Find the value(s) of k such that A is singular

$$\textcircled{1} \begin{bmatrix} k-1 & 3 \\ 2 & k-2 \end{bmatrix}$$

$$|A| = 0 \Rightarrow \begin{vmatrix} k-1 & 3 \\ 2 & k-2 \end{vmatrix} = (k-1)(k-2) - 2(3) = k^2 - 3k - 4 = 0 \Rightarrow k = -1, 4$$

$$\textcircled{2} \begin{bmatrix} 1 & k & 2 \\ -2 & 0 & -k \\ 3 & 1 & -4 \end{bmatrix}$$

$$|A| = 0 \Rightarrow \begin{vmatrix} 1 & k & 2 \\ -2 & 0 & -k \\ 3 & 1 & -4 \end{vmatrix} = -k \begin{vmatrix} -2 & -k \\ 3 & -4 \end{vmatrix} - 1 \begin{vmatrix} 1 & 2 \\ -2 & -k \end{vmatrix} = -k(8 + 3k) - (-k + 4) = 0$$

$$-3k^2 - 7k - 4 = 0 \Rightarrow k = -4/3, -1$$



Find (a) $|A^T|$, (b) $|A^2|$, (c) $|AA^T|$, (d) $|2A|$, and (e) $|A^{-1}|$

$$\textcircled{1} \quad A = \begin{bmatrix} 5 & 0 & 0 \\ 1 & -3 & 0 \\ 0 & -1 & 2 \end{bmatrix}$$

$$|A| = 5(-3)(2) = -30$$

$$\text{(a)} \quad |A^T| = |A| = -30$$

$$\text{(b)} \quad |A^2| = |A|^2 = (-30)^2 = 900$$

$$\text{(c)} \quad |AA^T| = |A||A^T| = (-30)(-30) = 900$$

$$\text{(d)} \quad |2A| = 2^3 |A| = 8(-30) = -240 \quad \text{(e)} \quad |A^{-1}| = \frac{1}{|A|} = \frac{-1}{30}$$



Use the determinant of the coefficient matrix to determine whether the system of linear equations has a unique solution

$$\textcircled{1} \quad \begin{aligned} -x + y + 2z &= 1 \\ 2x + 3y + z &= -2 \\ 5x + 4y + 2z &= 4 \end{aligned}$$

$$\textcircled{2} \quad \begin{aligned} x + 2y + 6z &= 1 \\ 2x + 5y + 15z &= 4 \\ 3x + y + 3z &= -6 \end{aligned}$$

$$\begin{vmatrix} -1 & 1 & 2 \\ 2 & 3 & 1 \\ 5 & 4 & 2 \end{vmatrix} = -15 \neq 0 \quad \text{The system has a unique solution}$$

$$\begin{vmatrix} 1 & 2 & 6 \\ 2 & 5 & 15 \\ 3 & 1 & 3 \end{vmatrix} = 0 \quad \text{The system does not have a unique solution}$$



Find the adjoint of the matrix A . Then use the adjoint to find the inverse of A (if possible)

$$\textcircled{1} \begin{bmatrix} -3 & -5 & -7 \\ 2 & 4 & 3 \\ 0 & 1 & -1 \end{bmatrix}$$

$$\text{adj}(A) = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} = \begin{bmatrix} \begin{vmatrix} 4 & 3 \\ 1 & -1 \end{vmatrix} & -\begin{vmatrix} -5 & -7 \\ 1 & -1 \end{vmatrix} & \begin{vmatrix} -5 & -7 \\ 4 & 3 \end{vmatrix} \\ -\begin{vmatrix} 2 & 3 \\ 0 & -1 \end{vmatrix} & \begin{vmatrix} -3 & -7 \\ 0 & -1 \end{vmatrix} & -\begin{vmatrix} -3 & -7 \\ 2 & 3 \end{vmatrix} \\ \begin{vmatrix} 2 & 4 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} -3 & -5 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} -3 & -5 \\ 2 & 4 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} -7 & -12 & 13 \\ 2 & 3 & -5 \\ 2 & 3 & -2 \end{bmatrix}$$



$$|A| = \begin{vmatrix} -7 & -12 & 13 \\ 2 & 3 & -5 \\ 2 & 3 & -2 \end{vmatrix} = -3 \Rightarrow A^{-1} = \frac{1}{\det A} \text{adj}(A) = \frac{-1}{3} \begin{bmatrix} -7 & -12 & 13 \\ 2 & 3 & -5 \\ 2 & 3 & -2 \end{bmatrix} = \begin{bmatrix} 7/3 & 4 & -13/3 \\ -2/3 & -1 & 5/3 \\ -2/3 & -1 & 2/3 \end{bmatrix}$$

$$\textcircled{2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 6 \\ 0 & -4 & -12 \end{bmatrix}$$

$$\text{adj}(A) = \begin{bmatrix} \begin{vmatrix} 2 & 6 \\ -4 & -12 \end{vmatrix} & -\begin{vmatrix} 0 & 0 \\ -4 & -12 \end{vmatrix} & \begin{vmatrix} 0 & 0 \\ 2 & 6 \end{vmatrix} \\ -\begin{vmatrix} 0 & 6 \\ 0 & -12 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 0 & -12 \end{vmatrix} & -\begin{vmatrix} 1 & 0 \\ 0 & 6 \end{vmatrix} \\ \begin{vmatrix} 0 & 2 \\ 0 & -4 \end{vmatrix} & -\begin{vmatrix} 1 & 0 \\ 0 & -4 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -12 & -6 \\ 0 & 4 & 2 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 2 & 6 \\ 0 & -4 & -12 \end{vmatrix} = 0 \Rightarrow$$

A has no inverse



Use expansion by cofactors to find the determinant of the matrix

$$\textcircled{1} \begin{bmatrix} 3 & -1 & 2 \\ 4 & 1 & 4 \\ -2 & 0 & 1 \end{bmatrix}$$

$$\textcircled{2} \begin{bmatrix} -3 & 0 & 0 \\ 7 & 11 & 0 \\ 1 & 2 & 2 \end{bmatrix}$$

$$\textcircled{3} \begin{bmatrix} \cos \theta & -r \sin \theta & 0 \\ \sin \theta & r \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Use elementary row or column operations to find the determinant

$$\textcircled{1} \begin{bmatrix} 1 & -3 & 2 \\ 5 & 2 & -1 \\ -1 & 0 & 6 \end{bmatrix}$$

$$\textcircled{2} \begin{bmatrix} 3 & 8 & -7 \\ 0 & -5 & 4 \\ 6 & 1 & 6 \end{bmatrix}$$

$$\textcircled{3} \begin{bmatrix} 3 & -1 & -3 \\ -1 & -4 & -2 \\ 3 & -1 & -1 \end{bmatrix}$$



Solve the system of linear equations by each of the methods listed below.

(a) Gaussian elimination with back-substitution

(b) Gauss-Jordan elimination

(c) Cramer's Rule

$$\textcircled{1} \begin{cases} 2x_1 + x_2 + 2x_3 = 6 \\ -x_1 + 2x_2 - 3x_3 = 0 \\ 3x_1 + 2x_2 - x_3 = 6 \end{cases}$$

$$\textcircled{2} \begin{cases} 2x_1 + 3x_2 + 5x_3 = 4 \\ 3x_1 + 5x_2 + 9x_3 = 7 \\ 5x_1 + 9x_2 + 13x_3 = 17 \end{cases}$$

Find the value(s) of k such that A is singular

$$\textcircled{1} \begin{bmatrix} 0 & k & 1 \\ k & 1 & k \\ 1 & k & 0 \end{bmatrix}$$

$$\textcircled{2} \begin{bmatrix} k & -3 & -k \\ -2 & k & 1 \\ k & 1 & 0 \end{bmatrix}$$



Find (a) $|A^T|$, (b) $|A^2|$, (c) $|AA^T|$, (d) $|2A|$, and (e) $|A^{-1}|$

① $\begin{bmatrix} -2 & 6 \\ 1 & 3 \end{bmatrix}$

② $\begin{bmatrix} 3 & 0 & 1 \\ -1 & 0 & 0 \\ 2 & 1 & 2 \end{bmatrix}$

Find the adjoint of the matrix A . Then use the adjoint to find the inverse of A (if possible)

① $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

② $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \\ 2 & 2 & 2 \end{bmatrix}$